INTEGRATION OF NON-MEASURABLE FUNCTIONS

Elias Zakon

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This is primarily an expository paper based on (and generalizing) some ideas of J. Pierpont [6], W.H. Young [8], R. L. Jeffery [4] and S. C. Fan [2]. Our aim is to give a simple and easily applicable theory of integration for arbitrary extended-real functions over arbitrary sets in a measure space. This will be achieved by using a generalized version of Pierpont's upper and lower integrals (with the upper integral playing the main role), and by appropriately defining the operations in the extended real number system, henceforth denoted by E^*, so as to make it a commutative semigroup under addition and multiplication. It will be seen that many theorems, usually proved for "integrable" functions only, remain valid for arbitrary functions on arbitrary sets; and many proofs and formulations become simpler and stronger because integrability or measurability need not be assumed. Some theorems (e.g. 5.3 and 5.4) are new in the proposed generality, as far as is known to the author.

The paper should be easily understood by first year graduate students or senior undergraduates.

§ 1. PRELIMINARIES. TERMINOLOGY AND NOTATION.

1. The operations and inequalities in E^* are defined as usual (cf. [5], pp. 8-9), with two additional conventions: $(\pm\infty) + (\mp\infty) = (\pm\infty) - (\pm\infty) = +\infty$, and $0.(\pm\infty) = (\pm\infty).0 = 0$. A finite or infinite sum is called orthodox if it does not involve the addition of $+\infty$ and $-\infty$. A non-orthodox sum is always equal to $+\infty$. As is easily seen, these conventions preserve the commutative and associative laws of addition and multiplication [whereas Saks' convention $(+\infty) + (-\infty) = 0$, ([7], p.6), fails to preserve the associativity of addition and hence also the general commutative law]. The distributive law, $x(y+z) = xy + xz$, holds if y and z have the same sign or if $0 \leq x < +\infty$, but may fail otherwise

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[e.g. $(-1)[(+\infty)+(-\infty)] = (-1).(+\infty) = -\infty$ but $(-1)(+\infty)+(-1)(-\infty) = (-\infty)+(+\infty) = +\infty$. Also, $-(x+y) = -x-y$ if $x+y$ is orthodox, and $|x(y+z)| = |xy+xz|$ (always)].

For any sequence $\{a_n\} \subseteq E^*$ we define its "starred sum": $\Sigma^* a_n = \Sigma a_n^+ - \Sigma a_n^-$ where $a_n^+ = \max(a_n, 0)$, $a_n^- = \max(-a_n, 0)$. If orthodox (i.e., if $\Sigma a_n^+ < +\infty$ or $\Sigma a_n^- < +\infty$), it equals the ordinary sum Σa_n since then $\Sigma^* a_n = \Sigma(a_n^+ - a_n^-) = \Sigma a_n$. For finite sequences this holds always, by the commutative, associative and distributive laws (for finite sums and products). Note that $\Sigma^* a_n$ exists always.

II. We denote by m a non-negative countably additive measure defined on a σ-ring $\mathcal{M}$ of subsets of a set S, henceforth fixed. By a σ-ring we mean a non-empty family of subsets of S which is closed under countable unions and differences. If in addition S belongs to it, we call it a σ-field in S (called σ-algebra in [3]). As is well known (cf. [5], p. 99, or [3], p. 42), m can be extended to an outer measure m^* on all sets $A \subseteq S$ by setting $m^*A = \inf\{mX \mid A \subseteq X \in \mathcal{M}\}$, with $m^*A = +\infty$ if A is not contained in any $X \in \mathcal{M}$. This m^* is countably additive when restricted to the σ-field $\mathcal{M}^*$ of all m^*-measurable (briefly, measurable) sets, i.e. sets $A \subseteq S$ such that $m^*X = \mathcal{M}^*(X \cup A) + m^*(X-A)$ for each $X \subseteq S$ (cf. [5], p. 85ff, or [3], p. 41ff). Note that, in our terminology, a measurable set, i.e. a member of $\mathcal{M}^*$, may not belong to $\mathcal{M}$; but $\mathcal{M} \subseteq \mathcal{M}^*$. Moreover, m^* is regular in the sense that every set $A \subseteq S$ has a measurable cover $\bar{A}$ such that $A \subseteq \bar{A} \in \mathcal{M}^*$ and $m^*A = m^*\bar{A}$. Two sets $A, B, \subseteq S$ are said to be separated if they have such covers $\bar{A}, \bar{B}$, with $m^*(\bar{A} \cap \bar{B}) = 0$.

III. By a measurable partition of a set $A \subseteq S$ we mean any finite or countable family P of (not necessarily measurable) disjoint and mutually separated sets A_n such that $A = \bigcup_n A_n$; in particular, P may consist of A alone. "Partition" will mean "measurable partition" unless otherwise stated. The intersection of two partitions $P' = \{A_n\}$ and $P'' = \{B_k\}$ is the family of all sets $A_n \cap B_k$ $(n, k = 1, 2, \ldots)$; we denote it by $P' \cap P''$. The following propositions are easily verified (by replacing all sets by suitable measurable covers); we omit the simple proofs.

1.1. *If* $P = \{A_n\}$ *is a measurable partition of a set* $A \subseteq S$, *then* $m^*A = \sum_n m^*A_n$. *Moreover, for every set* $B \subseteq A$, *the set family* $P = \{B \cap A_n\}$, $n = 1, 2, \ldots$, *is a measurable partition of* B.

1.2. *If* P', P'' *are measurable partitions of* A, *so is* $P' \cap P''$.

1.3. *Let* $P = \{A_n\}$ *be a measurable partition of* A. *For each* n, *let* $\{E_{ni}\}$, $i = 1, 2, \ldots$, *be such a partition of* A_n. *Then* $P' = \bigcup P_n = \{E_{ni}\}$, $n, i = 1, 2, \ldots$, *is a measurable partition of* A, *finer than* P, *i.e. such that each member of* P' *is a subset of some* $A_n \in P$.

IV. All point functions $f, g, h, \ldots$, with values in E^*, are supposed to be defined on *all* of S [otherwise, we always put $f(x_o) = 0$ if f was not originally defined at x_o]. Notation: $f: S \to E^*$. We write "$f \geq g$ on A" if $f(x) \geq g(x)$ for all $x \in A$, and put: $\sup f[A] = \sup_{x \in A} f(x)$, $\inf f[A] = \inf_{x \in A} f(x)$, $A(f > a) = \{x \in A \mid f(x) > a\}$, $A(f \geq a) = \{x \in A \mid f(x) \geq a\}$, $A(f \leq g) = \{x \in A \mid f(x) \leq g(x)\}$, etc. Also, $f \cup g = \max(f, g)$, $f \cap g = \min(f, g)$, $f^+ = f \cup 0$, $f^- = -f \cup 0$ (all defined *pointwise*).

V. Given a function $f: S \to E^*$ and a measurable partition $P = \{A_n\}$ of a set $A \subseteq S$, we define the *upper* and *lower Pierpont sums*, $\bar{S}(f, P)$ and $\underline{S}(f, P)$, as follows:

$$(1.4) \qquad \bar{S}(f, P) = \Sigma^* m^* A_n \cdot \sup f[A_n], \quad \underline{S}(f, P) = \Sigma^* m^* A_n \cdot \inf f[A_n].$$

As previously noted, Σ^* may be replaced by Σ if the sum is orthodox (e.g. if $f \geq 0$ on A) or if P is finite. It easily follows that

$$(1.5) \qquad \bar{S}(f, P) = \bar{S}(f^+, P) - \underline{S}(f^-, P) \geq \underline{S}(f, P) = \underline{S}(f^+, P) - \bar{S}(f^-, P).$$

1.6. *If* P, P' *are measurable partitions of* $A \subset S$, *with* P' *finer than* P, *then* $\underline{S}(f, P) \leq \underline{S}(f, P') \leq \bar{S}(f, P') \leq \bar{S}(f, P)$.

Proof. Let $P = \{A_n\}$ and $P' = \{B_{nk}\}$, with $A_n = \bigcup_k B_{nk}$

for each n. By 1.1, $m^* A_n = \sum_k m^* B_{nk}$. Thus, if $f \geq 0$ on A, $\underline{S}(f, P) = \sum_n m^* A_n \cdot \inf f[A_n] = \sum_{n,k} m^* B_{nk} \cdot \inf f[A_n] \leq \sum_{n,k} \cdot \inf f[B_{nk}] = \underline{S}(f, P')$. Similarly for $\bar{S}(f, P) \geq \bar{S}(f, P')$. The general case reduces to the case $f \geq 0$, by 1.5. Q.E.D.

1.7. <u>For any measurable partitions</u> P' <u>and</u> P'' <u>of a set</u> $A \subseteq S$, <u>and any function</u> $f: S \to E^*$, <u>we have</u> $\underline{S}(f, P') \leq \bar{S}(f, P'')$.

Indeed, by 1.2 and 1.6, $\underline{S}(P') \leq \underline{S}(P' \cap P'') \leq \bar{S}(P' \cap P'') \leq \bar{S}(P'')$.

In the sequel, we write m for m^* and often omit the multiplication sign (.) wherever confusion is ruled out.

§2. DEFINITION AND BASIC PROPERTIES OF THE INTEGRAL.

Given $f: S \to E^*$, with $f \geq 0$ on $A \subseteq S$, we define the <u>upper and lower Pierpont integrals</u>, $\bar{\int}$ and $\underline{\int}$ <u>of</u> f <u>over</u> A (with respect to the measure m) as follows:

$$(2.1) \quad \bar{\int}_A f\, dm = \bar{\int}_A f = \inf_P \bar{S}(f, P), \quad \underline{\int}_A f\, dm = \underline{\int}_A f = \sup_P \underline{S}(f, P)$$

where P ranges over all measurable partitions of A. Note that $\bar{\int}_A f \geq \underline{\int}_A f$, by 1.7. If f is not non-negative on A, we define:

$$(2.2) \quad \bar{\int}_A f = \bar{\int}_A f^+ - \underline{\int}_A f^-, \quad \underline{\int}_A f = \underline{\int}_A f^+ - \bar{\int}_A f^-.$$

EXAMPLE.* Let $f = \chi_E$ where E is a non-measurable subset of $A = [0, 1]$, with $mA = m^*E = m^*(A-E) = 1$. This implies that every set A_n ($mA_n \neq 0$) in any measurable partition $P = \{A_n\}$ of A meets both E and $A-E$, so that $\bar{S}(f, P) = 1$, $\underline{S}(f, P) = 0$. Hence, by (2.1), $\bar{\int}_A f = 1$, $\underline{\int}_A f = 0$. If, instead, $f = \chi_E - \chi_{A-E}$, then a similar argument shows that $\bar{\int}_A f^+ = 1$, $\underline{\int}_A f^- = 0$, $\underline{\int}_A f^+ = 0$, $\bar{\int}_A f^- = 1$. Hence, by (2.2), $\bar{\int}_A f = 1$, $\underline{\int}_A f = -1$.

Also, in the general case, $\bar{\int}_A f \geq \underline{\int}_A f$, since

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* Suggested by the referee.

$\overline{\int}_A f^+ \geq \underline{\int}_A f^+$ and $\overline{\int}_A f^- \geq \underline{\int}_A f^-$. If $\overline{\int}_A f = \underline{\int}_A f$, we say that f is *integrable* on A. If in addition $|\overline{\int}_A f| < +\infty$, we say that f is *strictly integrable* on A. An integral is called *orthodox* if it does not have the form $(+\infty) - (+\infty)$ when represented as in (2.2). This is always the case if the integral is less then $+\infty$ or if f does not change its sign on A.

As will be seen, integrability is not needed for the proof of most properties of the integral. The upper integral possesses them without certain restrictions which apply to the lower integral. Thus it is natural to call $\overline{\int}_A f$ simply "the integral" and denote it by $\int_A f$ [we find it unnecessary to restrict this notation to the integrable case alone]. However, wherever desirable, we shall also use the original notation $\overline{\int}_A f$, along with $\int_A f$. From our definitions we obtain at once:

2.3. *For any functions* $f, g : S \to E^*$ *and any set* $A \subseteq S$, *we have*:

(a) *If* $f \equiv c$ *(constant) on* A *then* $\int_A f = \underline{\int}_A f = m^*A.c$.

(b) *If* $f \equiv 0$ *on* A, *or* $m^*A = 0$, *then* $\int_A f = \underline{\int}_A f = 0$.

(c) *If* $f \geq g$ *on* A *then* $\int_A f \geq \int_A g$ *and* $\underline{\int}_A f \geq \underline{\int}_A g$. Hence:

(d) *If* $f \geq 0$ $(f \leq 0)$ *on* A *then* $\int_A f \geq 0$ *and* $\underline{\int}_A f \geq 0$ (resp. ≤ 0).

(e) *If* $0 \leq p < +\infty$ *then* $\int_A pf = p\int_A f$ *and* $\underline{\int}_A pf = p\underline{\int}_A f$.

(f) $\int_A(-f) = -\underline{\int}_A f$ *and* $\underline{\int}_A(-f) = -\int_A f$ *if one of the two integrals involved in each case is orthodox.* Otherwise, we only have:

(f') $|\int_A(-f)| = |\underline{\int}_A f|$, $|\underline{\int}_A(-f)| = |\int_A f|$, $-\int_A(-f) \leq \underline{\int}_A f$, $-\underline{\int}_A(-f) \leq \int_A f$.

(g) *If* $f \geq 0$ *on* A *and* $A \supseteq B$ *then* $\int_A f \geq \int_B f$. *If in addition* B *is separated from* A-B, *then also* $\underline{\int}_A f \geq \underline{\int}_B f$.

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(h) If $f \geq 0$ on A and $\overline{\int_A} f = 0$ (or if $f \leq 0$ and $\underline{\int_A} f = 0$) then $f = 0$ a.e. on A; i.e., $f \equiv 0$ on $A - D$ for some set D with $m^* D = 0$.

(i) $|\overline{\int_A} f| \leq \overline{\int_A} |f|$, $|\underline{\int_A} f| \leq \overline{\int_A} |f|$ (but not $|\underline{\int_A} f| \leq \underline{\int_A} |f|$, in general).

Indeed, for non-negative functions, all this follows by 2.1 and 1.1 - 1.4 from standard properties of sup and inf. The general case then follows by 2.2. In particular, for the first part of (g), use 2.1 noting that each Pierpont sum $\bar{S}(f, P)$ over A exceeds some $\bar{S}(f, P')$ over B, by 1.1 (2nd part). A somewhat different proof, utilizing the partition $\{B, A-B\}$, yields the 2nd part of (g). To prove (h) for $f \geq 0$, let $D = A(f > 0)$ and $A_n = A(f > 1/n)$, $n = 1, 2, \ldots$. Then $D = \bigcup A_n$. Also, by (g) and (c), $0 = \overline{\int_A} f \geq \overline{\int_{A_n}} f \geq \overline{\int_{A_n}} (1/n) = mA_n . (1/n)$. Thus, for all n, $mA_n = 0$, implying that also $mD = m(\bigcup A_n) = 0$, as required. The case $f \leq o$ in (h) reduces to the case $f \geq o$ by (f). Finally, (i) follows from (c) and (f) since $\pm f \leq |f|$ implies $\overline{\int_A} (\pm f) \leq \overline{\int_A} |f|$ and $\underline{\int_A} (\pm f) \leq \overline{\int_A} |f|$; also, $-\overline{\int_A} f \leq \underline{\int_A} (-f)$. Q. E. D.

Given a function $f : S \to E^*$, we can define two set functions $\bar{s}$ and $\underline{s}$ on all sets $X \subseteq S$ by setting $\bar{s}X = \overline{\int_X} f$ and $\underline{s}X = \underline{\int_X} f$. These set functions are called, respectively, the upper and lower indefinite integrals of f, denoted by $\overline{\int} f$ and $\underline{\int} f$. Our next theorem shows that $\bar{s}$ and $\underline{s}$ are, in a certain sense, countably additive (for measurable partitions). As will be seen in § 3, all members of measurable partitions of a set A constitute a σ-ring $\mathcal{A}$. Thus, if $f \geq 0$, $\bar{s}$ and $\underline{s}$ are measures when restricted to $\mathcal{A}$.

2.4. For any measurable partition $P = \{E_n\}$ of $A \subseteq S$ and any function $f : S \to E^*$, we have:

(a) $\overline{\int_A} f = \sum_n^* \overline{\int_{E_n}} f$ and (b) $\underline{\int_A} f = \sum_n^* \underline{\int_{E_n}} f$ where $\sum^* \overline{\int_{E_n}} f$ stands for $\sum \overline{\int_{E_n}} f^+ - \sum \underline{\int_{E_n}} f^-$, and $\sum^* \underline{\int_{E_n}} f$ stands

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for $\Sigma \underline{\int}_{E_n} f^+ - \Sigma \overline{\int}_{E_n} f^-$. *In both* (a) *and* (b) Σ^* *may be replaced by* Σ *if* P^* *is finite or* $\int_A f$ (*resp.* $\underline{\int}_A f$) *is orthodox*.

Proof. If $f \geq 0$ on A and $\varepsilon > 0$, use 2.1 to find for each E_n a partition $P_n = \{E_{ni}\}$, $i = 1, 2, \ldots,$ with $\bar{S}(f, P_n) \leq \int_{E_n} f + \varepsilon/2^n$. By 1.3, $P = \bigcup P_n$ is a measurable partition of A; thus $\int_A f \leq \bar{S}(f, P) = \sum_n (\sum_i mE_{ni} \cdot \sup f[E_{ni}]) \leq$ $\sum_n (\int_{E_n} f + \varepsilon/2^n) \leq \Sigma \int_{E_n} f + \varepsilon$. As ε is arbitrary, $\int_A f \leq \Sigma \int_{E_n} f$. The reverse inequality is obtained in much the same manner (without introducing an ε), by considering an arbitrary partition $\{A_k\} = P'$ of A and, for each E_n, the partition $P'_n = \{E_n \cap A_k\}$, $k = 1, 2, \ldots$. The proof of (b) for $f \geq 0$ is analogous on noting that all $\underline{\int}_{E_n} f$ may be assumed finite (otherwise, also $\underline{\int}_A f = +\infty$, by 2.3(g)). Finally, the general case easily follows by 2.2. Thus all is proved.

2.5. *The integrals* $\int_A f$ *and* $\underline{\int}_A f$ *do not change if* A *is modified by some set* D *with* $m^*D = 0$, *or* f *is arbitrarily re-defined on* D.

Indeed, D has a measurable cover $\bar{D}$, $m\bar{D} = 0$. Hence $\{A-D, A \cap D\}$ is a measurable partition. By 2.4, $\int_{A-D} f + \int_{A \cap D} f = \int_A f$, with $\int_{A \cap D} f = 0$. Q.E.D.

A function $f: S \to E^*$ is said to be *elementary* on A if f is constant on each set E_n of some measurable partition $\{E_n\} = P^*$ of A. If P^* is finite, f is called *simple*.

2.6. *If the functions* $f, g: S \to E^*$ *are elementary on* $A \subseteq S$, *with* $f \equiv a_n$ *on* A_n $(n = 1, 2, \ldots)$ *for some partition* $P^* = \{A_n\}$, *then:*

(a) $\int_A f = \underline{\int}_A f = \Sigma^* a_n mA_n$ (*thus* f *and* g *are integrable*);

(b) $\int_A f \pm \int_A g = \int_A (f \pm g) = \underline{\int}_A (f \pm g)$, *provided that* $\int_A f$ *or* $\int_A g$ *is finite, or that both integrals and their sum (resp.*

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difference) are orthodox.

Proof. Part (a) is immediate from 2.3(a) and 2.4. For (b), we may assume that $g = b_n$ on A_n, $n = 1, 2, \ldots$, for the same partition $P^* = \{A_n\}$ [otherwise replace P^* by the intersection of two given partitions; this also shows that, if f and g are elementary or simple, so are $f \pm g$, $f \cup g$ and $f \cap g$; hence also $f^+ = f \cup 0$ and $f^- = -f \cup 0$]. Thus $f \pm g = a_n \pm b_n$ on A_n. Also note that if $|\int_A f| < +\infty$ then the series $\Sigma a_n mA_n$ is absolutely convergent and its termwise addition to any other series does not affect the absolute convergence or divergence of the latter, i.e., the finiteness or infiniteness of its positive and negative parts; e.g. $\Sigma(a_n + b_n)^+ = +\infty$ iff $\Sigma b_n^+ = +\infty$. Thus if $\int_A g = \pm\infty$, then $\int_A (f+g) = \int_A g = \pm\infty = \int_A f + \int_A g$. If however both $|\int_A g|$, $|\int_A f| < +\infty$, all reduces to ordinary addition of convergent series. In the orthodox infinite case, a similar proof works on noting that either the positive or negative parts of both series are finite. Thus the theorem is proved.

2.7. For every function $f : S \to E^*$ and any set $A \subseteq S$, we have:

(a) If $\int_A f < q \in E^*$, there is an elementary function $h \geq f$ on A such that $\int_A f \leq \int_A h < q$.

(b) If $\int_A f > p \in E^*$ and if $\underline{\int}_A f$ is orthodox, there is an elementary function $g \leq f$ on A such that $\underline{\int}_A f \geq \int_A g > p$.

(c) Moreover, h and g can be so chosen that, for $x \in A$, $h(x) > f(x) > g(x)$ except if $h(x) = f(x) = g(x) = 0$ or $h(x) = f(x) = +\infty$ [resp. $f(x) = g(x) = -\infty$]; we then say that the inequalities $g \leq f \leq h$ are almost strict, and write $g \prec f \prec h$.

(d) If $f \geq 0$, then also $h \geq 0$ and $g \geq 0$.

Proof. (a) If $q > \int_A f = \int_A f^+ - \underline{\int}_A f^-$ then, by our conventions, $\int_A f^+ < +\infty$. Hence there are $u, v \in E^*$ ($|u| < +\infty$)

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such that $\int_A f^+ < u$ and $-\underline{\int}_A f^- < v \leq q-u$. By 2.1 there is a partition $P = \{A_n\}$ of A such that $u > \bar{S}(f^+, P) = \Sigma m A_n \sup f^+[A_n]$ and $-v < \underline{S}(f^-, P)$ [we can satisfy both conditions for <u>one and the same</u> P, by intersecting two partitions if necessary]. Define two elementary functions h', h'' on A by setting $h' = \sup f^+[A_n]$ and $h'' = \inf f^-[A_n]$ on A_n, $n = 1, 2, \ldots$. Then $h' \geq f^+$, $h'' \leq f^-$ on A and, by 2.6(a), $\int_A h' = \Sigma m A_n \sup f^+[A_n] = \bar{S}(f^+, P) < u$, $\int_A h'' = \underline{S}(f^-, P) > -v$. Then $h_o = h' - h''$ is the elementary function h of assertion (a) since $h_o \geq f^+ - f^- = f$ and, by 2.6(b), $\int_A h_o = \int_A h' - \int_A h'' < u+v < q$.

(c) Now let $C = A(h_o = -\infty)$, $B = A(-\infty < h_o < 0)$ and $D = A(h_o \geq 0)$. As h_o is elementary, $\{B, C, D\}$ is clearly a measurable partition of A. Hence we obtain for each n an elementary function $h_n \geq h_o \geq f$ on A by setting $h_n = -n$ on C, $h_n = (1 - 1/n)h_o$ on B, and $h_n = (1 + 1/n)h_o$ on D. Obviously, $h_n > h_o \geq f$ except where $h_o(x) = 0$ or $+\infty$; thus $h_n \succ f$. By 2.4 and 2.3(e, a), $\int_A h_n = \int_B h_n + \int_C h_n + \int_D h_n = (1 - 1/n) \int_B h_o + (-nmC) + (1 + 1/n) \int_D h_o$. Thus, since $\int_D h_o = \int_D h' - \int_D h'' \leq \int_D h' < +\infty$, all is orthodox, and we have

$$\lim_{n\to\infty} \int_A h_n = \int_B h_o + (-\infty) mC + \int_D h_o = \int_A h_o > q .$$

Hence, for a large n, $\int_A h_n > q$. As $h_n \succ f$, this h_n is the desired h in (c). Furthermore, if $f \geq 0$, we can replace h by $h^+ \geq 0$, with all inequalities only strengthened; this proves (d) for upper integrals. The dual proof for lower integrals works only if $\underline{\int}_A f$ is orthodox [instead, one can also use 2.3(f)].

NOTE. It follows from this proof that, if $f \geq 0$ and $h_n = c_{nk}$ on A_{nk}, $k = 1, 2, \ldots$, then $h_n > \sup f[A_{nk}]$, except where $h_n = c_{nk} = 0 = f$ or where all is $+\infty$. Similarly, for lower integrals, $g_n < \inf f[A_{nk}]$ except where $g_n = f = 0$ or $-\infty$.

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2.8. (a) *If* $\int_A f < +\infty$ *then* $f < +\infty$ *a.e. on* A;

(b) *If* $\underline{\int}_A f$ *is orthodox and* $\underline{\int}_A f > -\infty$ *then* $f > -\infty$ *a.e. on* A;

(c) *If both conditions hold then* $|f| < +\infty$ *a.e. on* A.

Proof. (a) If $\int_A f < +\infty$ then 2.7 yields an elementary function $h \geq f$ with $\int_A h < +\infty$. Let $h = a_n$ on A_n, $n = 1, 2, \ldots$, for some measurable partition of A. By 2.6, $\Sigma^* a_n mA_n = \Sigma a_n^+ mA_n - \Sigma a_n^- mA_n = \int_A h < +\infty$. Hence, by our conventions, we must have $a_n^+ < +\infty$ except if $mA_n = 0$. As $f \leq h \leq h^+$ on A_n, we have $f < +\infty$ except on sets A_n with $mA_n = 0$. This proves (a). The dual (b) follows by 2.3(f).

2.9. *For any functions* $f, g : S \to E^*$ *and any set* $A \subseteq S$, *we have*:

(a) $\int_A f + \int_A g \geq \int_A (f+g)$, *always*;

(b) $\int_A (f+g) \geq \int_A f + \underline{\int}_A g$ *if* $\underline{\int}_A g < +\infty$.

(c) $\int_A f + \underline{\int}_A g \geq \int_A (f+g)$ *if* $\underline{\int}_A f > -\infty$ *and* $\underline{\int}_A g > -\infty$;

(d) $\underline{\int}_A (f+g) \geq \underline{\int}_A f + \underline{\int}_A g$ *if* $\underline{\int}_A f$, $\underline{\int}_A g$ *and* $\underline{\int}_A f + \underline{\int}_A g$ *are orthodox*.

Proof. Suppose that $\int_A f + \int_A g < \int_A (f+g)$. Then there are two numbers $p > \int_A f$ and $q > \int_A g$, with $p + q < \int_A (f+g)$. By 2.7, there are elementary functions $f' \geq f$ and $g' \geq g$ with $p > \int_A f'$ and $q > \int_A g'$. As $f + g \leq f' + g'$, we have [by 2.3(c) and 2.6(b)] that $\int_A (f+g) \leq \int_A (f'+g') = \int_A f' + \int_A g' \leq p + q$, contrary to our choice of p and q. This proves (a).

(b) By our conventions we have $f \leq (f + G) + (-g)$. Hence, by 2.3(c, f) and 2.10(a), $\int_A f \leq \int_A (f+g) + \int_A (-g) = \int_A (f+g) - \underline{\int}_A g$. q.e.d.

(c) We may assume that $\int_A f < +\infty$ and $\underline{\int}_A g < +\infty$. Thus

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we have $-\infty < \underline{\int}_A f \leq \int_A f < +\infty$ and $|\underline{\int}_A g| < +\infty$ (all orthodox!). By 2.8, $|f| < +\infty$ on A, <u>a.e.</u>. By 2.5, we can make f finite on <u>all</u> of A so that $-(f+g) = (-f) + (-g)$ on A. Moreover, using 2.3(f), we have $-(\int_A f + \underline{\int}_A g) = -\int_A f - \underline{\int}_A g = \underline{\int}_A (-f) + \int_A (-g)$ $\leq \int_A (-f-g)$, by (b). Also, by (a), $\int_A (-f-g) \leq \int_A (-f) + \int_A (-g)$ $\leq |\underline{\int}_A f| + |\underline{\int}_A g| < +\infty$, so that $\int_A (-f-g) = -\underline{\int}_A (+f+g)$ by 2.3(f), all being orthodox. Combining, we have $-(\int_A f + \underline{\int}_A g) \leq -\underline{\int}_A (f+g)$, and (c) follows.

From (a) we deduce (d) by 2.3(f), noting that $-(f+g) \leq -f-g$.

2.10. *If the functions* $f, g: S \to E^*$ *are strictly integrable on* A, *so is* $pf + qg$ *for any finite numbers* p, q, *and* $\int_A (pf+qg) = p\int_A f + q\int_A g = \underline{\int}_A (pf+qg)$. *Non-strict integrability of* f, g *implies this formula too, if* $\int_A f$, $\int_A g$ *and the sum* $p\int_A f + q\int_A g$ *are orthodox*.

Indeed, this follows from 2.9(a, d) combined with 2.3(e, f).

2.11. *If* $f \geq 0$ *on* A *and* $A = G \cup H$, *then* $\int_A f \leq \int_G f + \int_H f$. [This fails for lower integrals, along with its generalization 4.3.]

<u>Proof</u>. We may assume that $G \cap H = \emptyset$ (otherwise replace H by H-G). Define two functions $g, h: S \to E^*$ by setting $g = f$ on G, $h = f$ on H, $g \equiv 0$ on S - G and $h \equiv 0$ on S - H. By 2.10(a), $\int_A f = \int_A (g+h) \leq \int_A g + \int_A h$, and it remains to show that $\int_A g = \int_G g = \int_G f$ and $\int_A h = \int_H h = \int_H f$. To achieve this, note that no upper Pierpont sum $\bar{S}(g, P)$, with P a partition of G, changes its value if P is replaced by a corresponding partition of a measurable cover $\bar{G}$ of G; for, since g vanishes outside G, this enlargement of the partition sets does not affect the supremum of g over such sets. By 2.1 and 2.3(g) it easily follows that $\int_G g = \int_{\bar{G}} g$. A similar argument shows that $\int_A g = \int_{\bar{A}} g = \int_{\bar{A}-\bar{G}} g + \int_{\bar{G}} g = 0 + \int_{\bar{G}} g = \int_G g$. Similarly, $\int_A h = \int_{\bar{H}} h = \int_H f$, as required.

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This completes the proof.

REMARKS (Alternative definitions). 1) From 2.3(c) and 2.7 we infer that $\int_A f = \inf_{h \geq f} \int_A h$ and, if $\underline{\int}_A f$ is orthodox, $\underline{\int}_A f = \sup_{g \leq f} \int_A g$, with g, h ranging over the set of elementary functions on A. This could serve as a definition (with 2.6a treated as a definition for elementary f).

2) The proof of 2.7 also shows that formulae 2.1 could be used as a definition also in the general case (instead of 2.2), except when $\underline{\int}_A f$ is not orthodox (then we put $\underline{\int}_A f = +\infty$). In particular, this definition can always be used for bounded functions on sets of finite outer measure. This procedure was adopted by Pierpont in [6], with unbounded functions treated separately by means of a limit process. It is less general since it makes the existence of the integral dependent on the existence of a limit. Pierpont considers only the Euclidean n-space, with m the Lebesgue measure and with separated sets defined in terms of approximating families of cells in E^n.

3) Fan [2] and Jeffery [4] likewise consider only sets of finite outer (or inner) measure. Fan's integral always exists for bounded functions (it is defined in terms of partitions of the y-axis), but it is, in general, neither linear nor monotone nor additive. Unbounded functions again require a separate treatment by means of a limit process.

§ 3. MEASURABILITY AND INTEGRABILITY. A function $f: S \to E^*$ is said to be A-measurable (on a set $A \subseteq S$) if the set $A(f \geq a)$ is separated from $A(f < a)$ for every $a \in E^*$, under the given measure m. A set $X \subseteq A$ is called A-measurable if X is separated from A-X (see § 1). Clearly, the A-measurable sets are exactly all members of the various measurable partitions of A. Instead of "A-measurable", Fan and Jeffery use such terms as "relatively measurable" or "separable". We shall show that this notion reduces to ordinary measurability under a new measure m_A depending on the set A. Our notation is as in § 1.

3.1. LEMMA. For every set $A \subseteq S$, the family $\mathcal{A}$ of all A-measurable sets is a σ-field in A which consists exactly of all sets of the form $\bar{A} \cap X$ $(X \in \mathcal{M}^*)$, with $A = \bar{A} \cap S \in \mathcal{A}$. Moreover, the restriction of m^* to $\mathcal{A}$ is a countably additive

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measure (we call it m_A).

Proof. If C is A-measurable then, by definition, C is separated from $A-C$; so there are sets $X, Y \in \mathcal{M}^*$ with $X \supseteq C$, $Y \supseteq A-C$ and $m(X \cap Y) = 0$. Let $Z = X \cap Y \in \mathcal{M}^*$, $mZ = 0$. Then $C-Z = (A \cap X) - Z$ whence $C = (C-Z) \cup (C \cap Z) = (A \cap X - Z) \cup (C \cap Z) = (A \cap X - Z) \cup (A \cap C \cap Z) = A \cap [(X-Z) \cup (C \cap Z)]$. Thus C has the desired form $A \cap X'$ where $X' = (X-Z) \cup (C \cap Z)$ is measurable since so are X, Z and $C \cap Z$ [the latter because $m^*(C \cap Z) \leq mZ = 0$]. Conversely, if $C = A \cap X$ $(X \in \mathcal{M}^*)$, let $\bar{A}$ be a measurable cover of A. Then $A - C = A - (A \cap X) = A - X \subseteq \bar{A} - X \in \mathcal{M}^*$ and $C = A \cap X \subseteq X \in \mathcal{M}^*$. Thus $A-C$ and C are contained in two disjoint measurable sets $A-X$ and X, respectively. It easily follows that $\bar{A}-C$ is separated from C. Thus, indeed, the A-measurable sets are exactly those of the form $A \cap X$ $(X \in \mathcal{M}^*)$. Such sets however form a σ-field (in A) because $\mathcal{M}^*$ is a σ-field in S. The fact that m^* is countably additive on that σ-field easily follows from 1.1. Q.E.D.

We thus have obtained a measure space $(A, \mathcal{A}, m_A)$, with m_A a measure in A. Since m_A is a restriction of m^*, we may write m^*X for $m_A(X)$ if $X \in \mathcal{A}$. It is also clear that a function $f: S \to E^*$ is A-measurable in the sense defined above if and only if its restriction to A is m_A-measurable (i.e. measurable in the ordinary sense under the measure m_A). It follows that all theorems on measurable functions apply to A-measurable functions, for any (fixed) set $A \subseteq S$. In particular, the sum and product of two A-measurable functions are themselves A-measurable; so also are the pointwise supremum, infimum, $\overline{\lim}$ and $\underline{\lim}$ of any sequence of A-measurable functions; so are all elementary functions on A.

3.2. If $f: S \to E^*$ is A-measurable, it is integrable on A $(A \subseteq S)$.

Proof. It suffices to consider the case $f \geq 0$. Fix any $\varepsilon > 0$ and let $A_\infty = A(f = +\infty)$, $A_n = A((1+\varepsilon)^n \leq f < (1+\varepsilon)^{n+1})$, $n = 0, \pm 1, \pm 2, \ldots$. By the A-measurability of f, all these sets together form a measurable partition of A. Thus we obtain two elementary functions g, h on A, setting $g = h = f = +\infty$ on A_∞, and $g = (1+\varepsilon)^n$, $h = (1+\varepsilon)^{n+1}$ on A_n for each n. Then

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$h = (1+\varepsilon)g$ and $g \leq f \leq h$ on A. Now, if $\int_A g = +\infty$, then also $\overline{\int}_A f = \underline{\int}_A f = +\infty$ and f is integrable, as required. If however $\int_A g < +\infty$ then, by 2.6(b) and 2.3(e), $|\int_A h - \int_A g| = \int_A (h-g)$ $= \varepsilon \int_A g$. Since $\underline{\int}_A g \leq \underline{\int}_A f \leq \overline{\int}_A f \leq \overline{\int}_A h$, we certainly have $|\overline{\int}_A f - \underline{\int}_A f| \leq \varepsilon \int_A g < +\infty$ whence, letting $\varepsilon \to 0$, $\overline{\int}_A f = \underline{\int}_A f$. Q.E.D.

Note that the converse to 3.2 fails. Counterexample: Let m be Lebesgue measure on the line. Let $A = [0, 2]$. Choose an unmeasurable set $B \subseteq [0, 1]$ and put $f \equiv 1$ on B and $f \equiv +\infty$ on A-B. Then $\overline{\int}_A f = \underline{\int}_A f = +\infty$ but f is not A-measurable. However, we have:

3.3 <u>For any</u> $f: S \to E^*$ <u>and</u> $A \subseteq S$, <u>the following are equivalent</u>:

(a) f <u>is strictly integrable on</u> A;

(b) f <u>is A-measurable and</u> $\int_A f$ <u>is finite</u>;

(c) <u>For every</u> $\varepsilon > 0$, <u>there are elementary functions</u> g, h <u>on</u> A <u>such that</u> $g \leq f \leq h$ <u>on</u> A <u>and</u> $\int_A h - \int_A g < \varepsilon$.

Indeed, (c) easily follows from 2.7. Condition (b) implies strict integrability by 3.2. In order to prove the converse, we first establish:

3.4. <u>For any function</u> $f: S \to E^*$ <u>and any set</u> $A \subseteq S$, <u>there is an A-measurable function</u> $h \geq f$ <u>on</u> A, <u>with</u> $\int_A h = \int_A f$. <u>If</u> $\underline{\int}_A f$ <u>is orthodox, there also is an A-measurable function</u> $g \leq f$ <u>on</u> A, <u>with</u> $\int_A g = \underline{\int}_A f$.

<u>Proof</u>. If $\int_A f = +\infty$, the constant function $h = +\infty$ is the required one. If $\int_A f < +\infty$, there is a decreasing sequence of reals $q_n \searrow \int_A f$. By 2.7, there is for each n an elementary function $h_n \geq f$ on A, with $q_n > \int_A h_n \geq \int_A f$. Clearly, all h_n are also A-measurable; hence so is the function $h = \inf h_n$. Moreover, the inequalities $q_n > \int_A h_n \geq \int_A h \geq \int_A f$ imply that

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$\int_A h = \lim q_n = \int_A f$. Thus h is the required A-measurable function. The second assertion is proved similarly. We now return to the proof of 3.3.

If f is strictly integrable then, by 3.4, there are two A-measurable functions g, h with $g \le f \le h$ and $\int_A g = \underline{\int_A} = \int_A f = \int_A h < +\infty$, so that $\int_A (h-g) = 0$ by 2.10 (since g and h are strictly integrable by 3.2). As $h - g \ge h - f \ge 0$, we have $\int_A (h-f) = 0$ whence, by 2.3(h), $h - f = 0$ a.e. on A. This implies that f is A-measurable (for so is h) and the proof of 3.3 is complete.

3.5. <u>If two functions</u> $f, g: S \to E^*$ <u>are strictly integrable on</u> A, <u>so also are</u> $f \cup g$, $f \cap g$, f^+, f^- <u>and</u> $|f|$ (<u>also</u> $f \pm g$, <u>by 2.10</u>).

Indeed, all these functions are A-measurable, hence integrable on A. As $|\int_A f| < +\infty$, $\int_A f^+$ and $\int_A f^-$ are finite; therefore $\int_A |f| \le \int_A f^+ + \int_A f^- < +\infty$ (by 2.9a). Also, $|\int_A (f \cup g)| \le \int_A |f \cup g| \le \int_A (|f| \cup |g|) \le \int_A |f| + \int_A |g| < +\infty$. Thus f^+, f^-, $|f|$ and $f \cup g$ are <u>strictly</u> integrable. Hence so is $f \cap g = -[(-f) \cup (-g)]$. Q.E.D.

Thus the family of all functions which are strictly integrable on A is closed under the operations $\pm$, $\cup$ and $\cap$. This fails for multiplication, but we have:

3.6. ("Mean value theorem"). <u>Let</u> f <u>be A-measurable and bounded on</u> A, <u>with</u> $p = \inf f[A]$, $q = \sup f[A]$. <u>Then, if</u> g <u>is strictly integrable on</u> A, <u>so is</u> fg ; <u>moreover,</u> $\int_A f|g| = a \int_A |g|$ <u>for some real</u> a ($p \le a \le q$). <u>If in addition</u> f <u>has the Darboux property on</u> A [<u>i.e. if the restriction of</u> f <u>to</u> A <u>takes on all intermediate values between</u> f(c) <u>and</u> f(b) <u>for every</u> b, c, $\in$ A], <u>then</u> $\int_A f|g| = f(x_0) \int_A |g|$ <u>for some</u> $x_0 \in A$.

<u>Proof</u>. By assumption, $|f| < k$ for some finite k. As f and g are A-measurable, so is fg; and, as $|\int_A fg| \le k \int_A |g| < +\infty$, fg is strictly integrable (by 3.3). Moreover, if

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$\int_A |g| \neq 0$, then the number $a = \int_A f|g| / \int_A |g|$ is the desired one. If however $\int_A |g| = 0$ then for any $a \in [p, q]$ we have $\int_A f|g| \leq k \int_A |g| = 0 = a \int_A |g|$. Finally, the last assertion is obvious if $p < a < q$. If, instead, $a = p = \inf f[A]$ then $(f-a)|g| \geq 0$ and $\int_A (f-a)|g| = \int_A f|g| - a\int_A |g| = 0$; so, by 2.3(h), $f - a = 0$ a.e. on A. Therefore there must be some $x_o \in A$ with $f(x_o) = a$ (unless $mA = 0$). The proof is similar in case $a = q = \sup f[A]$. Thus the theorem is proved.

§ 4. CONVERGENCE THEOREMS (B. Levi, Fatou, Lebesgue).

Throughout this section, f and f_n denote arbitrary (not necessarily measurable or integrable) functions from S to E^*, and $A \subseteq S$ is arbitrary, too. We write "$f_n \nearrow f$ (ptw.) on A" if the sequence $\{f_n\}$ is non-decreasing on A and if $f_n \to f$ pointwise on A, i.e. $f_n(x) \to f(x)$ for each $x \in A$. Similarly for non-increasing sequences ($f_n \searrow f$) and for a.e. convergence.

4.1. LEMMA. Given $f_n \nearrow f$ (ptw.) on A, there always are A-measurable functions $h_n \geq f$ and $h \geq f$ on A, with $h_n \nearrow h$ (ptw.) on A and $\int_A h_n = \int_A f_n$, $\int_A h = \int_A f$.

Proof. By 3.4, there are A-measurable functions $h' \geq f$ and $h'_n \geq f_n$, with $\int_A h' = \int_A f$ and $\int_A h'_n = \int_A f_n$, $n = 1, 2, \ldots$. Let $h_n = \inf_{k \geq n} (h' \cap h'_k)$ (ptw.) and $h = \sup_n h_n = \lim h_n$ (since $\{h_n\}$ is non-decreasing). Then all h_n and h are A-measurable. Moreover, $h_n \leq h'$ implies $h_n \leq h = \lim h_n \leq h'$ and, for $k \geq n$, $h'_k \geq f_k \geq f_n$ (since $\{f_n\}$ is non-decreasing). Thus $f_n \leq \inf_{k \geq n} (h' \cap h'_k) \leq h \leq h'$, whence $f = \lim f_n \leq h \leq h'$. By 2.3(c),

$$\int_A f_n \leq \int_A h_n \leq \int_A h'_n = \int_A f_n \quad \text{and} \quad \int_A f \leq \int_A h \leq \int_A h' = \int_A f. \text{ Thus}$$

$\int_A f_n = \int_A h_n$, $\int_A f = \int_A h$, Q.E.D.

4.2. (B. Levi). (i) *If* $f_n \nearrow f$ *(a.e.) on* A *and if* $\int_A f_n > -\infty$ *for some* n, *then* $\int_A f_n \nearrow \int_A f$. (ii) *If* $f_n \searrow f$ *(a.e.) on* A *and if* $\overline{\int}_A f_n < +\infty$ *for some* n, *then* $\overline{\int}_A f_n \searrow \overline{\int}_A f$.

Proof. (i) By 2.5, we may assume that $f_n \nearrow f$ on *all* of A and use 4.1 to replace f_n and f by *A-measurable* functions $h_n \nearrow h$, with $\int_A h = \int_A f$ and $\int_A h_n = \int_A f_n$. First, let $h \geq h_n \geq 0$ on A. By 2.3(c), $\int_A f = \int_A h \geq \sup \int_A h_n = \lim \int_A h_n = \lim \int_A f_n$. To prove the converse inequality, suppose that $\lim \int_A h_n = q < \int_A h$. As $\int_A h = \overline{\int}_A h$ is orthodox (for $h \geq 0$), 2.7(c) yields an elementary function $g \subset h$ with $q < \int_A g \leq \int_A h$. Let $A_n = A(h_n \geq g)$, $n = 1, 2, \ldots$. Then $A = \bigcup A_n$, $\{A_n\}$ is increasing, and the sets A_n are A-measurable (for so are the functions h_n and g). But, as we noted in § 2, the indefinite integral $\bar{s} = \int g$ is a measure on such sets. Therefore $\lim \int_{A_n} g = \lim \bar{s}A_n = \bar{s}(\bigcup A_n) = \bar{s}A = \int_A g$. Also, as $g \leq h_n$ on A_n, we have $\int_{A_n} g \leq \int_{A_n} h_n \leq \int_A h_n$. Thus $\int_A g \leq \lim \int_{A_n} h_n = q$ (contradiction!). Thus assertion (i) holds for $h_n \geq 0$.

In the general case, since $\{\int_A h_n\}\uparrow$, no generality is lost by assuming that $\int_A h_n > -\infty$ for *all* n (instead of *some* n), and that $\int_A h_n < +\infty$ (otherwise, $\int_A f = \int_A h \geq \sup \int_A h_n = +\infty$, and all is trivial). Thus, by 3.3(b), all h_n are strictly integrable on A. By 2.8 and 2.5, we may assume that all h_n are finite on A. Since $0 \leq (h_n - h_1) \nearrow (h - h_1)$, we have, by what was proved above and by 2.10,

4.2.1. $$\int_A (h - h_1) = \lim_n \int_A (h_n - h_1) = \lim_n \int_A h_n - \int_A h_1 .$$

Now, if $\int_A (h - h_1) < +\infty$, then $h - h_1$ is strictly integrable and so is $h = (h - h_1) + h_1$. We then have $\int_A (h - h_1) = \int_A h - \int_A h_1$,

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and 4.2.1 yields $\lim \int_A h_n = \int_A h$, i.e., $\int_A f = \lim \int_A f_n$. If however $\int_A (h-h_1) = +\infty$, then 4.2.1 yields $\lim \int_A h_n = +\infty \geq \int_A h$, while the converse inequality is obvious. Thus (i) is proved. Assertion (ii) follows from (i) by 2.3(f), on noting that $f_n \searrow f$ implies $-f_n \nearrow -f$. Q.E.D.

NOTES. 1) The restriction $\int_A f_n > -\infty$ (resp. $\underline{\int}_A f_n < +\infty$) is essential. Counterexample: Let A be the line interval (0, 1), $f \equiv 1$, $f_n = -\infty$ on $(0, 1/n)$ and $f_n = 1$ on $[1/n, 1)$, with m = Lebesgue measure; then $f_n \nearrow f$, $\int_A f_n = -\infty$, but $\int_A f = 1 \neq -\infty$.

2) Unless the functions f_n are integrable, (i) fails for lower integrals and (ii) fails for upper integrals. Example: Let A be the real axis and let $\mathcal{M} = \{A, \emptyset\}$. Define $mA = 1$ and $m\emptyset = 0$, $f_n \equiv 1$ on $(-n, n)$ and $f_n \equiv 0$ on $A-(-n, n)$, so that $f_n \nearrow f = 1$ on A. Then m is a measure on $\mathcal{M}$, $\underline{\int}_A f_n = 0$, $\int_A f_n = 1 = \underline{\int}_A f$. Thus $\lim \int_A f_n = \int_A f$, but $\lim \underline{\int}_A f_n \neq \underline{\int}_A f$.

3) From 4.2 it follows that if $f \geq 0$ on A then <u>the upper indefinite integral</u> $\bar{s} = \overline{\int} f$ <u>is a countably subadditive set function (i.e. an outer measure) when restricted to subsets of A.</u> Indeed, let $E = \bigcup E_n \subseteq A$. For each n define $f_n = f$ on $A_n = \bigcup_{k=1}^{n} E_k$ and $f_n \equiv 0$ on $S-A_n$. Then $f_n \nearrow f$ on E, whence $\bar{s}E = \int_E f = \lim_n \int_E f_n = \lim_n \int_{A_n} f = \lim \bar{s} \bigcup_{k=1}^{n} E_k$, and it remains to show that $\bar{s} \bigcup_{k=1}^{n} E_k \leq \sum_{k=1}^{n} \bar{s} E_k$. This however follows from 2.11. Thus we have obtained:

4.3. <u>If</u> $E = \bigcup_{k=1}^{\infty} E_k$ <u>and if</u> $f \geq 0$ <u>on</u> E <u>then</u> $\overline{\int}_E f \leq \sum_{k=1}^{\infty} \overline{\int}_{E_k} f$. <u>If further the sequence</u> $\{E_k\}$ <u>is increasing, then</u> $\overline{\int}_E f = \lim_{k \to \infty} \overline{\int}_{E_k}$.

4.4 (Fatou). <u>Given a sequence of functions</u> f_n, <u>let</u> $g_n = \inf_{f \geq n} f_k$, $h_n = \sup_{k \geq n} f_k$. <u>Then for any</u> $A \subseteq S$ <u>we have:</u>

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(a) $\overline{\int}_A (\underline{\lim}\, f_n) \leq \underline{\lim}\, \overline{\int}_A f_n \leq \sup_n \overline{\int}_A f_n$ if $\overline{\int}_A g_n > -\infty$ for some n;

(b) $\underline{\int}_A (\overline{\lim}\, f_n) \geq \overline{\lim}\, \underline{\int}_A f_n \geq \inf \underline{\int}_A f_n$ if $\underline{\int}_A h_n < +\infty$ for some n.

Proof. Since $\{g_n\}\uparrow$ and $\int_A g_n > -\infty$, 4.2 yields $\overline{\int}_A \lim g_n = \lim \int_A g_n \leq \underline{\lim}\, \overline{\int}_A f_n$ (for $g_n \leq f_n$). As $\lim g_n = \underline{\lim}\, f_n$, (a) follows. Similarly for part (b). Q.E.D.

4.5 (Lebesgue). If the functions f_n are integrable on A and if $f_n \to f$ (a.e.) on A, then $\overline{\int}_A f = \underline{\int}_A f = \lim_{n\to\infty} \int_A f_n$, provided that there is a function g, with $\underline{\int}_A g < +\infty$ and $g \geq |f_n|$, $n = 1, 2, \ldots,$ on A.

Proof. As $-g \leq f_n \leq g$, we have (with g_n, h_n as in 4.4) $g_n \geq -g$, $h_n \leq g$. Thus $\underline{\int}_A h_n \leq \underline{\int}_A g < +\infty$, $\int_A g_n \geq \int_A (-g) = -\underline{\int}_A g > -\infty$. As $\int_A f_n = \underline{\int}_A f_n$, and $\underline{\lim}\, f_n = \overline{\lim}\, f_n = f$ (a.e.) on A, we have by 4.4: $\underline{\lim} \int_A f_n \geq \int_A \underline{\lim}\, f_n = \overline{\int}_A f \geq \underline{\int}_A f = \underline{\int}_A \overline{\lim}\, f_n \geq \overline{\lim} \int_A f_n$. Q.E.D.

REMARKS. 1) As is well known, every measurable function $f \geq 0$ is the pointwise limit of a non-decreasing sequence of simple functions $f_n \geq 0$ (cf. [5], p. 155) so that $\int_A f = \lim \int_A f_n$, by 4.2(a). Now, $\int_A f_n$ as given in 2.4 coincides with the ordinary Lebesgue integral of f_n, and it follows that this is the case for any Lebesgue integrable function $f \geq 0$. By 2.2, this extends to arbitrary Lebesgue integrable functions: any such function is also integrable in the sense of § 2, and its Lebesgue integral coincides with its Pierpont integral.

2) The approximation by simple functions (from below) applies also to non-negative A-measurable functions. By 4.2 and 3.4, we thus have $\int_A f = \sup \int_A g$, with g ranging over simple functions $(0 \leq g \leq f)$ on A, even when f is not A-measurable.

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It follows that Proposition 2.7, for lower integrals of non-negative functions, holds also with elementary functions replaced by simple functions. This argument fails for upper integrals because an approximation by simple functions from above fails in general to preserve the restriction $\int_A f_n < +\infty$ contained in 4.2(b). However, this is certainly possible for bounded functions on sets of finite measure. This is why theories of integration based on simple functions (or on finite partitions) usually confine themselves to lower integrals of measurable functions or use some kind of "multi-stage" approach (starting with bounded functions). These complications do not occur in our theory.

§5. UNIFORM AND ALMOST UNIFORM CONVERGENCE

We shall now show that Theorem 4.2 holds also with upper and lower integrals interchanged, if the convergence of the functions f_n to f is uniform or "almost uniform". We say that $f_n \to f$ almost uniformly (a.unif.) on A if for every $\varepsilon > 0$ there is an A-measurable set D ($m^*D < \varepsilon$) such that $f_n \to f$ uniformly on $B = A-D$, i.e., $f_n \to f$ (unif.) in the ordinary sense on $B(|f| < +\infty)$ and $f_n \to \pm\infty$ (unif.) in a self-evident sense on $B(f = \pm\infty)$. This is tantamount to uniform convergence on B, with E^* suitably metrized (cf. e.g. [1], 3.3.2). We say that a set $C \subseteq A$ is $A\sigma$-finite if $C = \bigcup A_n$ for some sequence of A-measurable sets A_n with $m^*A_n < +\infty$, $n = 1, 2, \ldots$. Then, as follows from 3.1, the A_n can also be chosen disjoint and separated from each other.

5.1. LEMMA. If $\int_A f < +\infty$ and $f \geq 0$ on A, then $f \equiv 0$ on $A - C$ for some $A\sigma$-finite C.

Indeed, with the notation of the proof of 2.8, we see that none of the numbers mA_n can be infinite unless $h_n = a_n = 0$ on A_n (otherwise $\int_A h = +\infty$). As $h \geq f \geq 0$, f too must vanish, except on some sets A_n with $mA_n < +\infty$. Q.E.D.

5.2. LEMMA. If $f_n \searrow 0$ (unif.) on A, with $\int_A f_n < +\infty$, then $\int_A f_n \searrow 0$.

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Proof. By 5.1, $f_n \equiv 0$ on $A - C_n$ $(n=1, 2, \ldots)$ for some $A\sigma$-finite sets C_n. Thus all f_n vanish on the set $A_o = A - C$ where $C = \bigcup C_n$ is $A\sigma$-finite so that $C = \bigcup A_n$ $(mA_n < +\infty)$ for some disjoint separated A-measurable sets A_n. As $f_n \equiv 0$ on A_o (for all n), $\lim \int_{A_o} f_n = 0$, and it remains to show that $\lim \int_C f_n = 0$ as well. Here we may discard all A_n with $mA_n = 0$; thus we assume $0 < mA_n < +\infty$. Clearly, $\lim \int_C f_n \geq 0$. Seeking a contradiction, we suppose that $\lim \int_C f_n = q > 0$. Then define an elementary function h on C, setting $h = a_n = q/(2^{n+1} mA_n)$ on A_n, $n = 1, 2, \ldots$. Obviously $h > 0$ (strictly) on C and, by 2.6(a), $\int_C h = \sum_n (q/2^{n+1} mA_n) mA_n = \Sigma\, q/2^{n+1} < q$. Note that we can make $\int_C h$ less than any prescribed $q > 0$ by this method. The rest of the argument proceeds exactly as in the proof of 5.3 below, so we omit it here.

5.3. *If in 4.2 the convergence of the functions f_n to f is uniform or almost uniform, then:*

(a) *Part (i) of 4.2 holds also for lower integrals provided that $\underline{\int}_A f$ is orthodox;*

(b) *Part (ii) of 4.2 holds for upper integrals (always).*

Proof. We first consider the case $f_n \searrow f$ (unif.), with all $f_n \geq 0$ on A and $\int_A f \leq \int_A f_n < +\infty$. Clearly, $\int_A f \leq \lim \int_A f_n = q$. Suppose that $\int_A f < q$. Then 2.7 yields an elementary function $h \succ f \geq 0$ on A such that $h > f$ (strictly) except on $A(h = f = 0)$ and $\int_A h < q$. As in 5.2, we may discard the set A_o on which all f_n and h vanish, and then are left with an $A\sigma$-finite set $C \subseteq A$ on which still $h = f = 0$. Note that A_o and C can be made A-measurable because h is elementary. Proceeding as in 5.2, we can re-define h on C so as to make h strictly > 0, i.e., $h > f$ on C as well, with $\int_C h < \lim \int_C f_n$. Moreover, we can make $\int_C h$ so small that $\int_A h = \int_{A-C} h + \int_C h < q = \lim \int_A f_n$. Let $h = c_k$ on A_k $(k = 1, 2, \ldots)$ for some measurable partition $\{A_k\}$ of A. By the Note to 2.7, we may assume that

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$h > \sup f[A_k]$ on A_k. Then the uniform convergence $f_n \searrow f$ yields for each A_k a positive integer n_k $(n_k > n_{k-1})$ such that $f_n < c_k = h$ on A_k for $n > n_k$. By 2.4 and 2.3(c) we have, for $n > n_k$,

$$\int_A f_n = \sum_{i=1}^{\infty} \int_{A_i} f_n = \sum_{i=1}^{n_k - 1} \int_{A_i} f_n + \sum_{i=n_k}^{\infty} \int_{A_i} f_n \leq \sum_{i=1}^{n_k - 1} \int_{A_i} h + \sum_{i=n_k}^{\infty} \int_{A_i} f_1 .$$

Since $q = \lim_{n\to\infty} \int_A f_n \leq \int_A f_n$, we certainly have

$$q \leq \sum_{i=1}^{n_k - 1} \int_{A_i} h + \sum_{i=n_k}^{\infty} \int_{A_i} f_1 . \tag{5.3.1}$$

Now, as $\sum_{i=1}^{\infty} \int_{A_i} f_1 = \int_A f_1 < +\infty$, we have $\lim_{k\to\infty} \sum_{i=n_k}^{\infty} \int_{A_i} f_1 = 0$.

Thus, with $k \to +\infty$, (5.3.1) yields $q \leq \int_A h$, contrary to $\int_A h < q$. This proves (b) for the case $0 \leq f_n \searrow f$ (unif.). Similarly (but without having to use Lemma 5.2) one proves (a) for the case $0 \leq f_n \nearrow f$ (unif.). Replacing f_n and f by $-f_n$ and $-f$, and using 2.3(f), one reduces the case $f_n \leq 0$ to the positive case. Finally, if f_n and f are sign-changing, we use 2.2 to infer that $f_n \nearrow f$ (unif.) implies

$$\underline{\int}_A f_n = \underline{\int}_A f_n^+ - \bar{\int}_A f_n^- \nearrow \underline{\int}_A f^+ - \bar{\int}_A f^- = \underline{\int}_A f .$$

The passage to the limit is legitimate here because $\underline{\int}_A f$ is orthodox, by assumption. Thus (a) is proved for uniform convergence. Similarly for (b), noting that here orthodoxy need not be assumed since it follows from the fact that $\int_A f \leq \int_A f_n < +\infty$.

Next assume that $f_n \nearrow f$ (a.unif.) only. Then, for each integer $k > 0$, there is an A-measurable set $E_k \subseteq A$ such that $m(A - E_k) < 1/k$ and $f_n \nearrow f$ (unif.) on E_k. As $m(A - \bigcup E_i) \leq m(A - E_k) < 1/k$ for <u>all</u> k, we have $m(A - \bigcup E_i) = 0$. Thus, by 2.5, we may assume that $A = \bigcup_k E_k$. We may also assume that the E_k are mutually disjoint [otherwise replace E_k by $E_k - \bigcup_{i=1}^{k-1} E_i$ for $k > 1$]. Then the E_k are also mutually separated (being A-measurable), with $f_n \nearrow f$ (unif.) on each E_k. Hence $\lim_{n\to\infty} \underline{\int}_{E_k} f_n = \underline{\int}_{E_k} f$, $k = 1, 2, 3, \ldots$, by what was proved above. Now, by 2.4 and 2.3(c) we have for any integers $n, p > 0$

$$\underline{\int}_A f_n = \sum_{k=1}^{\infty} \underline{\int}_{E_k} f_n = \sum_{k=1}^{p} \underline{\int}_{E_k} f_n + \sum_{k=p+1}^{\infty} \underline{\int}_{E_k} f_n \geq \sum_{k=1}^{p} \underline{\int}_{E_k} f_n + \sum_{k=p+1}^{\infty} \underline{\int}_{E_k} f_1 .$$

Keeping p fixed, we let $n \to +\infty$ to obtain

$$\lim_{n\to\infty} \underline{\int}_A f_n \geq \sum_{k=1}^{p} \left(\lim_{n\to\infty} \underline{\int}_{E_k} f_n\right) + \sum_{k=p+1}^{\infty} \underline{\int}_{E_k} f_1 = \sum_{k=1}^{p} \underline{\int}_{E_k} f + \sum_{k=p+1}^{\infty} \underline{\int}_{E_k} f_1 .$$

Next, letting $p \to +\infty$ and noting that $\lim_{p\to\infty} \sum_{k=p+1}^{\infty} \underline{\int}_{E_k} f_1 = 0$, we find:

$$\lim_{n\to\infty} \underline{\int}_A f_n \geq \lim_{p\to\infty} \sum_{k=1}^{p} \underline{\int}_{E_k} f = \sum_{k=1}^{\infty} \underline{\int}_{E_k} f = \underline{\int}_A f .$$

Since the converse inequality is obvious the proof of (a) is complete; (b) follows dually by 2.3(f). Q.E.D.

5.4. <u>Let</u> $f_n : S \to E^*$ <u>be arbitrary (not necessarily measurable or integrable) functions with</u> $f_n \to f$ <u>(a.unif.) on a set</u> $A \subseteq S$. <u>Then:</u>

(a) $\lim_{n\to\infty} \int_A f_n = \int_A f$ and (b) $\lim_{n\to\infty} \underline{\int}_A f_n = \underline{\int}_A f$

<u>provided, in both cases, that</u> $|f_n| \leq g$, $n = 1, 2, \ldots,$ <u>on</u> A <u>for some function</u> g <u>with</u> $\int_A g < +\infty$.

<u>Proof.</u> Define g_n and h_n as in 4.4. Then, since $f_n \to f$ (a.unif.), it easily follows that the convergences $g_n \nearrow f$

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and $h_n \searrow f$ are likewise almost uniform on A. Hence by 5.3 the proof of Fatou's theorem 4.4 works also with upper and lower integrals interchanged. Therefore also 4.5 can be proved separately for upper and lower integrals without assuming integrability (the latter was only needed to replace lower integrals by the upper ones). This yields 5.4.

In conclusion we note that the upper integral "behaves" in many respects better than the lower one [cf. 2.3(g, i), 2.7, 2.8, 2.9, 2.11, 4.3, 3.4, 5.3]. It emerges as a convex (sublinear) functional on <u>all</u> extended real functions and becomes linear when restricted to strictly integrable functions. It seems to us that the above exposition is so simple that it can be easily adapted to any course in measure theory and that there is no necessity to limit the theory of integration to measurable functions.

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University of Windsor, Canada

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