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THE $\mathbb{F}_2$-COHOMOLOGY RINGS OF $\mathbb{S}\mathrm{ol}^3$-MANIFOLDS

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Abstract

We compute the rings $H^*(N;\mathbb{F}_2)$ for N a closed $\mathbb{S}\mathrm{ol}^3$-manifold, and then determine the Borsuk–Ulam indices $\mathrm{BU}(N,\phi)$ with $\phi \neq 0$ in $H^1(N;\mathbb{F}_2)$.

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The Borsuk–Ulam theorem states that any continuous function $f : S^n \to \mathbb{R}^n$ takes the same value at some antipodal pair of points. This may be put in a broader context as follows. Let N be an n-manifold and let N_ϕ be the double cover associated to an epimorphism $\phi : \pi \to Z/2Z$. Let t_ϕ be the covering involution. The *Borsuk–Ulam index* $\mathrm{BU}(N,\phi)$ is the maximal value of k such that for all maps $f : N_\phi \to \mathbb{R}^k$ there is an $x \in N_\phi$ with $f(x) = f(t_\phi(x))$. Then the Borsuk–Ulam theorem is equivalent to the assertion that $\mathrm{BU}(RP^n, \alpha) = n$, where $\alpha : \pi_1(RP^n) \to Z/2Z$ is the canonical epimorphism.

In low dimensions this invariant may be determined cohomologically, and is known for many pairs (N,ϕ), with N a Seifert fibred 3-manifold, including all those with geometry $\mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{E}^1$, $\mathbb{N}\mathrm{il}^3$ or $\mathbb{H}^2 \times \mathbb{E}^1$ [3, 1]. Here we shall determine this invariant for all such pairs with N a closed $\mathbb{S}\mathrm{ol}^3$-manifold. This follows easily once we know the mod-2 cohomology rings of such manifolds. We compute these using Poincaré duality and elementary properties of cup product in the low-degree cohomology of groups. (Our approach can also be applied to $\mathbb{E}^3$- and $\mathbb{N}\mathrm{il}^3$-manifolds.)

1. $\mathbb{S}\mathrm{ol}^3$-manifolds and their groups

Let M be a closed $\mathbb{S}\mathrm{ol}^3$-manifold. Then $\pi = \pi_1(M)$ has a unique maximal abelian normal subgroup $\sqrt{\pi}$, which is free abelian of rank two. (This subgroup is in fact the Hirsch–Plotkin radical [8] of π.) The quotient $\pi/\sqrt{\pi}$ is virtually $\mathbb{Z}$ (that is, has two ends), and so is an extension of $\mathbb{Z}$ or $D_\infty = Z/2Z * Z/2Z$ by a finite normal subgroup. The preimage of this finite normal subgroup is torsion-free, and so is either $\mathbb{Z}^2$ or $\mathbb{Z} \rtimes_{-1} \mathbb{Z}$ (the Klein bottle group). Since $\mathrm{Out}(\mathbb{Z} \rtimes_{-1} \mathbb{Z})$ is finite and π is not virtually abelian, this preimage must be $\sqrt{\pi}$. Hence $\pi/\sqrt{\pi} \cong \mathbb{Z}$ or D_∞.

Suppose first that $\pi/\sqrt{\pi} \cong \mathbb{Z}$. Then M is the mapping torus of a self-homeomorphism of $T = S^1 \times S^1$, and $\pi \cong \mathbb{Z}^2 \rtimes_\Theta \mathbb{Z}$, where $\Theta = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in GL(2, \mathbb{Z})$. Thus π has a presentation

$$\langle t, x, y \mid txt^{-1} = x^a y^b, tyt^{-1} = x^c y^d, xy = yx\rangle.$$

Let $\varepsilon = \det \Theta = \pm 1$ and $\tau = \operatorname{tr} \Theta = a + d$. Then M is orientable if and only if $\varepsilon = 1$, in which case $|\tau| > 2$, since π is not virtually nilpotent. Let θ be a root of $\det(\Theta - XI_2) = X^2 - \tau X + \varepsilon$, the characteristic polynomial of Θ. Then θ is a unit in the quadratic number field $\mathbb{Q}[\theta]$, and $\sqrt{\pi}$ is isomorphic to an ideal I in the ring $\mathbb{Z}[\theta]$. (The latter may not be the full ring of integers in $\mathbb{Q}[\theta]$!)

There is a converse. Let $[I]$ denote the isomorphism class of the ideal I. The Galois involution of the quadratic field $\mathbb{Q}[\theta]$ acts on the ring $\mathbb{Z}[\theta]$, since $\bar{\theta} = \tau - \theta \in \mathbb{Z}[\theta]$, and hence acts on the set of ideal classes.

Theorem 1.1. *Let α be a quadratic algebraic unit which is not a root of unity, and let J be a nonzero ideal in $\mathbb{Z}[\alpha]$. Let A be the automorphism of $J \cong \mathbb{Z}^2$ given by left multiplication by α, and let $\pi = J \rtimes_A \mathbb{Z}$. Then:*

(1) *π is a $\mathbb{S}\mathrm{ol}^3$-group;*

(2) *the groups corresponding to two such pairs (α, J) and (β, K) are isomorphic if and only if either $\beta = \alpha$ or α^{-1} and $[K] = [J]$, or $\beta = \bar{\alpha}$ or $\bar{\alpha}^{-1}$ and $[K] = \overline{[J]}$;*

(3) *given α, the number of isomorphism classes of such groups π is finite.*

Proof. The group π is the fundamental group of the mapping torus of a self-homeomorphism of T. If α is not a root of unity then this is a $\mathbb{S}\mathrm{ol}^3$-manifold.

Let

$$\pi = \langle J, t \mid tjt^{-1} = \alpha j \ \forall j \in J\rangle$$

and

$$\widetilde{\pi} = \langle k, \widetilde{t} \mid \widetilde{t}\widetilde{j}\widetilde{t}^{-1} = \beta k \ \forall k \in K\rangle$$

be two such groups. An isomorphism $f : \pi \cong \widetilde{\pi}$ restricts to an isomorphism $f_J : J = \sqrt{\pi} \cong \sqrt{\widetilde{\pi}} = K$. Hence it induces an isomorphism $\pi/\sqrt{\pi} \cong \widetilde{\pi}/\sqrt{\widetilde{\pi}}$, and so $f(t) = \widetilde{t}^{\eta} k$, for some $\eta = \pm 1$ and $k \in K$. We may assume that $f(t) = \widetilde{t}$, after replacing β by β^{-1}, if necessary. The characteristic polynomials of the automorphism of J and K induced by conjugation by t and $\widetilde{t}$ (respectively) must then agree. Thus either $\beta = \alpha$ and f_J is an isomorphism of $\mathbb{Z}[\alpha]$-modules, or $\beta = \bar{\alpha}$ and $\overline{f_J} : J \to \overline{K}$ is an an isomorphism of $\mathbb{Z}[\alpha]$-modules. The converse is similarly straightforward.

The group π is determined up to a finite ambiguity by α (equivalently, by the polynomial $t^2 - \tau t + \varepsilon$), since $\mathbb{Z}[\alpha]$ has finitely many ideal classes, by the Jordan–Zassenhaus theorem. □

If $\pi/\sqrt{\pi} \cong D_\infty$ then $\pi \cong B *_T C$, where B and C are torsion-free, $T \cong \mathbb{Z}^2$ and $[B : T] = [C : T] = 2$. Thus M is the union of two twisted I-bundles. Since $\beta_1(\pi; \mathbb{Q}) = 0$ and $\chi(M) = 0$, M is orientable, and so B and C must be copies of the Klein bottle group. Hence M is the union of two copies of the mapping cylinder of the double

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cover of the Klein bottle. The double cover of M corresponding to the preimage of $\sqrt{D_\infty}$ in π is a mapping torus.

In particular, π has a presentation

$$\langle u, v, y, z \mid uyu^{-1} = y^{-1}, vzv^{-1} = z^{-1}, yz = zy, v^2 = u^{2a}y^b, z = u^{2c}y^d\rangle,$$

where $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in GL(2, \mathbb{Z})$ corresponds to the identification of $\sqrt{C}$ with $T = \sqrt{B}$. This presentation simplifies immediately to

$$\langle u, v, y \mid uyu^{-1} = y^{-1}, v^2 = u^{2a}y^b, vu^{2c}y^dv^{-1} = u^{-2c}y^{-d}\rangle.$$

Hence $\pi^{ab} \cong Z/4cZ \oplus Z/4Z$ if b is odd, and $\pi^{ab} \cong Z/4cZ \oplus (Z/2Z)^2$ if b is even. Let $x = u^2$. Then conjugation by uv acts on $\langle x, y\rangle \cong \mathbb{Z}^2$ via $\Psi = \eta\left(\begin{smallmatrix} ad+bc & 2ac \\ 2bd & ad+bc \end{smallmatrix}\right)$, where $\eta = ad - bc = \pm 1$. Hence $\det \Psi = 1$, $\Psi \equiv I_2 \pmod 2$ and $\mathrm{tr}\,\Psi \equiv 2 \pmod 4$. (These conditions are not independent, for if $\Psi = I_2 + 2N$ then $\mathrm{tr}\,\Psi = 2 + 2\,\mathrm{tr}\,N$ and $\det \Psi \equiv 1 + 2\,\mathrm{tr}\,N \pmod 4$, so $\mathrm{tr}\,N$ is even and $\mathrm{tr}\,\Psi \equiv 2 \pmod 4$ if also $\det \Psi = 1$.) Moreover, $abcd \neq 0$, since M is not flat.

Conversely, any $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in GL(2, \mathbb{Z})$ with $abcd \neq 0$ gives rise to such a $\mathbb{S}\mathrm{ol}^3$-manifold, for then $|\mathrm{tr}\Psi| = 2|ad + bc| \geq 6$. Moreover, suppose that $P = \left(\begin{smallmatrix} 2k+1 & 2m \\ 2n & 2k+1 \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$, where $mn \neq 0$. Then $k(k + 1) = mn$, and so we may write $m = m_1m_2$ and $n = n_1n_2$, with $k = m_1n_1$ and $k + 1 = m_2n_2$. The $\mathbb{S}\mathrm{ol}^3$-rational homology sphere corresponding to $\left(\begin{smallmatrix} m_1 & -m_2 \\ -n_2 & n_1 \end{smallmatrix}\right) \in GL(2, \mathbb{Z})$ is doubly covered by the mapping torus associated to P.

The above matrix calculations show that a quadratic unit α is realised by such a $\mathbb{S}\mathrm{ol}^3$-manifold if and only if $\alpha\bar{\alpha} = 1$, $|\alpha + \bar{\alpha}| > 2$ and $\alpha + \bar{\alpha} \equiv 2 \pmod 4$. Determining the possible ideal classes represented by $\sqrt{\pi}$ is more complicated.

Theorem 1.2. *Let α be a quadratic unit which is not a root of unity, and let J be a nonzero ideal in $\mathbb{Z}[\alpha]$. Let A be the automorphism of $J \cong \mathbb{Z}^2$ given by left multiplication by α, and let $\kappa = J \rtimes_A \mathbb{Z}$. Then κ is a subgroup of index 2 in a $\mathbb{S}\mathrm{ol}^3$-group π with $\pi/\sqrt{\pi} \cong D_\infty$ if and only if $\alpha\bar{\alpha} = 1$, $\alpha \equiv 1 \pmod{2\,\mathbb{Z}[\alpha]}$ and there are $\lambda, \mu \neq 0 \in \mathbb{Z}[\alpha]$ and $v, w \in J$ such that $\lambda\bar{J} = \mu J$, $\lambda\bar{v} = \mu v$ and $\lambda\bar{w} = \bar{\alpha}\mu w$, but $\bar{\lambda}v \neq \bar{\lambda}j + \bar{\mu}\bar{j}$ and $\bar{\lambda}w \neq \bar{\lambda}j + \alpha\bar{\mu}\bar{j}$ for any $j \in J$.*

Given α, the number of isomorphism classes of such groups π is finite.

Proof. Suppose that $\pi = \langle \kappa, u\rangle$ with $\pi/\sqrt{\pi} \cong D_\infty$ and $[\pi : \kappa] = 2$, and that $t \in \kappa$ generates $\kappa \pmod{\sqrt{\pi}}$. Then t^{-1} is conjugate to t, and so A and A^{-1} have the same characteristic polynomial. Since $\mathrm{tr}\,A \neq 0$, $\alpha\bar{\alpha} = \det A = 1$.

Let $B(j) = uju^{-1}$ and $f(j) = \overline{B(j)}$, for all $j \in J$. Then B is an isomorphism of groups and $f : J \to \bar{J}$ is an isomorphism of $\mathbb{Z}[\alpha]$-modules. Let $v = u^2$ and $w = (tu)^2$. Then $B^2 = (AB)^2 = I$, $Bv = v$ and $ABw = w$. Since A has infinite order, $B \neq I$, and so $\det B = -1$. Moreover, $B \equiv AB \equiv I_2 \pmod 2$, since $\langle J, u\rangle$ and $\langle J, tu\rangle \cong \pi_1(Kb)$. Therefore $A \equiv I_2 \pmod 2$ also, and so $\alpha \equiv 1 \pmod{2\,\mathbb{Z}[\alpha]}$.

Since π is torsion-free, $(uj)^2$ and $(tuj)^2$ are nontrivial, for all $j \in J$. Equivalently, $v \notin (I + B)J$ and $w \notin (I + AB)J$.

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The isomorphism f extends to an automorphism $f_{\mathbb{Q}} = id_{\mathbb{Q}} \otimes f$ of $\mathbb{Q}[\alpha]$, as a vector space over itself. We may write $f_{\mathbb{Q}}(1) = \mu/\lambda$, for some nonzero $\lambda, \mu \in \mathbb{Z}[\alpha]$. (Note that $\mu\bar{\mu}/\lambda\bar{\lambda} = \det f_{\mathbb{Q}} = -\det B = 1$.) Then $\lambda f(j) = \mu j$, for all $j \in J$, since $\mathbb{Z}[\alpha]$ is an integral domain. The linear conditions on v and w become $\lambda\bar{v} = \mu v$ and $\lambda\bar{w} = \bar{\alpha}\mu w$, while $\bar{\lambda} v \neq \bar{\lambda} j + \bar{\mu}\bar{j}$ and $\bar{\lambda} w \neq \bar{\lambda} j + \alpha\bar{\mu}\bar{j}$ for any $j \in J$.

Conversely, suppose that these conditions hold. Let $Bj = \overline{(\mu/\lambda)j}$, for all $j \in J$, and let π be the group with presentation

$$\langle \kappa, u \mid u^2 = v, utu^{-1} = t^{-1}wv^{-1}, uju^{-1} = Bj\ \forall j \in J \rangle.$$

Then π is torsion-free and has κ as a subgroup of index 2. and so is a $\mathbb{S}ol^3$-group. Clearly $\pi/\sqrt{\kappa} \cong D_\infty$, and so $\sqrt{\kappa} \leq \sqrt{\pi} \leq \kappa$. Hence $\sqrt{\pi} = \sqrt{\kappa}$ and $\pi/\sqrt{\pi} \cong D_\infty$.

Since κ has trivial centre the extensions of $Z/2Z$ by κ are determined by the image in $\mathrm{Out}(\kappa)$ of the action of $Z/2Z$ on κ. Since there are finitely many groups κ realising α, by Theorem 1.1, and $\mathrm{Out}(\kappa)$ is finite, by [5, Theorem 8.10], there are finitely many such groups π. □

In particular, the ideal class $[J]$ must be fixed by the Galois involution. For example, if $\alpha\bar{\alpha} = 1$ and $\alpha \equiv 1$ (mod $2\,\mathbb{Z}[\alpha]$) then $J = \mathbb{Z}[\alpha]$, $v = 1$ and $w = \alpha$ satisfy the other conditions, with $\lambda = \mu = 1$.

Note that if α is a quadratic unit such that $\alpha\bar{\alpha} = 1$ and $\delta = \alpha - 1 \in 2\mathbb{Z}[\alpha]$ then $\bar{\delta} = -\alpha^{-1}\delta \in 2\mathbb{Z}[\alpha]$ also, and so $\alpha + \bar{\alpha} = 2 - \delta\bar{\delta} \equiv 2$ (mod 4). (This is equivalent to an earlier matrix argument.)

Every subgroup of finite index in π can be generated by three elements, while proper subgroups of infinite index need at most two generators. If a nontrivial normal subgroup N has infinite index in π then it has Hirsch length at most 2. Hence it is abelian, and so has finite index in $\sqrt{\pi}$. Thus proper quotients of a $\mathbb{S}ol^3$-group π either have two ends or are finite.

2. The mod-2 cohomology ring

Martins has constructed an explicit free resolution $P_* \to \mathbb{Z}$ of the augmentation $\mathbb{Z}[\pi]$-module, and a diagonal approximation $\Delta : P_* \to P_* \otimes P_*$, which he used to compute the integral and mod-p cohomology rings for semidirect products $\pi \cong \mathbb{Z}^2 \rtimes_\Theta \mathbb{Z}$ with $\Theta \in GL(2, \mathbb{Z})$ [6].

We shall take a somewhat different approach, first computing cup products into $H^2(\pi; \mathbb{F}_2)$ and then using Poincaré duality. Our strategy in determing relations in $H^2(\pi; \mathbb{F}_2)$ shall be to use restrictions to subgroups (such as $\sqrt{\pi}$) and epimorphisms to quotient groups (such as $\pi/\sqrt{\pi}$ or small finite 2-groups), with known cohomology rings.

We shall usually write $H_*(X)$ and $H^*(X)$ for the homology and cohomology of a space or group X, with coefficients $\mathbb{F}_2$, and denote the cup product by juxtaposition. In each case considered below, the given generators for a group G represent a basis for $H_1(G)$, and we shall use the corresponding Kronecker dual bases for $H^1(G) = \mathrm{Hom}(H_1(G), \mathbb{F}_2)$.

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Lemma 2.1. *Let $w = w_1(\pi)$. Then $w\alpha\beta = \alpha^2\beta + \alpha\beta^2$, for all $\alpha, \beta \in H^1(\pi)$. In particular, if $w = 0$ then $\alpha^2\beta = \alpha\beta^2$ and $(\alpha + \beta)^3 = \alpha^3 + \beta^3$.*

Proof. The first assertion follows from the Wu relation $Sq^1 z = wz$, for all $z \in H^{n-1}(X)$, which holds for any PD_n-complex X. The second follows easily. □

If G is a group let $X^n(G) = \langle g^n \mid g \in G\rangle$ be the subgroup generated by all nth powers. The next lemma is a refinement of [4, Theorem 2] (which is restated here as part (1) of the lemma).

Lemma 2.2. *Let G be a group, and $\rho, \phi, \psi \in H^1(G)$. Let $K = \mathrm{Ker}(\rho)$ and $L = K \cap \mathrm{Ker}(\phi)$. Then:*

(1) *the kernel of cup product from the symmetric product $\odot^2 H^1(G)$ to $H^2(G)$ is the dual of $X^2(G)/X^4(G)[G, X^2(G)]$;*
(2) *the canonical projections induce isomorphisms*

$$H^1(G/X^2(K)) \cong H^1(G/X^2(L)) \cong H^1(G/X^4(G)) \cong H^1(G);$$

(3) *$\rho\phi = 0$ in $H^2(G)$ if and only if $\rho\phi = 0$ in $H^2(G/X^2(K))$;*
(4) *$\phi^2 = \rho\phi + \rho\psi$ in $H^2(G)$ if and only if $\phi^2 = \rho\phi + \rho\psi$ in $H^2(G/X^2(L))$.*

Proof. Part (1) is [4, Theorem 2], while part (2) is clear.

If $\phi\psi = 0$ in $H^2(G)$ then there is a 1-cochain $F : G \to \mathbb{F}_2$ such that $\phi(g)\psi(h) = \delta F(g, h) = F(gh) + F(g) + F(h)$, for all $g, h \in G$. Part (3) follows easily, since F restricts to a homomorphism on K, and is constant on cosets of $X^2(K)$.

Part (4) is similar. □

In many of the cases considered here, the coefficients in the linear relations determining the kernel of cup product may be found by restricting to 2-generator subgroups. However, this is not always enough to determine the triple products in $H^3(\pi)$.

Lemma 2.3. *Let $\{T, Y\}$ be the basis for $H^1(D_8)$ corresponding to the presentation $D_8 = \langle t, y \mid t^2 = y^4 = 1, tyt^{-1} = y^{-1}\rangle$. Then $(T + Y)Y = 0$ in $H^2(D_8)$.*

Proof. Let D_∞ have the presentation $\langle u, v \mid u^2 = v^2 = 1\rangle$, and let U, V be the dual basis for $H^1(D_\infty)$. Then $H^*(D_\infty) = \mathbb{F}_2[U, V]/(UV)$. Let $f : D_\infty \to D_8$ be the epimorphism given by $f(u) = t$ and $f(v) = ty$. Then f induces an isomorphism $D_\infty/X^4(D_\infty) \cong D_8$, so $H^2(f)$ is injective. Since $f^*U = T + Y$ and $f^*V = Y$, we see that $(T + Y)Y = 0$ in $H^2(D_8)$. □

Let E be the 'almost extraspecial' 2-group with presentation

$$\langle t, u, v \mid t^2 = 1, u^2 = v^2, tut^{-1} = u^{-1}, tv = vt, uv = vu\rangle.$$

Lemma 2.4. *Let $\{T, U, V\}$ be the basis for $H^1(E)$ corresponding to the above presentation. Then $TU + U^2 + V^2 = 0$ in $H^2(E)$.*

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PROOF. Since $X^2(E) \cong Z/2Z$, the kernel of cup product from $\odot^2 H^1(G)$ to $H^2(G)$ has dimension one [4]. Thus there is a unique nontrivial linear relation $aT^2 + bU^2 + cV^2 + dTU + eTV + fUV = 0$ in $H^2(E)$. The coefficients can be determined by restriction to the subgroups $\langle t\rangle \cong Z/4Z$, $\langle t, u\rangle \cong D_8$, $\langle t, v\rangle \cong Z/4Z \oplus Z/2Z$, and $\langle u, v\rangle \cong Z/4Z \oplus Z/2Z$. □

3. Mapping tori

Suppose that $\pi \cong \mathbb{Z}^2 \rtimes_\Theta \mathbb{Z}$, where $\Theta = \left(\begin{smallmatrix} a & c \\ b & d\end{smallmatrix}\right) \in GL(2, \mathbb{Z})$. Let $\varepsilon = ad - bc = \pm 1$ and $\tau = a + d$. Let $\Delta_1 = \det(\Theta - I_2) = 1 - \tau + \varepsilon$ and $\Delta_2 = (a - 1, b, c, d - 1)$ be the elementary divisors of $\Theta - I_2$. Then Δ_2^2 divides Δ_1, and

$$\pi^{ab} \cong \mathbb{Z} \oplus Z/(\Delta_1/\Delta_2)Z \oplus Z/\Delta_2 Z.$$

Let $\beta = \beta_1(\pi; \mathbb{F}_2)$. Then $1 \le \beta \le 3$, and $\beta_2(\pi; \mathbb{F}_2) = \beta$, by Poincaré duality. Let $\rho : \pi \to Z/2Z$ be the unique epimorphism which factors through $\pi/\sqrt{\pi} \cong \mathbb{Z}$. If π is nonorientable then $\rho = w_1(M)$, and $K = \pi^+$, the maximal orientable subgroup of π.

(1) If τ is odd then Δ_1 is odd and $\pi^{ab} \cong \mathbb{Z} \oplus odd$. In this case ρ is the unique epimorphism from π to $Z/2Z$, and

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \Xi]/(\rho^2, \Xi^2),$$

where Ξ has degree two, by Poincaré duality.

(2) If $\tau \equiv \varepsilon - 1 \pmod 4$ then $\pi^{ab} \cong \mathbb{Z} \oplus Z/2Z \oplus odd$, and $\beta = 2$. Hence $H^1(\pi) = \langle \rho, \sigma\rangle$, where σ does not factor through $Z/4Z$. Moreover, if $G = \pi/X^4(\pi)$ then $X^2(G) \cong (Z/2Z)^2$ is central in G. Thus $\rho^2 = \rho\sigma = 0$, by Lemma 2.2, while $\sigma^2 \ne 0$. Hence $H^2(\pi) = \langle \sigma^2, \Xi\rangle$, for some Ξ of degree two. Duality then implies that $\sigma^3 = \rho\Xi \ne 0$. We may assume also that $\sigma\Xi = 0$, and so

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \Xi]/(\rho^2, \rho\sigma, \sigma\Xi, \rho\Xi + \sigma^3, \Xi^2).$$

(3) If $\tau \equiv \varepsilon + 1 \pmod 4$ and Δ_2 is odd then $\pi^{ab} \cong \mathbb{Z} \oplus Z/2^k Z \oplus odd$, for some $k \ge 2$. Hence $H^1(\pi) = \langle \rho, \sigma\rangle$, where $\sigma^2 = \rho^2 = 0$. Since $\rho\sigma = 0$, by the nondegeneracy of Poincaré duality,

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \Xi, \Omega]/(\rho^2, \rho\sigma, \sigma^2, \rho\Omega, \sigma\Xi, \rho\Xi + \sigma\Omega, \Xi^2, \Omega^2, \Xi\Omega),$$

where Ξ and Ω have degree two.

In all the remaining cases $\beta = 3$. For if $\tau \equiv \varepsilon + 1 \pmod 4$ and Δ_2 is even then a and d are odd and b and c are even. Hence $\Delta_1 = 2^k q$ and $\Delta_2 = 2^\ell q'$, where $0 < \ell \le k/2$ and q, q' are odd. In this case $\pi^{ab} \cong \mathbb{Z} \oplus Z/2^{k-\ell} Z \oplus Z/2^\ell Z \oplus odd$, so the images of $\{t, x, y\}$ form a basis for $H_1(\pi)$. Let $\{\rho, \sigma, \psi\}$ be the dual basis, so that

$$\sigma(x) = \psi(y) = 1 \quad \text{and} \quad \sigma(t) = \sigma(y) = \psi(t) = \psi(x) = 0.$$

If $G = \pi/X^4(\pi)$ then $X^2(G) = \langle t^2, x^2, y^2\rangle \cong (Z/2Z)^3$ is central in G, so the kernel of cup product from $\odot^2 H^1(\pi)$ to $H^2(\pi)$ has rank three. It then follows from Poincaré duality that $H^*(\pi)$ is generated as a ring by $H^1(\pi)$. In each case, $\rho\sigma^2 = \rho\rho\sigma = 0$ and $\rho\psi^2 = \rho\rho\psi = 0$, by Lemma 2.1. Hence $\rho\sigma\psi \neq 0$, by the nondegeneracy of Poincaré duality. It then follows easily that $\rho\sigma$, $\rho\psi$ and $\sigma\psi$ are linearly independent, and so form a basis for $H^2(\pi)$. We may write

$$\sigma^2 = m\rho\sigma + n\rho\psi + p\sigma\psi \quad \text{and} \quad \psi^2 = q\rho\sigma + r\rho\psi + s\sigma\psi,$$

for some $m, \ldots, s$. On restricting to $\sqrt{\pi}$, we see that $p = s = 0$, since $\sigma^2|_{\sqrt{\pi}} = \psi^2|_{\sqrt{\pi}} = 0$ and $\rho|_{\sqrt{\pi}} = 0$, while $\sigma\psi|_{\sqrt{\pi}} \neq 0$. Since $\rho\sigma^2 = \rho^2\sigma = \rho\psi^2 = \rho^2\psi = 0$, taking cup products with σ and ψ gives

$$\sigma^3 = n\rho\sigma\psi, \quad \sigma^2\psi = m\rho\sigma\psi, \quad \psi^3 = q\rho\sigma\psi \quad \text{and} \quad \sigma\psi^2 = r\rho\sigma\psi.$$

(4) If $\ell \geq 2$ then $a \equiv d \equiv 1$ and $b, c \equiv 0 \pmod 4$, so $\varepsilon \equiv 1 \pmod 4$ also, that is, π is orientable. In this case $\sigma^2 = \psi^2 = \rho^2 = 0$, and so

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \psi]/(\rho^2, \sigma^2, \psi^2).$$

Suppose now that $\ell = 1$.

(5) If π is orientable and $\Delta_1 \equiv 0 \pmod 8$ we may assume that one of σ, ψ or $\sigma + \psi$ factors through $Z/4Z$. Thus either $\sigma^2 = 0$, $\psi^2 = 0$ or $\sigma^2 = \psi^2$. We may assume that $\sigma^2 \neq 0$. Then $\rho\sigma^2 = \rho^2\sigma = 0$ and $\psi\sigma^2 = \psi^2\sigma = 0$, and so $\sigma^3 \neq 0$, by the nonsingularity of Poincaré duality. Hence

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \psi]/(\rho^2, \rho\psi + \sigma^2, \psi^2).$$

In this case we see that $\phi^3 = 0$ if and only if $\phi^2 = 0$.

If π is orientable and $\Delta_1 \equiv 4 \pmod 8$ then $\tau \equiv 6 \pmod 8$ and a, d are odd, and so $a \equiv d \pmod 4$. In this case $\psi^2 \neq 0$ and $(\sigma + \psi)^2 \neq 0$ also, and so $\sigma^2 = m\rho\sigma + n\rho\psi$ and $\psi^2 = q\rho\sigma + r\rho\psi$ are linearly independent. Hence $mr + nq = 1$ in $\mathbb{F}_2$. Since $w = 0$, $\sigma^2\psi = \sigma\psi^2$ and so $m = r$.

(6) Suppose first that $a \equiv 1 \pmod 4$. Then $bc \equiv 4 \pmod 8$, and so $b \equiv c \equiv 2 \pmod 4$. Let $L_\phi = \mathrm{Ker}(\rho) \cap \mathrm{Ker}(\phi)$. Then $\pi/X^2(L_\phi)$ has a presentation

$$\langle t, x, y \mid t^4 = x^4 = y^2 = 1, tx = xt, tyt^{-1} = x^2y, xy = yx\rangle.$$

Let $J = \langle t, x\rangle \cong (Z/4Z)^2$. Then $\sigma^2|_J = \rho\psi|_J = 0$, while $\rho\sigma|_J \neq 0$. Applying part (3) of Lemma 2.2, we see that $m = 0$, and so $\sigma^2 = \rho\psi$ and $\psi^2 = \rho\sigma$. (Note, however, that Lemma 2.2 does *not* assert that the relation $\psi^2 = q\rho\sigma + r\rho\psi$ also holds in $\pi/X^2(L_\phi)$! For this, we could use $L_\psi = \mathrm{Ker}(\rho) \cap \mathrm{Ker}(\psi)$ instead.) Hence

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \psi]/(\rho^2, \rho\psi + \sigma^2, \rho\sigma + \psi^2).$$

In particular, $\sigma^3 = \psi^3 = (\rho + \sigma)^3 = (\rho + \psi)^3 \neq 0$.

If $a \equiv -1 \pmod 4$ then $bc \equiv 0 \pmod 8$. If, say, $b \equiv 2 \pmod 4$ (so $c \equiv 0 \pmod 4$) then the change of basis $x' = x$, $y' = xy$ reduces this case to the one just considered. In terms of the given basis,

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \psi]/(\rho^2, \rho\sigma + \sigma^2, \rho\psi + \sigma^2 + \psi^2).$$

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In this case $\sigma^3 \neq 0$, but $\psi^3 = 0$. A similar result holds if $b \equiv 0 \pmod 4$ and $c \equiv 2 \pmod 4$.

(7) If, however, $a \equiv -1 \pmod 4$ and $b \equiv c \equiv 0 \pmod 4$ then $\pi/X^4(\pi)$ has a presentation

$$\langle t, x, y \mid t^4 = x^4 = y^4 = 1, txt^{-1} = x^{-1}, tyt^{-1} = y^{-1}, xy = yx\rangle.$$

In this case $J = \langle t, x\rangle$ is nonabelian, and $\sigma^2|_J \neq 0$, while $\rho\psi|_J = 0$. Hence we must have $m = r = 1$. It is clear from the symmetry of the presentation for $\pi/X^4(\pi)$ that we must also have $n = q$ in this case, and so $n = q = 0$. Thus

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \psi]/(\rho^2, \rho\sigma + \sigma^2, \rho\psi + \psi^2).$$

We now find that $\phi^3 = 0$ for all $\phi \in H^1(\pi)$.

If $\ell = 1$ and M is nonorientable then a and d are odd, and $\Delta_1 = -a - d \equiv 0 \pmod 4$. In this case $\rho = w_1(M)$, and so $\sigma^2\psi + \sigma\psi^2 = \rho\sigma\psi \neq 0$, by Lemma 2.1. After swapping x and y, if necessary, we may assume that $a \equiv 1 \pmod 4$.

(8) If $bc \equiv 0 \pmod 8$ then, after a further change of basis of the form $x' = x, y' = xy$ or $x' = xy, y' = y$, if necessary, we may assume that $b \equiv c \equiv 0 \pmod 4$. Then $\sigma^2 = 0$, and $\pi/\langle\langle t^2, x, y^4\rangle\rangle \cong D_8$, so $(\rho + \psi)\psi = 0$ also. Hence

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \psi]/(\rho^2, \sigma^2, \rho\psi + \psi^2).$$

In particular, $(\sigma + \psi)^3 = (\rho + \sigma + \psi)^3 \neq 0$, and all other classes have cube 0. In terms of the given bases, the other cases are as follows.

If $b \equiv 0$ and $c \equiv 2 \pmod 4$ then

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \psi]/(\rho^2, \sigma^2 + \psi^2, \rho\psi + \psi^2, \sigma^2\psi).$$

Here $\sigma^3 = (\rho + \sigma)^3 \neq 0$ and all other classes have cube 0.

If $b \equiv 2$ and $c \equiv 0 \pmod 4$ then

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \psi]/(\rho^2, \sigma^2, \psi^2 + \rho\sigma + \rho\psi).$$

Here $\psi^3 = (\rho + \psi)^3 \neq 0$ and all other classes have cube 0.

(9) If $b \equiv c \equiv 2 \pmod 4$ then σ^2 and ψ^2 are linearly independent. There are three distinct epimorphisms from π to the almost extraspecial group E, given by $f(x) = u^{-1}v$, $f(y) = u$; $g(x) = v$, $g(y) = uv^{-1}$; and $h(x) = v$, $h(y) = u$. Using these epimorphisms to pull back the relation given in Lemma 2.3, we find that

$$H^*(\pi) \cong \mathbb{F}_2[\rho, \sigma, \psi]/(\rho^2, \sigma^2 + \rho\psi, \psi^2 + \rho\sigma + \rho\psi).$$

In particular, every epimorphism $\phi \neq \rho$ has nonzero cube.

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4. Unions of twisted *I*-bundles

Suppose that $\pi/\sqrt{\pi} \cong D_\infty$. Then π is orientable, and has a presentation

$$\langle u, v, y \mid uyu^{-1} = y^{-1}, v^2 = u^{2a}y^b, vu^{2c}y^d v^{-1} = u^{-2c}y^{-d}\rangle,$$

where $ad - bc = \pm 1$ and $abcd \neq 0$. Let $B = \langle u, y\rangle$ and $C = \langle v, u^{2c}y^d\rangle$.

If b is odd then $\pi^{ab} \cong Z/4cZ \oplus Z/4Z$, where the summands are generated by u and $u^{-a}v$, respectively. Let $U(u) = V(v) = 1$, $U(v) = a$ and $V(u) = 0$. Then

$$H^*(\pi) \cong \mathbb{F}_2[U, V, \Xi, \Omega]/(U^2, UV, V^2, U\Xi + V\Omega, \Xi^2, \Omega^2, \Xi\Omega),$$

where Ξ and Ω have degree two.

If b is even then $\pi^{ab} \cong Z/4cZ \oplus (Z/2Z)^2$ and the images of u, v and y represent a basis for $H_1(\pi)$. Let $\{U, V, Y\} \in H^1(\pi)$ be the dual basis. Then U^2, V^2 and Y^2 are all nonzero, but $W = U + V$ lifts to a homomorphism from π to $Z/4Z$, and so $W^2 = 0$. Hence $U^2 = V^2$. Since U and V are induced from classes in $H^1(D_\infty)$ we have $UV = 0$. We also have $UY|_B = Y^2|_B$ and $VY|_C = Y^2|_C$, while $U|_C$, $V|_B$, $U^2|_B$ and $V^2|_C$ are all 0.

Suppose that $pU^2 + qY^2 + rUY + sVY = 0$ in $H^2(\pi)$. On restricting to the subgroups B and C, we find that $q + r = q + s = 0$. Since $U^2 \neq 0$ we must have $q = r = s = 1$. Multiplying by U and V, we find that $UY^2 + U^2Y = 0$ and $VY^2 + V^2Y = 0$. Poincaré duality for π now implies that $\{U^2, Y^2, UY\}$ is a basis for $H^2(\pi)$, while $UY^2 = U^2Y = VY^2$ generates $H^3(\pi)$. We see also that $U^3 = U^2V = UV^2 = V^3 = (U + V)^3 = 0$, while $(U + Y)^3 = (V + Y)^3 = (U + V + Y)^3 = Y^3$.

If $b \equiv 0 \pmod 4$ then $G = \pi/\langle\langle uv, u^2, y^4\rangle\rangle \cong D_8$. Hence $(U + V + Y)Y = 0$ in $H^3(\pi)$. It follows easily that $Y^3 = 0$, and so all cubes are 0 in $H^3(\pi)$.

If $b \equiv 2 \pmod 4$ then $\pi/\langle\langle u^2, (uv)^2, v^4, y^4\rangle\rangle$ has a presentation

$$\langle u, v, y \mid u^2 = (uv)^2 = v^4 = 1, uyu^{-1} = vyv^{-1} = y^{-1}, v^2 = y^2\rangle.$$

Hence there is an epimorphism $f : \pi \to E$, given by $f(u) = t$, $f(v) = u$ and $f(y) = u^{-1}t^{-1}v$. Since $f^*T = U + Y$, $f^*U = V + Y$, $f^*V = Y$ and $UV = 0$, it follows from Lemma 2.4 that $UY + VY + V^2 + Y^2 = 0$ in $H^2(\pi)$. Multiplying by Y, we find that $UY^2 + Y^3 = 0$ and so $Y^3 \neq 0$. In this case, only the cubes induced from $H^*(\pi/\sqrt{\pi})$ are zero.

5. The Borsuk–Ulam index

We may identify an epimorphism ϕ with a nonzero class in $H^1(N; \mathbb{F}_2)$. Then $\mathrm{BU}(N, \phi) = 1$ if and only if ϕ lifts to an integral class $\Phi \in H^1(N; \mathbb{Z})$, while $\mathrm{BU}(N, \phi) = n$ if and only if $\phi^n \neq 0$ in $H^n(N; \mathbb{F}_2)$. In general, $1 \leq \mathrm{BU}(N, \phi) \leq n$. See [3]. When $n = 3$ the remaining possibility is that $\mathrm{BU}(M, \phi) = 2$ if and only if $\phi^2 = 0$ but ϕ is not the reduction of an integral class.

Suppose first that $\pi/\sqrt{\pi} \cong \mathbb{Z}$. Then the following results are immediate from Section 3.

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(1) If $\rho : \pi \to Z/2Z$ is the unique epimorphism which factors through $\pi/\sqrt{\pi} \cong \mathbb{Z}$ then $\mathrm{BU}(M, \rho) = 1$.

(2) If $\tau \equiv \varepsilon - 1 \pmod 4$ then $\mathrm{BU}(M, \phi) = 3$ for all $\phi \neq \rho$.

(3) If $\tau \equiv \varepsilon + 1 \pmod 4$ and either Δ_2 is odd or $a \equiv d \equiv 1 \pmod 4$ and b, c are divisible by 4, then $\mathrm{BU}(M, \phi) = 2$ for all $\phi \neq \rho$.

(4) If $\varepsilon = 1$, $\Delta_1 \equiv 0 \pmod 8$ and $\Delta_2 \equiv 2 \pmod 4$ then $\mathrm{BU}(M, \phi) = 2$ for the two epimorphisms $\phi \neq \rho$ such that $\phi^2 = 0$ (that is, that factor through $Z/4Z$) and $\mathrm{BU}(M, \phi) = 3$ for the four such that $\phi^2 \neq 0$.

(5) If $\varepsilon = 1$, $\Delta_1 \equiv 4 \pmod 8$ and $\Theta \equiv -I_2 \pmod 4$ then $\mathrm{BU}(M, \phi) = 2$ for all $\phi \neq \rho$.

(6) If $\varepsilon = 1$ and $\Delta_1 \equiv 4 \pmod 8$, but $\Theta \not\equiv -I_2 \pmod 4$, then $\mathrm{BU}(M, \phi) = 2$ for the two epimorphisms $\phi \neq \rho$ such that $\phi^2 = 0$ and $\mathrm{BU}(M, \phi) = 3$ for the four such that $\phi^2 \neq 0$.

(7) If $\varepsilon = -1$, $\tau \equiv 0 \pmod 4$, $\Delta_2 \equiv 2 \pmod 4$ and $bc \equiv 0 \pmod 8$ then $\mathrm{BU}(M, \phi) = 2$ for the four epimorphisms $\phi \neq \rho$ such that $\phi^3 = 0$ and $\mathrm{BU}(M, \phi) = 3$ for the two such that $\phi^3 \neq 0$.

(8) If $\varepsilon = -1$, $\tau \equiv 0 \pmod 4$, $\Delta_2 \equiv 2 \pmod 4$ and $bc \equiv 4 \pmod 8$ then $\mathrm{BU}(M, \phi) = 3$ for all $\phi \neq \rho$.

Suppose now that $\pi/\sqrt{\pi} \cong D_\infty$. Then the following results are immediate from Section 4.

(9) If $\pi^{ab} \cong Z/4cZ \oplus Z/4Z$ then $\mathrm{BU}(M, \phi) = 2$ for all ϕ.

(10) If $\pi^{ab} \cong Z/4cZ \oplus (Z/2Z)^2$ and $b \equiv 0 \pmod 4$ then $\mathrm{BU}(M, \phi) = 2$ for all ϕ.

(11) If $\pi^{ab} \cong Z/4cZ \oplus (Z/2Z)^2$ and $b \equiv 2 \pmod 4$ then $\mathrm{BU}(M, \phi) = 2$ for epimorphisms ϕ which factor through $\pi/\sqrt{\pi}$, while $\mathrm{BU}(M, \phi) = 3$ otherwise.

6. Other geometries

We remark finally that similar arguments may be used to determine the $\mathbb{F}_2$-cohomology rings and Borsuk–Ulam invariants for pairs (N, ϕ) with N a closed $\mathbb{E}^3$- or $\mathbb{N}\mathrm{il}^3$-manifold. These manifolds are all Seifert fibred over flat 2-orbifolds. Since they have been covered in [1], we shall confine ourselves to some brief observations.

The ten closed flat 3-manifolds may be easily treated individually. The only one admitting a class ϕ with $\phi^3 \neq 0$ has group G_4, with holonomy $Z/4Z$ and abelianisation $\mathbb{Z} \oplus Z/2Z$. Thus $H^1(\pi) = \langle T, X\rangle$, where $T^2 = 0$ and $X^2 \neq 0$. We may deduce that $TX = 0$ also, by mapping G_4 onto D_8. It follows easily that

$$H^*(G_4) \cong \mathbb{F}_2[T, X, \Omega]/(T^2, TX, X\Omega, T\Omega + X^3, \Omega^2),$$

where Ω has degree two. (Thus $X^3 = (T + X)^3 \neq 0$. These classes correspond to the two epimorphisms without integral lifts.)

The possible Seifert bases B of closed $\mathbb{N}\mathrm{il}^3$-manifolds are the seven flat 2-orbifolds with no reflector curves: $B = T$, Kb, $S(2, 2, 2, 2)$, $S(2, 4, 4)$, $S(2, 3, 6)$, $S(3, 3, 3)$ or $P(2, 2)$. Let $\beta = \pi_1^{orb}(B)$ be the orbifold fundamental group of the base. Then π^{ab} is an extension of β^{ab} by a finite cyclic group Z/qZ, if the base is orientable ($B \neq Kb$ or $P(2, 2)$), and by $Z/(2, q)Z$ otherwise. The ring $H^*(\pi)$ depends only on the base B and the residue of $q \pmod 4$.

If $B = T$ or Kb then $\pi \cong \mathbb{Z}^2 \rtimes_\Theta \mathbb{Z}$, for some $\Theta \in GL(2, \mathbb{Z})$. These are in fact the cases requiring most effort. In all other cases π^{ab} is finite, and the projection of π onto β induces an isomorphism $H_1(\pi) \cong H_1(\beta)$. When $B = S(2, 3, 6)$ or $S(3, 3, 3)$ this group is cyclic. (In particular, such $\mathbb{N}\mathrm{il}^3$-manifolds are neither mapping tori nor unions of twisted I-bundles.) When $B = S(2, 4, 4)$ we have $\pi/X^4(\pi) \cong \beta/X^4(\beta) \cong G_4/X^4(G_4)$. The cases of $S(2, 2, 2, 2)$ and $P(2, 2)$ are related to those of the flat 3-manifolds G_2 and B_4, respectively.

The Borsuk–Ulam theorem and its applications and extensions are treated in detail in the book [7].

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