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ACTION OF FINITE GROUPS ON REES ALGEBRAS AND GORENSTEINNESS IN INVARIANT SUBRINGS

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Let G be a finite group of order N and assume that G acts on a Cohen-Macaulay local ring A as automorphisms of rings. Let N be a unit in A. For a given G-stable ideal I in A we denote by $\mathcal{R}(I) = \oplus_{n\geq 0} I^n$ and $\mathcal{G}(I) = \oplus_{n\geq 0} I^n/I^{n+1}$ the Rees algebra and the associated graded ring of I, respectively. Then G naturally acts on $\mathcal{R}(I)$ and $\mathcal{G}(I)$ too. In this paper the conditions under which the invariant subrings $\mathcal{R}(I)^G$ of $\mathcal{R}(I)$ are Cohen-Macaulay and/or Gorenstein rings are described in connection with the corresponding ring-theoretic properties of $\mathcal{G}(I)^G$ and the a-invariants $a(\mathcal{G}(I)^G)$ of $\mathcal{G}(I)^G$. Consequences and some applications are discussed.

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1. Introduction

Let A be a commutative ring and G a finite group of order N. We assume G acts on A as automorphisms of rings. Let t be an indeterminable over A. For each ideal I ($I \neq A$) in A we put

$$\begin{aligned} \mathcal{R}(I) &= A[It] \subseteq A[t], \\ \mathcal{R}'(I) &= A[It, t^{-1}] \subseteq A[t, t^{-1}], \text{ and} \\ \mathcal{G}(I) &= \mathcal{R}'(I)/t^{-1}\mathcal{R}'(I) \end{aligned}$$

and call them the Rees algebra, the extended Rees algebra, and the associated graded ring of I, respectively. Now let us extend the action of G on A to that on the Laurent polynomial ring $B = A[t, t^{-1}]$, letting $\sigma(t) = t$ for all $\sigma \in G$. Then if the ideal I is G-stable that is $\sigma(I) \subseteq I$ for any $\sigma \in G$, the algebras $\mathcal{R}(I)$ and $\mathcal{R}'(I)$ remain stable in B under this action of G, so that our group G naturally acts on the associated graded ring $\mathcal{G}(I)$ too. In this paper we are interested in the question how and why certain ring-theoretic properties of $\mathcal{R}(I)^G$ are determined by those of $\mathcal{G}(I)^G$. And our starting point for this research is the following.

Theorem 2.4. *Let A be a Cohen-Macaulay local ring and let I ($\neq A$) be a G-stable ideal in A. Assume the order N of G is invertible in A. Then if* $\mathrm{ht}_A I \geq 1$ (*resp.* $\mathrm{ht}_A I \geq 2$), *the following two conditions are equivalent.*

(1) $\mathcal{R}(I)^G$ *is a Cohen-Macaulay (resp. Gorenstein) ring.*

(2) $\mathcal{G}(I)^G$ *is a Cohen-Macaulay (resp. Gorenstein) ring and* $a(\mathcal{G}(I)^G) < 0$ *(resp.* $a(\mathcal{G}(I)^G) = -2$*).*

Here $a(\mathcal{G}(I)^G)$ *denotes the a-invariant of* $\mathcal{G}(I)^G$.

Now let A be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A \geq 2$. We assume the order N of G is invertible in A. Then in general the ring A contains numerous G-stable ideals and eventually our question is very subtle to handle. Therefore to go farther, in this paper we would like to restrict our attention mainly to the case where $I = \mathfrak{m}$. Let

$$\mathcal{R} = \mathcal{R}(\mathfrak{m}) \quad \text{and} \quad \mathcal{G} = \mathcal{G}(\mathfrak{m}).$$

And with this notation we have from Theorem 2.4 the following.

Theorem 3.4. *Let* $\mathcal{G}$ *be a Gorenstein ring and suppose that* G *trivially acts on the residue class field* $k = A/\mathfrak{m}$ *of* A. *Consider the following three conditions.*

(1) $\mathcal{R}^G$ *is a Gorenstein ring.*

(2) $\mathcal{G}^G$ *is a Gorenstein ring of* $a(\mathcal{G}^G) = -2$.

(3) $\chi_{G,\mathcal{G}} = 1$ *and* $a(\mathcal{G}) = -2$.

Then one has the implications (1) $\Leftrightarrow$ (2) $\Leftarrow$ (3). *Furthermore, if* $a(\mathcal{G}) \leq -2$ *or if* $\mathcal{G}$ *is a normal ring and the extension* $\mathcal{G}/\mathcal{G}^G$ *is divisorially unramified, then the above three conditions are equivalent to each other.*

We shall briefly recall in Section 3 the definition and some basic properties of the canonical character $\chi_{G,\mathcal{G}}$ stated in condition (3) in Theorem 3.4. Instead let us note here two consequences of Theorem 3.4.

Corollary 3.5. *Assume that* $\mathcal{R}$ *is a Gorenstein ring and that* G *trivially acts on the residue class field* $k = A/\mathfrak{m}$ *of* A. *Then the following two conditions are equivalent.*

(1) $\mathcal{R}^G$ *is a Gorenstein ring.*

(2) $\chi_{G,\mathcal{G}} = 1$.

Corollary 3.6. *Suppose that* A *is a regular local ring and* G *trivially acts on the residue class field* $k = A/\mathfrak{m}$. *Let* $\rho : G \to GL(\mathfrak{m}/\mathfrak{m}^2)$ *be the representation of* G *over* k *which is induced from the action on* A. *Then the following two conditions are equivalent.*

(1) $\mathcal{R}^G$ *is a Gorenstein ring.*

(2) $\dim A = 2$ *and* $\rho(\mathcal{G}) \subseteq SL(\mathfrak{m}/\mathfrak{m}^2)$.

We will prove Theorem 2.4 in Section 2. The proof is directly based on the recent progress [4] due to Goto and Nishida in the theory of Rees algebras associated to filtrations of ideals. To check the implications stated in Theorem 3.4 we need a part of the theory of canonical characters $\chi_{G,\mathcal{G}}$, that we shall briefly recall in Section 3. The proof of Theorem 3.4 and its consequences also shall be given in Section 3. In Section 4 we will explore a few examples to illustrate our theorems.

In what follows let G be a finite group of order N which acts on a commutative ring A as automorphisms of rings. We extend the action of G to that on the Laurent polynomial ring $A[t, t^{-1}]$ with $\sigma(t) = t$ for all $\sigma \in G$.

2. Proof of Theorem 2.4

Let I be a G-stable ideal in A. We put $B = A[t, t^{-1}]$, $\mathcal{R} = \mathcal{R}(I)$, $\mathcal{R}' = \mathcal{R}'(I)$, and $\mathcal{G} = \mathcal{G}(I)$. Then G acts on the rings $B, \mathcal{R}, \mathcal{R}'$, and $\mathcal{G}$. For each $i \in \mathbb{Z}$ let $F_i = I^i \cap A^G$. Then the family $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ of ideals in A^G satisfies the following.

Lemma 2.1. (1) $F_i = A^G$ *for* $i \leq 0$.
(2) $F_iF_j \subseteq F_{i+j}$ *for all* $i, j \in \mathbb{Z}$.

We put $\mathcal{R}(\mathcal{F}) = \sum_{i \geq 0} F_i t^i \subseteq A^G[t]$ and $\mathcal{R}'(\mathcal{F}) = \sum_{i \in \mathbb{Z}} F_i t^i \subseteq A^G[t, t^{-1}]$. Then $\mathcal{R}(\mathcal{F})$ and $\mathcal{R}'(\mathcal{F})$ are graded A^G-subalgebras of $A^G[t, t^{-1}]$. Let $\mathcal{G}(\mathcal{F}) = \mathcal{R}'(\mathcal{F})/t^{-1}\mathcal{R}'(\mathcal{F})$.

Proposition 2.2. (1) $\mathcal{R}^G = \mathcal{R}(\mathcal{F})$ *and* $\mathcal{R}'^G = \mathcal{R}'(\mathcal{F})$ *as graded* A^G*-algebras.*
(2) *Suppose that N is invertible in A. Then there is a natural isomorphism*

$$\mathcal{G}^G \cong \mathcal{G}(\mathcal{F})$$

of graded A^G*-algebras.*

Proof. (1) This follows from the fact that $\mathcal{R}^G = \mathcal{R} \cap A^G[t]$ and $\mathcal{R}'^G = \mathcal{R}' \cap A^G[t, t^{-1}]$.

(2) Since N is invertible in A, from the exact sequence $0 \to \mathcal{R}'(1) \xrightarrow{t^{-1}} \mathcal{R}' \xrightarrow{\varepsilon} \mathcal{G} \to 0$ of graded $\mathcal{R}'$-modules we get the exact sequence

$$0 \to \mathcal{R}'^G(1) \xrightarrow{t^{-1}} \mathcal{R}'^G \xrightarrow{\varepsilon} \mathcal{G}^G \to 0$$

of graded $\mathcal{R}'^G$-modules, where ε denotes the canonical epimorphism. (For $x \in \mathcal{R}'$ let $x^* = x \bmod t^{-1}\mathcal{R}'$. Then for each $x^* \in \mathcal{G}^G$ the element $[\sum_{\sigma \in G} \sigma(x)]/N$ of $\mathcal{R}'^G$ is chosen to be the inverse image of x^*.) Hence $\mathcal{G}^G \cong \mathcal{R}'(\mathcal{F})/t^{-1}\mathcal{R}'(\mathcal{F}) = \mathcal{G}(\mathcal{F})$ by (1).

Lemma 2.3. (1) *A in integral over A^G. Hence* $\dim A = \dim A^G$ *and if A is a local ring, then so is A^G.*
(2) *Let A be a Cohen-Macaulay local ring and assume that N is invertible in A. Then* $\mathrm{ht}_A I = \mathrm{ht}_{A^G} I^G$.

Proof. (1) For each $a \in A$ let $f_a(t) = \prod_{\sigma \in G}(t - \sigma(a))$. Then $f_a(t) \in A^G[t]$ and $f_a(a) = 0$. Hence A is integral over A^G and so $\dim A = \dim A^G$. See [**2**, (5.8)] for the second assertion.

(2) By [**8**, Proposition 13] the ring A^G is Cohen-Macaulay. Since our ideal I is G-stable, the group G acts on the local ring A/I as automorphisms of rings. We have $(A/I)^G \cong A^G/I^G$, because N is invertible in A. Hence

$$\dim A = \dim A^G \quad \text{and} \quad \dim A/I = \dim(A/I)^G = \dim A^G/I^G$$

by (1). On the other hand, since both the local rings A and A^G are Cohen-Macaulay, we get

$$\dim A = \dim A/I + \mathrm{ht}_A I \quad \text{and} \quad \dim A^G = \dim A^G/I^G + \mathrm{ht}_{A^G} I^G.$$

Thus $\mathrm{ht}_A I = \mathrm{ht}_{A^G} I^G$.

The next result plays a key role in this paper.

Theorem 2.4. *Let A be a Cohen-Macaulay local ring and I ($\neq A$) a G-stable ideal of A. Assume the order N of G is invertible in A. Then if* $\mathrm{ht}_A I \geq 1$ (*resp.* $\mathrm{ht}_A I \geq 2$), *the following conditions are equivalent.*

(1) *$\mathcal{R}^G$ is a Cohen-Macaulay (resp. Gorenstein) ring.*

(2) *$\mathcal{G}^G$ is a Cohen-Macaulay (resp. Gorenstein) ring of $a(\mathcal{G}^G) < 0$ (resp. $a(\mathcal{G}^G) = -2$).*

Here $a(\mathcal{G}^G)$ denotes the a-invariant of $\mathcal{G}^G$.

Proof. By [**8**, Proposition 13] and (2.3) A^G is a Cohen-Macaulay local ring and $\mathrm{ht}_A I = \mathrm{ht}_{A^G} I^G$. Since $F_1 = I^G$, we get $\mathrm{ht}_{A^G} F_1 \geq 1$ (resp. $\mathrm{ht}_{A^G} F_1 \geq 2$) if $\mathrm{ht}_A I \geq 1$ (resp. $\mathrm{ht}_A I \geq 2$). Therefore by [**4**, Part II, (1.2) and (1.4)], provided $\mathrm{ht}_A I \geq 1$ (resp. $\mathrm{ht}_A I \geq 2$), $\mathcal{R}(\mathcal{F})$ is a Cohen-Macaulay (resp. Gorenstein) ring if and only if $\mathcal{G}(\mathcal{F})$ is a Cohen-Macaulay (resp. Gorenstein) ring and $a(\mathcal{G}(\mathcal{F})) < 0$ (resp. $a(\mathcal{G}(\mathcal{F})) = -2$). Hence the required equivalence follows, since $\mathcal{R}(\mathcal{F}) = \mathcal{R}^G$ and $\mathcal{G}(\mathcal{F}) \cong \mathcal{G}^G$ by (2.2).

3. The case where $I = \mathfrak{m}$

We begin with a survey on canonical characters [**3**]. For a while let G be a finite group of order N and let k be a field. Let $R = \oplus_{n \geq 0} R_n$ be a Noetherian graded ring with $R_0 = k$. We assume the following three conditions.

(i) R is a Gorenstein ring.

(ii) G acts on R as automorphisms of graded k-algebras.

(iii) $N \neq 0$ in k.

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Then by (iii) the graded k-algebra R^G is a Cohen-Macaulay ring. Let K_R and K_{R^G} denote respectively the canonical modules of R and R^G (cf. [6, Sect. 2]). Since the extension R/R^G is module-finite, we have an isomorphism $K_R \cong \mathrm{Hom}_{R^G}(R, K_{R^G})$ of graded R-modules (cf. [6, (2.2.9)]). Therefore $R(a) \cong \mathrm{Hom}_{R^G}(R, K_{R^G})$, because $K_R = R(a)$ by assumption (i) (here $a = a(R)$ denotes the a-invariant of R (cf. [6, (3.1.4)])). Let $L = \mathrm{Hom}_{R^G}(R, K_{R^G})$ and let $\xi \in L_{-a}$ be a generator for the R-module L. Let the group G act on L, setting $\sigma(f) = f \circ \sigma^{-1}$ for $\sigma \in G$ and $f \in L$. Choose a character Ψ of G over k satisfying the equality

$$\sigma(\xi) = \Psi(\sigma)\xi \quad \text{for all } \sigma \in G.$$

Then if one defines the action $*$ of G on $K_R = R(a)$ so that $\sigma * x = \Psi(\sigma)\sigma(x)$ for $x \in K_R = R(a)$ and $\sigma \in G$, any isomorphism $K_R \cong \mathrm{Hom}_{R^G}(R, K_{R^G})$ of graded R-modules is compatible also with G-action. Hence this character Ψ is independent of the choice of the elements $\xi \in L_{-a}$.

Definition 3.1. We put $\chi_{G,R} = \Psi^{-1}$ and call it the *canonical character of G with respect to the action on R.*

Let us summarize below some basic results in [3] on canonical characters. The proof is standard and follows from the fact that any isomorphism $K_R \cong \mathrm{Hom}_{R^G}(R, K_{R^G})$ of graded R-modules is compatible with G-action.

Proposition 3.2 ([3]). *Let $a = a(R)$.*

(1) $K_{R^G} \cong R_{\chi_{G,R}}(a)$ *as graded R^G-modules, where* $R_{\chi_{G,R}} = \{f \in R \mid \sigma(f) = \chi_{G,R}(\sigma)f$ for all $\sigma \in G\}$ *denotes the semi-invariants in R of weight $\chi_{G,R}$.*

(2) $a \geq a(R^G)$.

(3) $\chi_{G,R} = 1$ *if and only if $a(R^G) = a$. When this is the case, R^G is a Gorenstein ring.*

(4) *Let $f \in [R^G]_n$ $(n > 0)$ be R-regular. Then $\chi_{G,R/fR} = \chi_{G,R}$.*

Proposition 3.3 ([10]). *Assume $R = k[X_1, X_2, \ldots, X_d]$ $(d \geq 1)$ is a polynomial ring with $\deg X_i = 1$ for all $1 \leq i \leq d$. Let $V = R_1$ and let $\rho: G \to GL(V)$ denote the representation of G induced from the action on R. Then*

$$\chi_{G,R}(\sigma) = 1/\det(\rho(\sigma))$$

for all $\sigma \in G$.

Now we assume that A is a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A \geq 2$. Let the order N of G be invertible in A and assume G trivially acts on the residue class field $k = A/\mathfrak{m}$ of A. We put $\mathcal{R} = \mathcal{R}(\mathfrak{m})$ and $\mathcal{G} = \mathcal{G}(\mathfrak{m})$.

Firstly we shall prove the following, in which the equivalence of conditions (1) and (2) directly follows from Theorem 2.4.

Theorem 3.4. *Let $\mathcal{G}$ be a Gorenstein ring. Consider the following three conditions:*

(1) *$\mathcal{R}^G$ is a Gorenstein ring.*

(2) *$\mathcal{G}^G$ is a Gorenstein ring of $a(\mathcal{G}^G) = -2$.*

(3) *$\chi_{G,\mathcal{G}} = 1$ and $a(\mathcal{G}) = -2$.*

Then one has the implications (1) $\Leftrightarrow$ (2) $\Leftarrow$ (3). Furthermore, if $a(\mathcal{G}) \leq -2$ or if $\mathcal{G}$ is a normal ring and the extension $\mathcal{G}/\mathcal{G}^G$ is divisorially unramified, the above three conditions are equivalent to each other.

Proof. (3) $\Rightarrow$ (2). If $\chi_{G,\mathcal{G}} = 1$, by (3.2) (3) $\mathcal{G}^G$ is a Gorenstein ring with $a(\mathcal{G}^G) = a(\mathcal{G})$. Hence $a(\mathcal{G}^G) = -2$.

(2) $\Rightarrow$ (3). Firstly assume that $a(\mathcal{G}) \leq -2$. Then as $a(\mathcal{G}) \geq a(\mathcal{G}^G)$ by (3.2) (2), we get $a(\mathcal{G}) = a(\mathcal{G}^G) = -2$ whence $\chi_{G,\mathcal{G}} = 1$ by (3.2) (3). Therefore condition (3) is satisfied. Assume that $\mathcal{G}$ is a normal ring and $\mathcal{G}$ is divisorially unramified over $\mathcal{G}^G$. Then $\chi_{G,\mathcal{G}} = 1$ because $\mathcal{G}^G$ is a Gorenstein ring (see the proof of [**11**, Theorem 2]), so that $a(\mathcal{G}) = a(\mathcal{G}^G) = -2$ by (3.2) (3).

Corollary 3.5. *Suppose that $\mathcal{R}$ is a Gorenstein ring. Then the following two conditions are equivalent.*

(1) *$\mathcal{R}^G$ is a Gorenstein ring.*

(2) $\chi_{G,\mathcal{G}} = 1$.

Proof. Since A is Cohen-Macaulay and $\mathcal{R}$ is Gorenstein, by [**9**, (3.6)] $\mathcal{G}$ is a Gorenstein ring with $a(\mathcal{G}) = -2$, whence the equivalence follows from (3.4).

Let $\rho : G \to GL(\mathfrak{m}/\mathfrak{m}^2)$ be the representation of G induced from the G-action on A.

Corollary 3.6. *Assume that A is a regular local ring. Then the following two conditions are equivalent.*

(1) *$\mathcal{R}^G$ is a Gorenstein ring.*

(2) $\dim A = 2$ *and* $\rho(G) \subseteq SL(\mathfrak{m}/\mathfrak{m}^2)$.

Proof. Let $d = \dim A (\geq 2)$. Then since A is a regular local ring, $\mathcal{G} = \mathcal{G}(\mathfrak{m})$ is a polynomial ring in d variables over $k = A/\mathfrak{m}$. Therefore $a(\mathcal{G}) = -d \leq -2$ ([**6**, (3.1.6)]). Consequently by (3.4) $\mathcal{R}^G$ is a Gorenstein ring if and only if $d = 2$ and $\chi_{G,\mathcal{G}} = 1$. According to (3.3), the later condition $\chi_{G,\mathcal{G}} = 1$ is equivalent to saying that $\rho(G) \subseteq SL(\mathfrak{m}/\mathfrak{m}^2)$.

4. Examples

In what follows, let k be a field and let $R = k[X_1, X_2, \ldots, X_n]$ $(n \geq 1)$ be the polynomial ring in n variables over k. We consider R to be a graded ring with $R_0 = k$ and $\deg X_i = 1$ for all $1 \leq i \leq n$. Let G be a finite group of order N with $N \neq 0$ in k and assume that G acts on R as automorphisms of graded k-algebras. Let $\mathfrak{a}(\mathfrak{a} \neq R)$ be a G-stable graded ideal in R. We put $R^* = R/\mathfrak{a}$ and $\mathfrak{M} = [R^*]_+$. Let $A = R^*_{\mathfrak{M}}$ and $\mathfrak{m} = \mathfrak{M}R^*_{\mathfrak{M}}$. Then the group G acts on A as automorphisms of rings, because the ideal $\mathfrak{M}$ is G-stable. We have

Lemma 4.1. *The natural isomorphisms $R^* \cong \mathcal{G}(\mathfrak{M}) \cong \mathcal{G}(\mathfrak{m})$ are compatible with G-actions. Hence $R^{*G} \cong \mathcal{G}(\mathfrak{m})^G$ as graded k-algebras.*

Thanks to (4.1) we may apply Theorem 2.4 to the local ring $(A, \mathfrak{m})$ and get

Proposition 4.2. *Assume that R^* is a Cohen-Macaulay ring of $d = \dim R^* \geq 1$. Then*

(1) *$\mathcal{R}(\mathfrak{m})^G$ (resp. $\mathcal{R}(\mathfrak{m})$) is a Cohen-Macaulay ring if and only if $a(R^{*G}) < 0$ (resp. $a(R^*) < 0$).*

(2) *Let $d \geq 2$. Then $\mathcal{R}(\mathfrak{m})^G$ (resp. $\mathcal{R}(\mathfrak{m})$) is a Gorenstein ring if and only if R^{*G} (resp. R^*) is a Gorenstein ring with $a(R^{*G}) = -2$ (resp. $a(R^*) = -2$).*

Proof. Recall that R^{*G} is a Cohen-Macaulay ring and that A is a Cohen-Macaulay local ring with $d = \mathrm{ht}_A \mathfrak{m}$. And the assertions on $\mathcal{R}(\mathfrak{m})^G$ follow from (2.4) and (4.1). The assertions on $\mathcal{R}(\mathfrak{m})$ are due to [5, (1.1) and (1.2)], since $R^* \cong \mathcal{G}(\mathfrak{m})$.

Let us now apply Proposition 4.2 to the following examples.

Example 4.3. Let $G = \mathfrak{S}_n$ be the symmetric group and $1 \leq q \in \mathbb{Z}$. Assume that $\mathrm{ch}\, k = 0$. Let G act on the polynomial ring $R = k[X_1, X_2, \ldots, X_n]$ so that $\sigma(X_i) = X_{\sigma(i)}$ for all $\sigma \in G$ and $1 \leq i \leq n$. We put $f = X_1^q + X_2^q + \ldots + X_n^q$ and $\mathfrak{a} = fR$. Then $\mathfrak{a}$ is a G-stable graded ideal in R, since $f \in R^G$. Let $R^* = R/\mathfrak{a}$, $\mathfrak{M} = [R^*]_+$, and $A = R^*_{\mathfrak{M}}$. Let $\mathfrak{m} = \mathfrak{M}R^*_{\mathfrak{M}}$. Then we have

Theorem 4.4. (1) *Let $n \geq 2$. Then the following assertions hold true.*

(a) *$\mathcal{R}(\mathfrak{m})$ is a Cohen-Macaulay ring if and only if $q < n$.*

(b) *$\mathcal{R}(\mathfrak{m})^G$ is a Cohen-Macaulay ring if and only if $q < n(n+1)/2$.*

(2) *Let $n \geq 3$. Then the following assertions hold true.*

(a) *$\mathcal{R}(\mathfrak{m})$ is a Gorenstein ring if and only if $q = n - 2$.*

(b) *$\mathcal{R}(\mathfrak{m})^G$ is a Gorenstein ring if and only if $q = [n(n+1) - 4]/2$.*

Proof. The ring R^* is Gorenstein with $\dim R^* = n - 1$ and $a(R^*) = q - n$ (cf. [6, (3.1.6)]). Since R^G is the polynomial ring in n variables over k and $R^{*G} \cong R^G/fR^G$, we see that $a(R^G) = -n(n+1)/2$ and that R^{*G} is a Gorenstein ring of $a(R^{*G}) = q - n(n+1)/2$. Hence from (4.2) the assertions (a) and (b) in (4.4) follow.

If we take $n = q \geq 2$ in (4.4), the ring $\mathcal{R}(\mathfrak{m})$ is not Cohen-Macaulay but $\mathcal{R}(\mathfrak{m})^G$ is. Letting $n \geq 3$ and $q = [n(n+1) - 4]/2$, we get examples of non-Cohen-Macaulay rings $\mathcal{R}(\mathfrak{m})$ for which the invariant subrings $\mathcal{R}(\mathfrak{m})^G$ are Gorenstein.

Example 4.5. Let $n \geq 3$ and let $R = \mathbb{C}[X_1, X_2, \ldots, X_n]$ be the polynomial ring. Let ζ denote a primitive $(n-2)$-th root of unity. Let $\sigma : R \to R$ be the automorphism of $\mathbb{C}$-algebras defined by $\sigma(X_i) = \zeta X_i$ for $1 \leq i \leq n-1$ and $\sigma(X_n) = \zeta^{-1} X_n$. Let G be the subgroup of $\mathrm{Aut}_{\mathbb{C}} R$ generated by σ. We take $f = \sum_{1 \leq i \leq n} X_i^{n-2}$ and $\mathfrak{a} = fR$. Then $\mathcal{R}(\mathfrak{m})^G$ is a Gorenstein ring.

Proof. The ring R^* is Gorenstein with $\dim R^* = n - 1$ and $a(R^*) = -2$. Therefore by (4.2) (2) $\mathcal{R}(\mathfrak{m})$ is a Gorenstein ring too. On the other hand since $f \in R^G$, by (3.2) (4) we have $\chi_{G,R^*} = \chi_{G,R}$. Let τ be the linear transformation of $V = R_1$ induced from the action of σ on V. Then $\det \tau = 1$ whence $\chi_{G,R} = 1$ by (3.3), so that $\chi_{G,R^*} = 1$. Because the canonical isomorphism $R^* \cong \mathcal{G}(\mathfrak{m})$ is compatible with G-action, from (3.5) that the ring $\mathcal{R}(\mathfrak{m})^G$ is Gorenstein follows.

Example 4.6. Let A be an arbitrary Noetherian local ring of $\dim A \geq 2$. Let G be a finite group of order N and assume that G acts on A as automorphisms of rings. Let N be invertible in A. We choose elements a, b in A^G so that a, b forms a subsystem of parameter for the local ring A^G. Let $I = (a, b)A$. Then $\mathcal{R}(I)^G$ is a Gorenstein ring, if so is A^G.

Proof. Let $J = (a, b)A^G$. Then since the extension A/A^G is pure (cf. [**8**]), we have $I^i \cap A^G = J^i$ for all $i \in \mathbb{Z}$. Hence $\mathcal{R}(I)^G = \mathcal{R}(J) \cong A^G[X, Y]/(aX - bY)$ (here $A^G[X, Y]$ is the polynomial ring in two variables over A^G). Thus the ring $\mathcal{R}(I)^G$ is Gorenstein, if so is A^G.

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