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STRONG AND QUASISTRONG DISCONJUGACY

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ABSTRACT. A complex linear homogeneous differential equation of the nth order is called strong disconjugate in a domain G if, for every n points $z_1, \ldots, z_n$ in G and for every set of positive integers, $k_1, \ldots, k_l$, $k_1 + \ldots + k_l = n$, the only solution $y(z)$ of the equation which satisfies

$$y(z_1) = \cdots = y(z_{k_1}) = y^{(k_1)}(z_{k_1+1}) = \cdots = y^{(k_1)}(z_{k_1+k_2}) = \cdots = y^{(k_1+\cdots+k_{l-1})}(z_n) = 0$$

is the trivial one $y(z) \equiv 0$. The equation $y^{(n)}(z) = 0$ is strong disconjugate in the whole plane and for every other set of conditions of the form $y^{(m_k)}(z_k) = 0$, $k = 1, \ldots, n$, $m_1 \leq m_2 \leq \cdots \leq m_n$, there exist, in any given domain, points $z_1, \ldots, z_n$ and nontrivial polynomials of degree smaller than n, which satisfy these conditions. An analogous results holds also for real disconjugate differential equations.

1. Introduction. Let the functions $a_0(z), \ldots, a_{n-1}(z)$ be regular in a simply connected domain G. The differential equation

$$L_n y = y^{(n)}(z) + a_{n-1}(z) y^{(n-1)}(z) + \cdots + a_0(z) y(z) = 0 \tag{1}$$

is called *disconjugate in* G if no (nontrivial) solution has more than $n-1$ zeros in G. (The zeros are counted by their multiplicities.) In [4] a more restrictive notion was introduced. Equation (1) was called *strong disconjugate in* G if, for every choice of n (not necessarily distinct) points $z_1, \ldots, z_n$ in G and every set of positive integers $k_1, \ldots, k_l$ such that $k_1 + \cdots + k_l = n$, the only solution of (1) which satisfies

$$\begin{aligned} y(z_1) = \cdots = y(z_{k_1}) = y^{(k_1)}(z_{k_1+1}) = \cdots = y^{(k_1)}(z_{k_1+k_2}) \\ = \cdots = y^{(k_1+\cdots+k_{l-1})}(z_{k_1+\cdots+k_{l-1}+1}) = \cdots = y^{(k_1+\cdots+k_{l-1})}(z_n) = 0, \end{aligned} \tag{2}$$

is the trivial one $y(z) \equiv 0$. (If a point z^* appears m times as the argument of the same derivative of order $k_1 + \cdots + k_p$, then this point z^* is a zero of $y^{(k_1+\cdots+k_p)}(z)$ of at least multiplicity m.) Strong disconjugacy implies disconjugacy ($k_1 = n$), but in general disconjugacy does not imply strong disconjugacy. For example, the equation $y''(z) + y(z) = 0$ is disconjugate in $|z| < \pi/2$ but is strong disconjugate only in $|z| < \pi/4$. Sufficient conditions for strong disconjugacy were given in [3, 4, 7, 8].

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We write (2) in the form

$$y^{(m_k)}(z_k)=0, \qquad k=1,\ldots,n, \tag{3}$$

where the (ordered) n-tuple $(m_1,\ldots,m_n)$ is given by

$$\begin{gathered} 0=m_1=\cdots=m_{k_1}, \qquad k_1=m_{k_1+1}=\cdots=m_{k_1+k_2},\ldots, \\ k_1+\cdots k_{l-1}=m_{k_1+\cdots+k_{l-1}+1}=\cdots=m_n\left(k_i>0,\, i=1,\ldots,l,\, \sum_{i=1}^{l} k_i=n\right) \end{gathered} \tag{4}$$

A n-tuple $(m_1,\ldots,m_n)$ of nonnegative integers, satisfying (4), will be called *admissible*. There are 2^{n-1} admissible n-tuples. Every other n-tuple of nonnegative integers, satisfying

$$0\le m_1\le m_2\le\cdots\le m_n, \tag{5}$$

will be called *nonadmissible*.

The definition of strong disconjugacy is natural because of the following reasons: (a) the equation $y^{(n)}(z)=0$ is strong disconjugate in the whole plane; (b) for any given nonadmissible n-tuple $(m_1,\ldots,m_n)$ and any given domain G, there exist points $z_1,\ldots,z_n$ in G and a nontrivial solution $y(z)$ of $y^{(n)}(z)=0$ such that (3) holds. Part (a) was proved in [4], but part (b) was only stated there. In Section 2 we prove both parts (Theorem 1). In terms of polynomial interpolation, part (a) means that for any given admissible n-tuple $(m_1,\ldots,m_n)$ and arbitrary (not necessarily distinct) points $z_1,\ldots,z_n$ and values $b_1,\ldots,b_n$, there exists a unique polynomial $y(z)$ of degree at most $n-1$ satisfying

$$y^{(m_k)}(z_k)=b_k, \qquad k=1,\ldots,n. \tag{3'}$$

Part (b) means that for any given nonadmissible n-tuple this assertion is wrong.

In Section 3 we obtain an analogue of Theorem 1 for general real disconjugate equations (Theorem 2).

2. The equation $y^{(n)}(z)=0$. Admissible and nonadmissible n-tuples $(m_1,\ldots,m_n)$ were defined in the introduction. We denote the set of all polynomials of degree not larger than k by P_k.

THEOREM 1. (a) *The equation $y^{(n)}(z)=0$ is strong disconjugate in the whole plane. That means, let $(m_1,\ldots,m_n)$ be admissible and let $z_1,\ldots,z_n$ be an arbitrary set of (not necessarily distinct) points in the plane. If $y(z)\in P_{n-1}$ and*

$$y^{(m_k)}(z_k)=0, \qquad k=1,\ldots,n, \tag{3}$$

then $y(z)\equiv 0$.

(b) *Let the n-tuple $(m_1,\ldots,m_n)$ be nonadmissible and let G be any given domain. Then there exists points $z_1,\ldots,z_n$ in G and a polynomial $y(z)\in P_{n-1}$, $y(z)\not\equiv 0$, such that (3) holds.*

Proof. To prove part (a), let $y(z)\in P_{n-1}$ and assume that (2) holds (i.e. (3) holds for an admissible n-tuple $(m_1,\ldots,m_n)$ and for a given set of, not necessarily distinct, points $z_1,\ldots,z_n$). To prove that $y(z)\equiv 0$, we use induction on n. As $k_1=n$ implies $y(z)\equiv 0$, we may assume that $k_1<n$. We then use the $n-k_1$ last equations of (2) with respect to $y^{(k_1)}(z)\in P_{n-k_1-1}$ and we note that the corresponding $(n-k_1)$-tuple is also admissible. It thus follows by the induction hypothesis that $y^{(k_1)}(z)\equiv 0$. Hence, $y(z)\in P_{k_1-1}$. Using now the k_1 first equations of (2), $y(z_1)=\cdots=y(z_{k_1})=0$, it follows that $y(z)\equiv 0$ and thus we proved part (a).

We remark now that for admissible n-tuples $(m_1,\ldots,m_n)$ the definition (4) implies

$$m_k\le k-1, \qquad k=1,\ldots,n. \tag{6}$$

For the proof of part (b) we use again induction on n. It is easily seen that (b) holds for $n=1$ (and $n=2$). (At the end of this section we bring a list of all n-tuples for $n=1$, 2 and 3.) We thus assume that (b) holds for all r, $r<n$, and prove it now for n. We partition the (infinite) set of non-admissible n-tuples $(m_1,\ldots,m_n)$ into three subsets A, B and C.

$(m_1,\ldots,m_n)$ belongs to set A if $m_n\ge n$. (Such a n-tuple is nonadmissible as (6) does not hold for $k=n$.) We may now choose arbitrary points $z_1,\ldots,z_{n-1}$ in G, and there will always exist $y(z)\in P_{n-1}$, $y(z)\not\equiv 0$, which satisfies the first $n-1$ equations of (3) for the chosen points $z_1,\ldots,z_{n-1}$. As $m_n\geqslant n$, $y^{(m_n)}(z)\equiv 0$, and we thus proved part (b) for the set A (which contains an infinite number of nonadmissible n-tuples.)

For the remaining nonadmissible n-tuples we have

$$0\le m_1\le m_2\le\cdots\le m_n\le n-1. \tag{5'}$$

(As the total number of n-tuples satisfying (5′) is $\binom{2n-1}{n}$, we remain with $\binom{2n-1}{n}-2^{n-1}$ nonadmissible n-tuples satisfying (5′).)

The set B consists of all n-tuples $(m_1,\ldots,m_n)$, satisfying (5′), for which each value $k-1$, $k=1,\ldots,n$, appears at most $n-k$ times in the n-tuple. We choose $n-1$ points $z_1^*,\ldots,z_{n-1}^*$ in our given domain G, so that also their convex hull belongs to G. Let $y^*(z)=\prod_{i=1}^{n-1}(z-z_i^*)$. It follows by the Gauss-Lucas theorem [5] that the $n-k$ zeros of $(y^*)^{(k-1)}$ lie also in G, $k=1,\ldots,n-1$. In case B we thus choose the points z_l, appearing in equation (3) as arguments of $y^{(k-1)}(z_l)=0$, from the set of the $n-k$ zeros of $(y^*)^{(k-1)}$. So $y(z)=y^*(z)$ and the just chosen points $z_1,\ldots,z_n$ satisfy (3), and we thus proved part (b) for the set B. (As we have already proved part (a), this shows that all n-tuples of this set are nonadmissible.)

There remains thus the set C of all nonadmissible n-tuples, satisfying (5′), for which at least one value $r-1$, $r=1,\ldots,n$, appears at least $n-r+1$ times in $(m_1,\ldots,m_n)$. Let us assume that $r-1$ is the largest of these values. (This assumption serves only to define the subsets C_s uniquely.) By (5′) the first number m_k in the n-tuple $(m_1,\ldots,m_n)$ which equals $r-1$ must be m_{r-s}, with $0\le s\le r-1$, since otherwise there are not enough m_k's left which equal $r-1$. We partition the set C into subsets C_s according to these values s.

If $s=0$, then

$$m_r=m_{r+1}=\cdots=m_n=r-1. \tag{7}$$

The case $r=1$ cannot occur, as then (7) would yield the admissible n-tuple $(0,\ldots,0)$. Hence $2\le r\le n$. But then the complementary $(r-1)$-tuple $(m_1,\ldots,m_{r-1})$ is nonadmissible. Indeed, if it were admissible, (7) would imply that the given n-tuple is also admissible. By our induction hypothesis, there exist points $z_1,\ldots,z_{r-1}$ in G and a polynomial $y(z)\in P_{r-2}$, $y(z)\not\equiv 0$, such that

$$y^{(m_k)}(z_k)=0, \qquad k=1,\ldots,r-1. \tag{8}$$

As $y^{(r-1)}(z)\equiv 0$, it follows by (7) and (8) that this $y(z)$ satisfies (3) (for arbitrary $z_r,\ldots,z_n$) and we thus proved part (b) for the subset C_0.

Assume now that $1\le s\le r-1$ (hence $r\ge 2$). As $m_{r-s}=r-1$, it follows that the $(r-s)$-tuple $(m_1,\ldots,m_{r-s})$ is nonadmissible (as (6) does not hold for its last element). By the induction hypothesis, there exist points $z_1,\ldots,z_{r-s}$ in G and a polynomial $y(z)\in P_{r-s-1}$, $y(z)\not\equiv 0$, such that

$$y^{(m_k)}(z_k)=0, \qquad k=1,\ldots,r-s. \tag{8′}$$

As $y^{(m_k)}(z)\equiv 0$ for $k=r-s+1,\ldots,n$, we proved part (b) also for all subsets C_s, $1\le s\le r-1$. This completes the proof of Theorem 1.

We add here the list of n-tuples for $n=1$, 2 and 3. We include only the n-tuples satisfying (5′), so the infinite subset A is missing.

$n=1$: (0) adm.
$n=2$: (0, 0) adm., (0, 1) adm., (1, 1) C_1.
$n=3$: (0, 0, 0) adm., (0, 0, 1) B, (0, 0, 2) adm., (0, 1, 1) adm., (0, 1, 2) adm., (0, 2, 2) C_1, (1, 1, 1) C_1, (1, 1, 2) C_0, (1, 2, 2) C_1, (2, 2, 2) C_2.

We remark that the assertion (b) of the theorem remains correct if the domain G in the plane is replaced by an interval I of the real line and the complex polynomial $y(z)\in P_{n-1}$ by a real polynomial.

3. Real disconjugate equations. Let now $a_0(x),\ldots,a_{n-1}(x)$ be real continuous functions in a compact interval I of the real line. We assume that the differential equation

$$L_n y=y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_0(x)y(x)=0, \tag{9}$$

is disconjugate in I. Disconjugacy of (9) in I is equivalent to the existence of positive functions $v_k(x)$, $k=1,\ldots,n$, such that $v_k\in C^{n-k+1}$ in I and such that the given operator L_ny has the factorization

$$L_ny=v_1\cdots v_n\,D\frac{1}{v_n}D\cdots D\frac{1}{v_2}D\frac{1}{v_1}y,\qquad \left(Dy=\frac{dy}{dx}\right), \tag{10}$$

[6], [1, pp. 91–94].

Given such a factorization of L_ny, we define the kth quasiderivative [2] L_ky by the differential operators

$$L_ky=v_1\cdots v_k\,D\frac{1}{v_k}D\cdots D\frac{1}{v_1}y,\qquad k=1,\ldots,n-1.$$

We also set

$$L_0y=y.$$

Finally, for solutions $y(x)$ of the equation $L_ny=0$, we set

$$L_ky=0,\qquad k\geqslant n,$$

i.e. for such functions y the operator L_ky, $k\geqslant n$, is the null operator.

Using these definitions and conventions, we define: the disconjugate equation $L_ny=0$ is called *quasistrong disconjugate in I* if, for every choice of n (not necessarily distinct) points $x_1,\ldots,x_n$ in I and for every admissible n-tuple $(m_1,\ldots,m_n)$, the only solution of $L_ny=0$ which satisfies

$$L_{m_k}y(x_k)=0,\qquad k=1,\ldots,n, \tag{11}$$

is the trivial one $y(x)\equiv 0$.

THEOREM 2. *Let the differential equation* (9) *be disconjugate in the compact interval I and let* (10) *be a factorization of* L_n.

(a) *The equation* (9) *is quasistrong disconjugate in I. That means, let* $(m_1,\ldots,m_n)$ *be admissible and let* $x_1,\ldots,x_n$ *be arbitrary points in I. If* $y(x)$ *is a solution of* (9) *satisfying* (11) *then* $y(x)\equiv 0$.

(b) *Let the n-tuple* $(m_1,\ldots,m_n)$ *be nonadmissible and let J be any given subinterval of I. Then there exist points* $x_1,\ldots,x_n$ *in J and a nontrivial solution* $y(x)$ *of* (9) *such that* (11) *holds.*

As the proof of Theorem 2 is similar to the proof of Theorem 1, we omit it.

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