

THE TOPOLOGICAL STABILITY OF DIFFEOMORPHISMS

KAZUMINE MORIYASU

§ 1. Introduction

The present paper is concerned with the stability of diffeomorphisms of C^∞ closed manifolds. Let M be a C^∞ closed manifold and $\text{Diff}^r(M)$ be the space of C^r diffeomorphisms of M endowed with the C^r topology (in this paper we deal with only the case $r = 0$ or 1). Let us define

$$\mathcal{F}(M) = \left\{ f \in \text{Diff}^1(M) \left| \begin{array}{l} \text{there exists a } C^1 \text{ neighborhood } \mathcal{U}(f) \text{ of } \\ f \text{ such that all periodic points of every } \\ g \in \mathcal{U}(f) \text{ are hyperbolic} \end{array} \right. \right\}.$$

Then every C^1 structurally stable and Ω -stable diffeomorphism belongs to $\mathcal{F}(M)$ (see [3]). In light of this result Mañé solved in [5] the C^1 Structural Stability Conjecture by Palis and Smale. After that Palis [9] obtained, in proving that every diffeomorphism belonging to $\mathcal{F}(M)$ is approximated by Axiom A diffeomorphisms with no cycle, the C^1 Ω -Stability Conjecture. Recently Aoki [2] proved that every diffeomorphism belonging to $\mathcal{F}(M)$ is Axiom A diffeomorphisms with no cycle (a conjecture by Palis and Mañé). For the topological stability Walters [14] proved that every Anosov diffeomorphism is topologically stable. In [7] Nitecki showed that every Axiom A diffeomorphism having strong transversality is topologically stable, and that every Axiom A diffeomorphism having no cycle is Ω -topologically stable.

Thus it will be natural to ask whether topologically stable diffeomorphisms belonging to $\text{Diff}^1(M)$ satisfy Axiom A and strong transversality.

Let $f \in \text{Diff}^1(M)$. Then $f: M \rightarrow M$ is topologically stable if and only if given $\varepsilon > 0$ there exists $\delta > 0$ such that for any $g \in \text{Diff}^0(M)$ with $d(f, g) < \delta$ there exists a continuous map $h: M \rightarrow M$ satisfying $h \circ g = f \circ h$ and $d(h, \text{id}) < \varepsilon$ (where id is the identity). Note that if ε is sufficiently small then the above continuous map h is surjective since h is homotopic to id . We denote by $\Omega(f)$ the set of nonwandering points of f . A diffeo-

Received July 20, 1990.

morphism f is Ω -topologically stable if and only if given $\varepsilon > 0$ there exists $\delta > 0$ such that for any $g \in \text{Diff}^0(M)$ with $d(f, g) < \delta$ such that there exists a continuous map $h : \Omega(g) \rightarrow \Omega(f)$ ($h(\Omega(g)) \subset \Omega(f)$) satisfying $h \circ g = f \circ h$ on $\Omega(g)$ and $d(h(x), x) < \varepsilon$ for all $x \in \Omega(g)$.

A sequence $\{x_i | i \in (a, b)\}$ ($-\infty \leq a < b \leq \infty$) of points is called a δ -pseudo orbit for f if $d(f(x_i), x_{i+1}) < \delta$ for $i \in (a, b - 1)$. Given $\varepsilon > 0$ a pseudo orbit $\{x_i\}$ is said to be ε -traced by a point $x \in M$ if $d(f^i(x), x_i) < \varepsilon$ for $i \in (a, b)$. We say that f has the pseudo orbit tracing property (abbrev. POTP) if for $\varepsilon > 0$ there is $\delta > 0$ such that every δ -pseudo orbit for f can be ε -traced by some point of M .

For compact spaces the notions stated above are independent of the compatible metric used. It is known that if $f : M \rightarrow M$ is topologically stable then f has POTP and all the periodic points of f are dense in $\Omega(f)$ (see [6], [15]), and that if $f : M \rightarrow M$ has POTP then so is $f|_{\Omega(f)} : \Omega(f) \rightarrow \Omega(f)$ (see [1]).

To mention precisely our aim let us define the subsets of $\text{Diff}^1(M)$ as

$$\text{AxS}(M) = \{f | f \text{ satisfies Axiom A and strong transversality}\},$$

$$\text{AxN}(M) = \{f | f \text{ satisfies Axiom A and no cycle}\},$$

$$\text{POTP}(M) = \text{int}\{f | f \text{ has POTP}\},$$

$$\Omega\text{-POTP}(M) = \text{int}\{f|_{\Omega(f)} \text{ has POTP}\},$$

$$\text{TS}(M) = \text{int}\{f | f \text{ is topologically stable}\},$$

$$\Omega\text{-TS}(M) = \text{int}\{f | f \text{ is } \Omega\text{-topologically stable}\}.$$

Here $\text{int } E$ denotes the interior of E . Among these sets exist the following

$$\begin{aligned} \text{POTP}(M) &\subset \Omega\text{-POTP}(M) \text{ ([1])}, & \text{TS}(M) &\subset \Omega\text{-TS}(M), \\ \text{TS}(M) &\subset \text{POTP}(M) \text{ ([6] or [15])}, & \text{AxS}(M) &\subset \text{TS}(M) \text{ ([7] or [12])}, \\ \text{AxN}(M) &\subset \Omega\text{-TS}(M) \text{ ([7])}, & \text{AxN}(M) &= \mathcal{F}(M) \text{ ([2])}. \end{aligned}$$

For the question mentioned above we shall show the following

THEOREM 1. *Under the above notations, the following holds.*

$$(1) \quad \Omega\text{-TS}(M) = \mathcal{F}(M),$$

$$(2) \quad \text{TS}(M) = \text{AxS}(M).$$

By Theorem 1 the following is concluded.

$$\Omega\text{-TS}(M) = \text{AxN}(M) = \mathcal{F}(M) \subset \text{TS}(M) = \text{AxS}(M).$$

We have the following theorem as an easy conclusion of Theorem 1.

THEOREM 2. *Let $f \in \text{POTP}(M)$. If $\dim W^s(x, f) = 0$ or $\dim M$ or $\dim M - 1$ for $x \in M$, then f belongs to $\text{AxS}(M)$.*

The proof of Theorem 2 will be given in § 5.

The conclusions of Theorem 1 will be obtained in proving the following three propositions.

PROPOSITION 1. $\Omega\text{-POTP}(M) \subset \mathcal{F}(M)$.

The proof will be based on the techniques of the proof of Theorem 1 of Franks [3].

If we establish Proposition 1, then we have that $\Omega\text{-POTP}(M) = \mathcal{F}(M)$ by the fact mentioned above.

PROPOSITION 2. $\Omega\text{-TS}(M) \subset \mathcal{F}(M)$.

For the proof we need the methods in [6] or [15], in which it is proved that topological stability implies POTP, and the facts used in the proof of Proposition 1.

Since Proposition 2 shows $\Omega\text{-TS}(M) = \mathcal{F}(M)$, (1) of Theorem 1 is concluded.

PROPOSITION 3. $\text{TS}(M) \subset \text{AxS}(M)$.

A result that Axiom A diffeomorphisms satisfying structural stability have strong transversality was proved in Robinson [11]. However every diffeomorphism dealt with in Proposition 3 is Axiom A and topologically stable. Thus it does not follow from Robinson's result that the diffeomorphism satisfies strong transversality.

Proposition 3 ensures that $\text{TS}(M) = \text{AxS}(M)$ and therefore (2) of Theorem 1 is concluded.

§ 2. Proof of Proposition 1

Let $P(f)$ denote the set of periodic points of $f \in \Omega\text{-POTP}(M)$. If $p \in P(f)$ with the prime period k , then $T_p M$ splits into the direct sum $T_p M = E^u(p) \oplus E^s(p) \oplus E^c(p)$ where $E^u(p)$, $E^s(p)$ and $E^c(p)$ are $D_p f^k$ -invariant subspaces corresponding to the absolute values of the eigenvalues of $D_p f^k$ with greater than one, less than one and equal to one.

To obtain Proposition 1 it suffices to prove that each $p \in P(f)$ is hyperbolic: i.e. $E^c(p) = \{0\}$. On the contrary suppose that $p \in P(f)$ is non-hyperbolic and let $k > 0$ be the prime period of p . Then, for every $\varepsilon > 0$ there exists a linear automorphism $\mathcal{O} : T_p M \rightarrow T_p M$ such that

$$(2.1) \quad \begin{cases} \text{(i)} & \|\mathcal{O}\| \leq \varepsilon, \\ \text{(ii)} & \mathcal{O}(E^\sigma(p)) = E^\sigma(p) \text{ for } \sigma = s, u, c, \\ \text{(iii)} & \text{all eigenvalues of } \mathcal{O} \circ D_p f^k|_{E^c(p)} \text{ are of a root of unity.} \end{cases}$$

Making use of the following Franks’s lemma, we can find $\delta_0 > 0$ and a diffeomorphism $g \in \Omega\text{-POTP}(M)$ such that

$$(2.2) \quad \begin{cases} \text{(i)} & B_{\delta_0}(f^i(p)) \cap B_{\delta_0}(f^j(p)) = \emptyset \text{ for } 0 \leq i \neq j \leq k - 1, \\ \text{(ii)} & g(x) = f(x) \text{ for } x \in \{p, f(p), \dots, f^{k-1}(p)\} \cup \{M - \bigcup_{i=0}^{k-1} B_{\delta_0}(f^i(p))\}, \\ \text{(iii)} & g(x) = \exp_{f^{i+1}(p)} \circ D_{f^i(p)} f \circ \exp_{f^i(p)}^{-1}(x) \\ & \qquad \qquad \qquad \text{for } x \in B_{\delta_0}(f^i(p)) (0 \leq i \leq k - 2), \\ \text{(iv)} & g(x) = \exp_p \circ \mathcal{O} \circ D_{f^{k-1}(p)} f \circ \exp_{f^{k-1}(p)}^{-1} \text{ for } x \in B_{\delta_0}(f^{k-1}(p)). \end{cases}$$

FRANKS’S LEMMA. For $f \in \text{Diff}^1(M)$ let F be a finite set of distinct points in M . If $\varepsilon > 0$ is sufficiently small and $G_x : T_x M \rightarrow T_{f(x)} M$ is an isomorphism such that $\|G_x - D_x f\| < \varepsilon/10$ ($x \in F$), then there exist $\delta > 0$ and a diffeomorphism $g : M \rightarrow M$, ε close to f in the C^1 topology, such that $B_{\delta}(x) \cap B_{\delta}(y) = \emptyset$ for $x, y \in F$ with $x \neq y$ and $g(z) = \exp_{f(x)} \circ G_x \circ \exp_x^{-1}(z)$ if $z \in B_{\delta}(x)$ and $g(z) = f(z)$ if $z \notin B_{\delta}(x)$ ($x \in F$).

Define $G = \mathcal{O} \circ D_p f^k$. Then there exists $m > 0$ such that $G^m|_{E^c(p)}$ is the identity by (2.1), and $\delta_1 > 0$ such that

$$(2.3) \quad g^{m \cdot k}|_{\exp_p T_p M(\delta_1)} = \exp_p \circ G^m \circ \exp_p^{-1} \text{ (by (2.2))}$$

where $T_p M(\delta_1) = \{v \in T_p M \mid \|v\| \leq \delta_1\}$. Put $E^c(p, \delta_1) = E^c(p) \cap T_p M(\delta_1)$, then it is clear that

$$(2.4) \quad g^{m \cdot k}|_{\exp_p E^c(p, \delta_1)} = \text{id}|_{\exp_p E^c(p, \delta_1)}.$$

Since $g \in \Omega\text{-POTP}(M)$, we see that $g^{m \cdot k}|_{\Omega(g)}$ has POTP. Then, for $0 < \varepsilon < \delta_1/4$ there exists $0 < \delta < \varepsilon$ such that every δ -pseudo orbit is ε -traced by some point in $\Omega(g)$. Now take and fix $y \in \exp_p E^c(p, \delta_1)$ with $d(p, y) = \frac{3}{4}\delta_1$. From (2.4) we can construct a cyclic δ -pseudo orbit $\{x_i\}$ of $g^{m \cdot k}$ satisfying

$$(2.5) \quad \begin{cases} \text{(i)} & \{x_i\} \subset \exp_p E^c(p, \delta_1), \\ \text{(ii)} & x_0 = p \text{ and } x_s = y \text{ for some } s > 0, \\ \text{(iii)} & B_\varepsilon(x_i) \subset \exp_p T_p M(\delta_1) \text{ for } i \in \mathbf{Z}. \end{cases}$$

For the pseudo orbit $\{x_i\}$ there is $z \in \Omega(g)$ such that $d(g^{m \cdot ki}(z), x_i) < \varepsilon$ for $i \in \mathbf{Z}$ as explained above. By (2.5) (iii) we have $\exp_p^{-1} \circ g^{m \cdot ki}(z) \in T_p M(\delta_1)$ and letting $u = \exp_p^{-1} z$, $\|G^{m \cdot i}(u)\| = \|\exp_p^{-1} \circ g^{m \cdot ki}(z)\| \leq \delta_1$ for $i \in \mathbf{Z}$. Thus

$u \in E^c(p)$ and so $z \in \exp_p E^c(p, \delta_1)$. From (2.4) we have that $d(p, z) \geq d(p, x_s) - d(x_s, z) = d(p, y) - d(x_s, g^{mks}(z)) \geq \frac{3}{4}\delta_1 - \varepsilon > \frac{1}{2}\delta_1 > \varepsilon$, which shows a contradiction.

§ 3. Proof of Proposition 2

Let $f \in \Omega\text{-TS}(M)$. For $\varepsilon > 0$ there exists $\delta > 0$ such that for $g \in \text{Diff}^0(M)$, $d(f(x), g(x_s)) \leq \delta$ ($x \in M$) implies that there exists a continuous map $h : \Omega(g) \rightarrow \Omega(f)$ satisfying $h \circ g = f \circ h$ and $d(h(x), x) \leq \varepsilon$ for $x \in \Omega(g)$. Note that g does not belong to $\Omega\text{-TS}(M)$.

The proof is divided into two the cases $\dim M = 1$ and $\dim M \geq 2$. For the case $\dim M = 1$ we know that the set of all Morse-Smale diffeomorphisms is open dense in $\text{Diff}^1(M)$. Choose a Morse-Smale diffeomorphism as the diffeomorphism g . Then, it is easily checked that $h(P(g)) = P(f)$. Thus $\#P(f) \leq \#P(g) < \infty$, which implies that $f|_{P(f)}$ has POTP. Therefore $f \in \mathcal{F}(M)$ by the same proof as Proposition 1.

For the case $\dim M \geq 2$, we prove directly that $f \in \mathcal{F}(M)$. To do this it suffices to show that every $x \in P(f)$ is hyperbolic. Suppose that $p \in P(f)$ is non-hyperbolic and let $k > 0$ be a prime period of p . As in the proof of Proposition 1, for $\varepsilon > 0$ take a linear automorphism $\mathcal{O} : T_p M \rightarrow T_p M$ satisfying (2.1) and after that take $\delta_0 > 0$ and $g \in \Omega\text{-TS}(M)$ satisfying (2.2). Moreover let $m > 0$ be a minimal integer such that $G^m_{|E^c(p)}$ is the identity map on $E^c(p)$. We put $I_0 = E^c(p)$ when $m = 1$. If $m \geq 2$ then we take $v \in E^c(p)$ with $\|v\| = 1$ such that, letting $I_0 = \{tv \mid t \geq 0\}$ and $I_l = G^l(I_0)$ ($0 \leq l \leq m - 1$)

$$(3.1) \quad \begin{cases} \text{(i)} & I_l \cap I_{l'} = \{0\} \quad (0 \leq l \neq l' \leq m - 1), \\ \text{(ii)} & G^m(I_l) = I_l \quad (0 \leq l \leq m - 1). \end{cases}$$

Let $\delta_1 > 0$ be as in (2.3) and take $y \in \exp_p(I_0 \cap T_p M(\delta_1))$ with $d(p, y) = \frac{3}{4}\delta_1$. Since $G^{mk}_{|\exp_p E^c(p, \delta_1)}$ is the identity on $\exp_p E^c(p, \delta_1)$ by (2.4), for $0 < \varepsilon < \frac{1}{4}\delta_1$ and every $\delta > 0$ we can find a finite sequence $\{x_i\}_{i=0}^{2s}$ of M such that

$$(3.2) \quad \begin{cases} \text{(i)} & \{x_i\}_{i=0}^{2s} \subset \exp_p\{(I_0 \cap T_p M(\delta_1)) - \{0\}\}, \\ \text{(ii)} & d(x_0, p) < \varepsilon, \\ \text{(iii)} & x_{2s} = x_0 \quad \text{and} \quad x_s = y, \\ \text{(iv)} & x_i \neq x_j \quad \text{for} \quad 0 \leq i \neq j \leq 2s - 1, \\ \text{(v)} & d(x_i, x_{i+1}) < \delta \quad \text{for} \quad 0 \leq i \leq 2s - 1, \\ \text{(vi)} & B_\varepsilon(x_i) \in \exp_p T_p M(\delta_1) \quad \text{for} \quad 0 \leq i \leq 2s. \end{cases}$$

Now define $p_{mki} = x_i$ and $q_{mki} = x_{i+1}$ and $p_{mki+j} = q_{mki+j} = g^j(x_{i+1})$ for $0 \leq i \leq 2s - 1$ and $1 \leq j \leq mk - 1$. Then we have that $d(p_n, q_n) < \delta$, $p_n \neq p_{n'}$ and $q_n \neq q_{n'}$ for $0 \leq n \neq n' \leq 2smk - 1$. Thus, by Lemma 13 of Nitecki and Shub [8] we have that there exists $\varphi \in \text{Diff}^1(M)$ such that $d(\varphi(x), x) < 2\pi\delta$ for $x \in M$ and $\varphi(p_n) = q_n$ for $0 \leq n \leq 2smk - 1$. Define $\tilde{g} = g \circ \varphi$. Since δ is arbitrary, we can take \tilde{g} such that \tilde{g} is small C^0 near to g . Thus there exists a continuous map $h : \Omega(\tilde{g}) \rightarrow \Omega(g)$ satisfying $h \circ \tilde{g} = g \circ h$ and $d(h(x), x) < \varepsilon$ for $x \in \Omega(\tilde{g})$. Moreover $\tilde{g}^{2smk}(x_0) = x_0$. Thus $d(p, h(x_0)) \leq d(p, x_0) + d(x_0, h(x_0)) \leq 2\varepsilon$ and

$$\begin{aligned} d(y, g^{smk}(h(x_0))) &= d(y, h(\tilde{g}^{smk}(x_0))) \\ &= d(y, h(x_s)) \leq d(y, x_s) + d(x_s, h(x_s)) \leq \varepsilon. \end{aligned}$$

Therefore we have $\frac{3}{4}\delta_1 = d(p, y) \leq d(p, h(x_0)) + d(g^{smk}(h(x_0)), y) \leq 3\varepsilon < \frac{3}{4}\delta_1$, since $g^{smk}(h(x_0)) = h(x_0)$. We arrived at a contradiction.

§ 4. Proof of Proposition 3

Since $TS(M) \subset \mathcal{F}(M)$ by Propositions 1 and 2, it is clear that $f \in TS(M)$ satisfies Axiom A and no cycle. Thus f is Ω -stable. On the other hand, since f is topologically stable, for $\varepsilon > 0$ small enough we can find a small neighborhood $\mathcal{U}(f)$ of f in $\text{Diff}^1(M)$ such that for $g \in \mathcal{U}(f)$ there exists a continuous surjection $h : M \rightarrow M$ such that $h \circ g(x) = f \circ h(x)$ and $d(h(x), x) < \varepsilon$ for all $x \in M$ and moreover $h_{|\Omega(g)} : \Omega(g) \rightarrow \Omega(f)$ is bijective.

Thus we have that for $x \in M$

$$(4.1) \quad h^{-1}W^\sigma(h(x), f) = W^\sigma(x, g) \quad (\sigma = s, u)$$

where

$$\begin{aligned} W^s(x, g) &= \{y \in M \mid d(g^n(x), g^n(y)) \rightarrow 0 \text{ as } n \rightarrow \infty\}, \\ W^u(x, g) &= \{y \in M \mid d(g^{-n}(x), g^{-n}(y)) \rightarrow 0 \text{ as } n \rightarrow \infty\}. \end{aligned}$$

Indeed, (4.1) is checked as follows. Since $f \in \mathcal{F}(M)$ and $\mathcal{U}(f)$ is a sufficiently small neighborhood, we can take it as $\mathcal{U}(f) \subset \mathcal{F}(M)$. Thus

$$(4.2) \quad M = \bigcup_{x \in \Omega(g)} W^\sigma(x, g) \quad \text{for } g \in \mathcal{U}(f) \text{ } (\sigma = s, u).$$

Since $hW^\sigma(x, g) \subset W^\sigma(h(x), f)$ for $x \in M$, we have $W^\sigma(x, g) \subset h^{-1} \circ hW^\sigma(x, g) \subset h^{-1}W^\sigma(h(x), f)$. To obtain (4.1) suppose that $W^\sigma(x, g) \neq h^{-1}W^\sigma(h(x), f)$. Then $y \notin W^\sigma(x, g)$ and $h(y) \in W^\sigma(h(x), f)$ for some $y \in M$. By (4.2) there exist $x', y' \in \Omega(g)$ such that $W^\sigma(x', g) = W^\sigma(x, g)$ and $W^\sigma(y', g) = W^\sigma(y, g)$. Then

we have

$$h(y) \in W^\sigma(h(x), f) \cap W^\sigma(h(y), f) = W^\sigma(h(x'), f) \cap W^\sigma(h(y'), f)$$

and so $W^\sigma(h(x'), f) = W^\sigma(h(y'), f)$. For $\sigma = s$ we have $d(h \circ g^n(x'), h \circ g^n(y')) = d(f^n \circ h(x'), f^n \circ h(y')) \rightarrow 0$ as $n \rightarrow \infty$. Since $h_{1, \Omega(g)}$ is a homeomorphism, it follows $d(g^n(x'), g^n(y')) \rightarrow 0$ as $n \rightarrow \infty$ and hence $y' \in W^s(x', g) = W^s(x, g)$. Therefore $y \in W^s(x, g)$ which is a contradiction. Similarly we can derive a contradiction for $\sigma = u$.

Next we check that for $x \in M$

$$(4.3) \quad \dim W^s(x, f) + \dim W^u(x, f) \geq \dim M.$$

Since $h_{1, \Omega(g)}$ is bijective, for $p, q \in P(f)$ with $W^s(p, f) \cap W^u(q, f) \neq \emptyset$ there exist $p', q' \in P(g)$ satisfying $h(p') = p$ and $h(q') = q$. From (4.1) we have

$$\begin{aligned} W^s(p', g) \cap W^u(q', g) &= h^{-1}[W^s(h(p'), f) \cap W^u(h(q'), f)] \\ &= h^{-1}[W^s(p, f) \cap W^u(q, f)] \neq \emptyset. \end{aligned}$$

Use here the fact that the set of all Kupka-Smale diffeomorphisms is residual in $\text{Diff}^1(M)$. Then we can take a Kupka-Smale diffeomorphism as the diffeomorphism g . Thus $\dim W^s(p', g) + \dim W^u(q', g) \geq \dim M$. Since g is C^1 near to f , we have that $\dim W^\sigma(x, g) = \dim W^\sigma(h(x), f)$ for $x \in \Omega(g)$ ($\sigma = s, u$). Therefore (4.3) was obtained for this case.

Since f satisfies Axiom A, there exists $\varepsilon > 0$ such that $\bigcap_{n \in \mathbb{Z}} f^n(U_\varepsilon(A_i)) = A_i$ for each basic set A_i of $\Omega(f)$. Since topological stability derives POTP, for the number $\varepsilon > 0$ let $\delta > 0$ be a number satisfying properties in the definition of POTP. Since $M = \bigcup_{y \in \Omega(f)} W^\sigma(y, f)$ for $\sigma = s, u$, for $x \in M$ there exist $y_i \in A_i$ and $y_j \in A_j$ such that $x \in W^s(y_i, f) \cap W^u(y_j, f)$. Take $m > 0$ so large that $d(f^m(x), f^m(y_i)) < \delta$ and $d(f^{-m}(x), f^{-m}(y_j)) < \delta$. Since $A_k \cap P(f)$ is dense in A_k for each basic set A_k , we can choose periodic points $p_i \in A_i$ and $p_j \in A_j$ satisfying $d(f^m(x), p_i) \leq \delta$ and $d(f^{-m}(x), p_j) \leq \delta$. Then a δ -pseudo orbit $\mathcal{O} = \{\dots, f^{-2}(p_j), f^{-1}(p_j), f^{-m}(x), \dots, x, \dots, f^{m-1}(x), p_i, f(p_i), \dots\}$ is ε -traced by a point z in M . Obviously $z \in W^s(f^{-m}(p_i), f) \cap W^u(f^m(p_j), f)$, and hence $\dim W^s(f^{-m}(p_i), f) + \dim W^u(f^m(p_j), f) \geq \dim M$ as above. Therefore we have

$$\begin{aligned} \dim W^s(x, f) + \dim W^u(x, f) &= \dim W^s(p_i, f) + \dim W^u(p_j, f) \\ &= \dim W^s(f^{-m}(p_i), f) + \dim W^u(f^m(p_j), f) \\ &\geq \dim M. \end{aligned}$$

We now are ready to prove Proposition 3.

For $x \in M - \Omega(f)$ it suffices to prove that $W^s(x, f)$ and $W^u(x, f)$ meet transversally. Since $M = \bigcup_{y \in \Omega(f)} W^\sigma(y, f)$ for $\sigma = s, u$, there exist $y_1, y_2 \in \Omega(f)$ such that

$$W^s(x, f) = W^s(y_1, f) \quad \text{and} \quad W^u(x, f) = W^u(y_2, f).$$

We know (cf. see [4]) that there is $\varepsilon_1 > 0$ with $B_{\varepsilon_1}(x) \cap B_{\varepsilon_1}(\Omega(f)) = \emptyset$ such that for $0 < \varepsilon < \varepsilon_1$ and $y \in \Omega(f)$

$$(4.4) \quad \begin{cases} \text{(i)} & W_\varepsilon^\sigma(y, f) \text{ is a } C^1\text{-disk for } \sigma = s, u, \\ \text{(ii)} & W^s(y, f) = \bigcup_{n \geq 0} f^{-n}(W_\varepsilon^s(f^n(y), f)), \\ \text{(iii)} & W^u(y, f) = \bigcup_{n \geq 0} f^n(W_\varepsilon^u(f^{-n}(y), f)). \end{cases}$$

Thus, for $0 < \varepsilon_2 < \varepsilon_1$ there exist $n_1, n_2 > 0$ satisfying

$$(4.5) \quad \begin{cases} \text{(i)} & f^{n_1}(x) \in \text{int}W_{\varepsilon_2}^s(f^{n_1}(y_1), f), \\ \text{(ii)} & f^{-n_2}(x) \in \text{int}W_{\varepsilon_2}^u(f^{-n_2}(y_2), f) \end{cases}$$

where $\text{int}W_{\varepsilon_2}^\sigma(y, f)$ denotes the interior of $W_{\varepsilon_2}^\sigma(y, f)$ in $W_{\varepsilon_1}^\sigma(y, f)$, and $\delta_0 > 0$ satisfying

$$(4.6) \quad \begin{cases} \text{(i)} & B_{\delta_0}(f^n(x)) \cap B_{\delta_0}(f^m(x)) = \emptyset \quad \text{for } -n_2 \leq n \neq m \leq -n_1, \\ \text{(ii)} & f^{-1}[B_{\delta_0}(f^{-n_2}(x))] \cap B_{\delta_0}(f^n(x)) = \emptyset \quad \text{for } -n_2 \leq n \leq n_1, \\ \text{(iii)} & f^m[B_{\delta_0}(f^n(x))] \cap B_{\delta_0}(f^n(x)) = \emptyset \quad \text{for } -n_2 \leq n \leq n_1 \text{ and } m \neq 0. \end{cases}$$

Denote by $C_\delta^\sigma(y, f)$ the connected component of y in $B_\delta(y) \cap W^\sigma(y, f)$ for $\sigma = s, u$. From (4.4) and (4.5) it follows that there is $0 < \delta_1 < \delta_0$ such that for $0 < \delta \leq \delta_1$

$$\begin{aligned} \text{int}W_{\varepsilon_2}^s(f^{n_1}(y_1), f) \cap B_\delta(f^{n_1}(x)) &= W_{\varepsilon_1}^s(f^{n_1}(y_1), f) \cap B_\delta(f^{n_1}(x)) \\ &= C_\delta^s(f^{n_1}(x), f), \\ \text{int}W_{\varepsilon_2}^u(f^{-n_2}(y_2), f) \cap B_\delta(f^{-n_2}(x)) &= W_{\varepsilon_1}^u(f^{-n_2}(y_2), f) \cap B_\delta(f^{-n_2}(x)) \\ &= C_\delta^u(f^{-n_2}(x), f). \end{aligned}$$

Let $\mathcal{U}(f)$ be a small neighborhood of f in $\text{TS}(M)$. Given a sufficiently small $0 < \delta_2 < \delta_1$ we can construct diffeomorphisms φ_i ($i = 1, 2$), C^1 near to the identity, such that

$$\begin{cases} \varphi_1(f^{n_1}(x)) = f^{n_1}(x), \\ \varphi_1(C_{\delta_1}^s(f^{n_1}(x), f) \cap B_{\delta_2}(f^{n_1}(x))) = \exp_{f^{n_1}(x)}(Df^{n_1}E_1)(\delta_2), \\ \varphi_1 = \text{id} \quad \text{on } M - B_{\delta_1}(f^{n_1}(x)), \\ f \circ \varphi_1^{-1} \in \mathcal{U}(f) \end{cases}$$

where E_1 denotes the tangent space at x of $W^s(x, f)$, and

$$\begin{cases} \varphi_2(f^{-n_2}(x)) = f^{-n_2}(x), \\ \varphi_2(C_{\delta_2}^u(f^{-n_2}(x), f) \cap B_{\delta_2}(f^{-n_2}(x))) = \exp_{f^{-n_2}(x)}(Df^{-n_2}E_2)(\delta_2). \\ \varphi_2 = \text{id} \text{ on } M - B_{\delta_1}(f^{-n_2}(x)), \\ \varphi_2 \circ f \in \mathcal{U}(f) \end{cases}$$

where $E_2 = T_x W^u(x, f)$. In general φ_1 and φ_2 can be constructed as follows. For $y \in \Omega(f)$ let $F_1 = T_y W^s(y, f)$ and write $F_2 = F_1^\perp$. Since there exists a C^1 map $\gamma : F_1(\delta) \rightarrow F_2$ such that $\text{graph}(\gamma) = \exp_y^{-1}(C_\delta^s(y, f))$, we can define a C^1 embedding $Q : T_y M(\delta) \rightarrow T_y M$ satisfying $Q(z) = Q(z_1, z_2) = (z_1, z_2 + \gamma(z_1))$ for $z = (z_1, z_2) \in (F_1 \oplus F_2) \cap T_y M(\delta)$. Clearly $D_0 Q = \text{id}$ and so Q is C^1 near to $\text{id}_{|T_y M(\delta)}$ when δ is small enough. As usual define a C^∞ bump function $\alpha : \mathbf{R} \rightarrow [0, 1]$ such that $\alpha(t) = 0$ if $|t| \leq 1$, $\alpha(t) = 1$ if $|t| \geq 2$ and $|\alpha'(t)| < 2$. Then, for a sufficiently small δ' with $0 < 2\delta' < \delta$ we set

$$\varphi(z) = \begin{cases} z \text{ if } z \notin B_\delta(y) \\ \exp_y\{k \cdot \exp_y^{-1}z + (1 - k)Q^{-1}(\exp_y^{-1}z)\} \text{ if } z \in B_\delta(y) \end{cases} \text{ where } k = \alpha\left(\frac{\|\exp_y^{-1}z\|}{\delta'}\right).$$

Then $\varphi : M \rightarrow M$ is a diffeomorphism C^1 near to id such that $\varphi(y) = y$ and $\varphi(C_\delta^s(y, f)) = F_1(\delta')$.

As the finite set F and the isomorphism G_x of Franks's lemma (mentioned above), we set $F = \{f^{-n_2}(x), f^{-n_2+1}(x), \dots, f^{n_1-1}(x)\}$ and $G_{f^n(x)} = D_{f^n(x)}f$ ($-n_2 \leq n \leq n_1 - 1$). Then we see that for $0 < \delta_2 < \delta_1$ small enough there is $g_3 \in \mathcal{U}(f)$ satisfying

$$\begin{cases} g_3(f^n(x)) = f^{n+1}(x) \text{ for } -n_2 \leq n \leq n_1 - 1, \\ g_3 = f \text{ on } M - \bigcup_{n=-n_2}^{n_1-1} B_{\delta_1}(f^n(x)), \\ g_3 = \exp_{f^{n+1}(x)} \circ D_{f^n(x)}f \circ \exp_{f^n(x)}^{-1} \\ \text{on } B_{\delta_2}(f^n(x)) \text{ for } -n_2 \leq n \leq n_1 - 1. \end{cases}$$

Thus by (4.6) we can define a diffeomorphism g belonging to $\mathcal{U}(f)$ by

$$g(y) = \begin{cases} f \circ \varphi_1^{-1}(y) & \text{if } y \in B_{\delta_1}(f^{n_1}(x)) \\ \varphi_2 \circ f(y) & \text{if } y \in f^{-1}(B_{\delta_1}(f^{-n_2}(x))) \\ g_3(y) & \text{otherwise.} \end{cases}$$

Then it is easily checked that for $\delta_3 > 0$ small enough

$$\begin{aligned} d(h(\tilde{g}^n(z)), \tilde{g}^n(z)) &= d(g^n(h(z)), \tilde{g}^n(z)) \\ &\geq d(\tilde{g}^n(z), g^n(y_1)) - d(g^n(y_1), g^n(h(z))) > \varepsilon_1 - \varepsilon_2 > \varepsilon. \end{aligned}$$

This is inconsistent with the property of h . For the case

$$z \notin \tilde{g}^{n_2}(W_{\varepsilon_1}^u(\tilde{g}^{-n_2}(y_2), \tilde{g}))$$

we obtain a contradiction by the same way. Therefore $W^s(x, f)$ is transversal to $W^u(x, f)$ for all $x \in M$. The proof of Proposition 3 is complete.

§ 5. Proof of Theorem 2

As in the proof of Proposition 3, we can construct $g \in \text{POTP}(M)$ satisfying (4.7). Now assume that $\dim W^s(x, f) = \dim M - 1$ and $W^s(x, f)$ is not transversal to $W^u(x, f)$. Then $T_x W^s(x, f) \supset T_x W^u(x, f)$ and so $E = T_x W^s(x, f) \cap T_x W^u(x, f) = T_x W^u(x, f)$. Take $\delta_3 > 0$ small enough, then there exist $\varepsilon' > 0$ and $0 < \varepsilon < \varepsilon'$ such that $W_\varepsilon^u(x, g) \subset \exp_x(E(\delta_3)) \subset W_\varepsilon^s(x, g)$ and $W_{2\varepsilon}^s(x, g) \subset W^s(x, g)$. Since g has POTP, there exists $\delta > 0$ such that if $d(y, z) \leq \delta$ ($y, z \in M$) then $W_\varepsilon^s(y, g) \cap W_\varepsilon^u(z, g) \neq \emptyset$. Thus we have $W_\varepsilon^s(y, g) \cap W_\varepsilon^u(x, g) \neq \emptyset$ for all $y \in B_\delta(x)$, and so $W_\varepsilon^s(y, g) \cap W_\varepsilon^s(x, g) \neq \emptyset$. Therefore $y \in W_{\varepsilon+\varepsilon}^s(x, g) \subset W_{2\varepsilon}^s(x, g) \subset W^s(x, g)$, and so $B_\delta(x) \subset W^s(x, g)$. This contradicts $\dim W^s(x, f) = \dim W^s(x, g) = \dim M - 1$.

REFERENCES

- [1] N. Aoki, On homeomorphisms with pseudo orbit tracing property, Tokyo J. Math., **6** (1983), 329–334.
- [2] ———, The set of Axiom A diffeomorphisms with no cycle, preprint.
- [3] J. Franks, Necessary conditions for stability of diffeomorphisms, Trans. A.M.S., **158** (1971), 301–308.
- [4] M. Hirsch and C. Pugh, Stable manifolds and hyperbolic sets, in Global Analysis, Proc. Sympos. Pure Math., A.M.S., **14** (1970), 133–163.
- [5] R. Mañé, A proof of the C^1 stability conjecture, Publ. Math. I.H.E.S., **66** (1987), 161–210.
- [6] A. Morimoto, Stochastically stable diffeomorphisms and Takens conjecture, Suriken Kokyuroku, **303** (1977), 8–24.
- [7] Z. Nitecki, On semi-stability for diffeomorphisms, Invent. Math., **14** (1971), 83–122.
- [8] Z. Nitecki and M. Shub, Filtrations, decompositions and explosions, Amer. J. Math., **97** (1976), 1029–1047.
- [9] J. Palis, On the C^1 Ω -stability conjecture, Publ. Math. I.H.E.S., **66** (1987), 211–215.
- [10] M. Peixoto, Structural stability on two-dimensional manifolds, Topology, **1** (1962), 101–120.
- [11] C. Robinson, C^r structural stability implies Kupka-Smale, In: Dynamical Systems, edited by Peixoto. Academic Press (1973).

- [12] —, Stability theorems and hyperbolicity in dynamical systems, *Rocky Mountain J. Math.*, **7** (1977), 425–437.
- [13] S. Smale, The Ω -stability theorem, in *Global Analysis*, Proc. Sympos. Pure Math., A.M.S., **14** (1970), 289–297.
- [14] P. Walters, Anosov diffeomorphisms are topologically stable, *Topology*, **9** (1970), 71–78.
- [15] —, On the pseudo-orbit tracing property and its relationship to stability, *Springer L.N.*, No. **668** (1979), 231–244.

Department of Mathematics
Tokyo Metropolitan University
Minami-Ohsawa 1-1, Hachioji-shi
Tokyo 192-03
Japan