J. Austral. Math. Soc. Ser. B 26 (1985), 422–429

NONLINEAR PROGRAMMING DUALITY AND MATRIX GAME EQUIVALENCE

S. CHANDRA[1,2], B. D. CRAVEN[2] AND B. MOND[3]

(Received 13 February 1984)

Abstract

Certain well known results on linear programming duality and matrix game equivalence are extended to nonlinear and fractional programming problems.

1. Introduction

Consider the linear programming problem (LP) together with its dual (LD) as follows:

(LP) Minimize $c^T x$, subject to $Ax \geqslant b$, $x \geqslant 0$,

(LD) Maximize $b^T y$, subject to $A^T y \leqslant c$, $y \geqslant 0$,

where $c \in \mathbf{R}^n$, $x \in \mathbf{R}^n$, $b \in \mathbf{R}^m$, $y \in \mathbf{R}^m$, $A = [a_{ij}]$ is an $(m \times n)$ real matrix and the symbol ^{T} denotes the transpose.

Now consider the matrix game associated with the following $(n + m + 1) \times (n + m + 1)$ skew symmetric matrix B:

$$B = \begin{bmatrix} 0 & A^T & -c \\ -A & 0 & b \\ c^T & -b^T & 0 \end{bmatrix}.$$

Since B is skew symmetric, the value of the matrix game associated with B is zero and both players have the same optimal strategies. In the following, matrix game

[1] Mathematics Department, Indian Institute of Technology, Hauz Khas, New Delhi-110016, India.
[2] Mathematics Department, University of Melbourne, Parkville, Victoria 3052.
[3] Department of Mathematics, La Trobe University, Bundoora, Victoria 3083.

B will mean the matrix game associated with B and indices i and j will run from 1 to m and 1 to n respectively.

The following results due to Dantzig [5] are well known (Karlin [9], Gass [8]):

THEOREM 1. *Let* $\bar{x}$ *and* $\bar{y}$ *be optimal solutions to* (LP) *and* (LD) *respectively. Let* $z^* = 1/(1 + \Sigma_j \bar{x}_j + \Sigma_i \bar{y}_i)$, $x^* = z^*\bar{x}$, $y^* = z^*\bar{y}$. *Then* (x^*, y^*, z^*) *solves the matrix game* B.

THEOREM 2. *Let* (x^*, y^*, z^*) *be an optimal strategy of the matrix game* B *with* $z^* > 0$. *Let* $\bar{x}_j = (x_j^*/z^*)$, $\bar{y}_i = (y_i^*/z^*)$. *Then* $\bar{x}$ *and* $\bar{y}$ *are optimal solutions to* (LP) *and* (LD) *respectively.*

Thus Theorems 1 and 2 above give complete equivalence between linear programming duality and the matrix game B. It is natural then to ask whether such results hold for nonlinear programming problems as well. To the best of our knowledge, there is no explicit mention of such results beyond linearity in the literature. Of course, there are such results for infinite dimensions, *e.g.*, Forgó [7] for continuous linear programming and Tijs [13] for semi-infinite linear programming.

The purpose of this note is to present analogues of Theorems 1 and 2 for certain classes of nonlinear programming problems. These problems are finite dimensional and satisfy certain generalized convexity requirements. Nonlinear programming problems with linear constraints are studied in Section 2, while those with nonlinear constraints are treated in Section 3. Finally, certain remarks and conclusions about fractional programming and matrix games are included in Section 4.

2. Linear constraints case

Let us consider the linearly constrained nonlinear programming problem (NP), together with its dual (ND), as follows:

(NP):
$$\text{Minimize } f(x)$$
$$\text{subject to } Ax \geqslant b,\ x \geqslant 0;$$

(ND):
$$\text{Maximize } f(u) - y^T(Au - b)$$
$$\text{subject to } \nabla f(u) - A^T y \geqslant 0,$$
$$u^T[\nabla f(u) - A^T y] \leqslant 0,$$
$$y \geqslant 0.$$

https://doi.org/10.1017/S033427000000463X Published online by Cambridge University Press

Here x, y and A are as introduced in Section 1; $u \in \mathbf{R}^n$, f: $\mathbf{R}^n \to \mathbf{R}$ is differentiable and $\nabla f(u)$ denotes the gradient (column) vector of f at u.

Now it is to be observed that (ND) is the Mond and Weir [12] type dual of (NP) (in the presence of $x \geqslant 0$) and therefore if $f(x) - y^T(Ax - b)$ is a pseudoconvex function of x for all feasible x, u and y, then there is weak duality between (NP) and (ND). This will happen in particular if f is convex. Other examples when f is not convex are given in [3].

Now similarly to B, we define the $(n + m + 1) \times (n + m + 1)$ skew symmetric matrix $B_1(x)$ (depending on x) and study relations between the primal-dual pair (NP)–(ND) and the matrix game $B_1(x)$, where

$$B_1(x) = \begin{bmatrix} 0 & A^T & -\nabla f(x) \\ -A & 0 & b \\ \nabla f(x)^T & -b^T & 0 \end{bmatrix}.$$

The Mond and Weir dual [12] of (NP) is taken here for convenience. We could also start with the Wolfe dual (*e.g.* Craven [4]), and prove similar equivalence results.

LEMMA 1. *For a matrix game described by a skew symmetric matrix B, the value of the game is zero, and $\bar{y}$ is an optimal strategy for player* II (*or* I) *if and only if* $B\bar{y} \leqslant 0$.

PROOF. Let B be $n \times n$. If $(\bar{x}, \bar{y})$ is optimal for the game B, then $x^TB\bar{y} \leqslant \bar{x}^TB\bar{y} \leqslant \bar{x}^TBy$ for all x, $y \in S$, the set of probability vectors in $\mathbf{R}^n$. Setting $y = \bar{x}$ shows that $x^TB\bar{y} \leqslant \bar{x}^TB\bar{x} = 0$, for all $x \in S$, hence $B\bar{y} \leqslant 0$. Note that skew symmetry of B gives $\bar{x}^TB\bar{x} = 0$, hence zero value for the game. Conversely, assume that $B\bar{y} \leqslant 0$. Then $x^TB\bar{y} \leqslant 0$ for each $x \in S$. So $\bar{y}^TB\bar{y} = 0 \geqslant x^TB\bar{y} = -\bar{y}^TBx$ since B is skew symmetric. Hence $\bar{y}^TBx \geqslant 0 = \bar{y}^TB\bar{y} \geqslant x^TB\bar{y}$. Then $(\forall x \in S) x^TB\bar{y} \leqslant \bar{y}^TB\bar{y}$ and $(\forall y \in S) \bar{y}^TBy \geqslant \bar{y}^TB\bar{y}$, which show that $(\bar{y}, \bar{y})$ is optimal for the game.

THEOREM 3. *Let $\bar{x}$ and $(\bar{x}, \bar{y})$ be feasible solutions to* (NP) *and* (ND) *respectively, such that $\bar{y}^T(A\bar{x} - b) = 0$. Let $z^* = 1/(1 + \Sigma_j\bar{x}_j + \Sigma_i\bar{y}_i)$, $x^* = z^*\bar{x}$ and $y^* = z^*\bar{y}$. Then (x^*, y^*, z^*) solves the matrix game $B_1(\bar{x})$.*

PROOF. By the conditions of the theorem we have the following:

$$A\bar{x} \geqslant b, \tag{1}$$

$$\nabla f(\bar{x}) - A^T\bar{y} \geqslant 0, \tag{2}$$

$$\bar{x}^T\left[\nabla f(\bar{x}) - A^T\bar{y}\right] \leqslant 0, \tag{3}$$

$$\bar{y}^T(A\bar{x} - b) = 0, \tag{4}$$

$$\bar{x} \geqslant 0, \quad \bar{y} \geqslant 0. \tag{5}$$

But $z^* > 0$ by (5), and hence, expressing relations (1) to (5) in terms of x^*, y^* we get:

$$-Ax^* + z^*b \leqslant 0, \tag{6}$$

$$A^Ty^* - z^*\nabla f(\bar{x}) \leqslant 0, \tag{7}$$

$$x^{*T}\left[z^*\nabla f(\bar{x}) - A^Ty^*\right] \leqslant 0, \tag{8}$$

$$y^{*T}(Ax^* - z^*b) = 0, \tag{9}$$

$$x^* \geqslant 0, \quad y^* \geqslant 0, \tag{10}$$

$$x^* + y^* + z^* = 1. \tag{11}$$

Now (8) and (9) give

$$z^*\left[x^{*T}\nabla f(\bar{x}) - y^{*T}b\right] \leqslant 0,$$

i.e.,

$$\left[\nabla f(\bar{x})\right]^T x^* - b^Ty^* \leqslant 0. \tag{12}$$

But (6), (7) and (12) imply

$$B_1(\bar{x})\xi^* \leqslant 0, \tag{13}$$

with $\xi^* = \mathrm{col}(x^*, y^*, z^*)$, where col denotes column vector. Now (13), together with (10) and (11), implies by Lemma 1 that ξ^* is an optimal strategy for Player II in the matrix game $B_1(\bar{x})$. Since $B_1(\bar{x})$ is skew symmetric, the value of the matrix game $B_1(\bar{x})$ is zero and ξ^* is an optimal strategy to Player I as well. Thus (x^*, y^*, z^*) solves the matrix game $B_1(\bar{x})$.

THEOREM 4. *Let* (x^*, y^*, z^*) *with* $z^* > 0$, *solve the matrix game* $B_1(\bar{x})$, *where* $\bar{x} = x^*/z^*$. *Let* $\bar{y} = y^*/z^*$. *Then* $\bar{x}$ *and* $(\bar{x}, \bar{y})$ *are feasible solutions to* (NP) *and* (ND) *respectively with* $f(\bar{x}) = f(\bar{x}) - \bar{y}^T(A\bar{x} - b)$. *In addtion, if there is weak duality between* (NP) *and* (ND) *then* $\bar{x}$ *and* $(\bar{x}, \bar{y})$ *are optimal to the respective problems.*

PROOF. Let $\xi^* = \mathrm{col}(x^*, y^*, z^*)$. Then, by Lemma 1, $B_1(\bar{x})\xi^* \leqslant 0$, thus

$$-Ax^* + z^*b \leqslant 0, \tag{14}$$

$$A^Ty^* - z^*\nabla f(\bar{x}) \leqslant 0, \tag{15}$$

$$\left[\nabla f(\bar{x})\right]^T x^* - b^Ty^* \leqslant 0, \tag{16}$$

$$x^* + y^* + z^* = 1, \tag{17}$$

$$x^* \geqslant 0, \quad y^* \geqslant 0, \quad z^* > 0. \tag{18}$$

Since $z^* > 0$, (14) and (18) give the feasibility of $\bar{x}$ to (NP). Similarly (15), (16) and (18) show that $(\bar{x}, \bar{y})$ is feasible to (ND). Also, using (15) and (16),

$$\bar{y}^T(A\bar{x} - b) \leqslant \bar{x}^T \nabla f(\bar{x}) - \bar{y}^T b \leqslant 0. \tag{19}$$

But $\bar{y}^T(A\bar{x} - b) \geqslant 0$ by (14). Therefore (19) implies that $\bar{y}^T(A\bar{x} - b) = 0$. This proves the equality of primal and dual objective functions and hence the theorem.

REMARK. The hypothesis of Theorem 4 seems to require a knowledge of $\bar{x}$ in order to find $\bar{x}$, but such results are not uncommon in nonlinear programming (for reference see Karlin [9], Kortanek and Evans [10]). Sometimes these may even be exploited for solving (NP) by developing certain systematic procedures of suitably adjusting the parameter $\bar{x}$, *e.g.* Frank and Wolfe [6], Mangasarian [11], Bhat [2] and Bector and Bhat [1].

3. Nonlinear constraints case

Let us consider the general nonlinear programming problem (NP1) together with its dual (ND1) as follows:

(NP1)
$$\text{Minimize } f(x)$$
$$\text{subject to } g(x) \geqslant b,\ x \geqslant 0;$$

(ND1)
$$\text{Maximize } f(u) - y^T[g(u) - b]$$
$$\text{subject to } \nabla f(u) - \nabla(y^T g)(u) \geqslant 0,$$
$$u^T[\nabla f(u) - \nabla(y^T g)(u)] \leqslant 0,$$
$$y \geqslant 0.$$

Here x, u, b, y and f are the same as in Section 2, and g: $\mathbf{R}^n \to \mathbf{R}^m$ is continuously differentiable. The gradient $\nabla f(x)$ is an $n \times 1$ matrix, and $\nabla g(x)$ is an $n \times m$ matrix. The problem (ND1) is the Mond and Weir type dual [12] of (NP1), and therefore under suitable assumptions, strong and converse duality theorems hold between (NP1) and (ND1).

Define now the $(n + m + 1) \times (n + m + 1)$ skew symmetric matrix $B_2(x)$ by

$$B_2(x) = \begin{bmatrix} 0 & \nabla g(x) & -\nabla f(x) \\ -\nabla g(x)^T & 0 & b - g(x) + (\nabla g(x))^T x \\ \nabla f(x)^T & -(b - g(x))^T - x^T \nabla g(x) & 0 \end{bmatrix}.$$

THEOREM 5. *Let $\bar{x}$ and $(\bar{x}, \bar{y})$ be feasible solutions to* (NP1) *and* (ND1) *respectively, with $\bar{y}^T(g(\bar{x}) - b) = 0$. Let $z^* = 1/(1 + \Sigma_j \bar{x}_j + \Sigma_i \bar{y}_i)$, $x^* = z^* \bar{x}$ and $y^* = z^* \bar{y}$. Then (x^*, y^*, z^*) solves the matrix game $B_2(\bar{x})$.*

PROOF. From the hypotheses of the theorem, we have the following

$$g(\bar{x}) \geqslant b, \tag{20}$$

$$\nabla f(\bar{x}) - (\nabla g(\bar{x}))\bar{y} \geqslant 0, \tag{21}$$

$$\bar{x}^T[\nabla f(\bar{x}) - (\nabla g(\bar{x}))\bar{y}] \leqslant 0, \tag{22}$$

$$\bar{y}^T(g(\bar{x}) - b) = 0, \tag{23}$$

$$\bar{x} \geqslant 0, \quad \bar{y} \geqslant 0. \tag{24}$$

From Lemma 1, it suffices to prove that $B_2(\bar{x})\xi^* \leqslant 0$, where $\xi^* = \text{col}(x^*, y^*, z^*)$, namely that

$$-(\nabla g(\bar{x}))^T x^* + \left(b - g(\bar{x}) + (\nabla g(\bar{x}))^T \bar{x}\right) z^* \leqslant 0, \tag{25}$$

$$\nabla(y^{*T} g)(\bar{x}) - z^* \nabla f(\bar{x}) \leqslant 0, \tag{26}$$

$$x^{*T} \nabla f(\bar{x}) - y^{*T}\left(b - g(\bar{x}) + (\nabla g(\bar{x}))^T \bar{x}\right) \leqslant 0, \tag{27}$$

$$x^* + y^* + z^* = 1, \tag{28}$$

$$x^* \geqslant 0, \quad y^* \geqslant 0, \quad z^* > 0. \tag{29}$$

Here (28) and (29) follow from (24) and the definitions x^*, y^* and z^*; (26) follows from (21) after multiplication by z^*; (25) follows from (20), with $x^* = z^*\bar{x}$; and (27) follows from (22) and (23), after multiplication by z^*.

THEOREM 6. *Let* (x^*, y^*, z^*) *with* $z^* > 0$ *solve the matrix game* $B_2(\bar{x})$, *where* $\bar{x} = x^*/z^*$. *Let* $\bar{y} = y^*/z^*$. *Then* $\bar{x}$ *and* $(\bar{x}, \bar{y})$ *are feasible solutions for* (NP1) *and* (ND1) *respectively, with the two objective functions having equal values. If also weak duality holds between* (NP1) *and* (ND1), *then* $\bar{x}$ *is optimal for* (NP1) *and* $(\bar{x}, \bar{y})$ *is optimal for* (ND1).

PROOF. Let $\xi^* = \text{col}(x^*, y^*, z^*)$. Since ξ^* solves the game $B_2(\bar{x})$ and the matrix is skew symmetric, $B_2(\bar{x})\xi^* \leqslant 0$ by Lemma 1. Thus (25), (26), (27), (28) and (29) hold. Dividing (25) by $z^* > 0$ gives $b - g(\bar{x}) \leqslant 0$, so that $\bar{x}$ is feasible for (NP1). Dividing (26) by z^* gives $\nabla f(\bar{x}) \geqslant \nabla(\bar{y}^T g)(\bar{x})$, hence, since $x^* \geqslant 0$, $\bar{x}^T \nabla(f - \bar{y}^T g)(\bar{x}) \geqslant 0$. Dividing (27) by z^* gives

$$\bar{x}^T \nabla\left(f - \bar{y}^T g\right)(\bar{x}) \leqslant \bar{y}^T(b - g(\bar{x})).$$

Hence $\bar{y}^T(g(\bar{x}) - b) \leqslant 0$. But $\bar{y} \geqslant 0$ from (29), and $g(\bar{x}) - b \geqslant 0$ has been proved; so $\bar{y}^T(g(\bar{x}) - b) \geqslant 0$. Hence $\bar{y}^T(g(\bar{x}) - b) = 0$, and also $\bar{x}^T \nabla(f - \bar{y}^T g)(\bar{x}) = 0$. Thus $(\bar{x}, \bar{y})$ is feasible for (ND1), and the objective function of (NP1) at $\bar{x}$ equals the objective function of (ND1) at $(\bar{x}, \bar{y})$. This, with weak duality, proves that $\bar{x}$ is optimal for (NP1) and $(\bar{x}, \bar{y})$ is optimal for (ND1).

4. Fractional programming

Consider the following fractional programming problem (FP):

(FP) Minimize $[f(x)/h(x)]$ subject to $Ax \geqslant b, \quad x \geqslant 0.$

Here $h\colon \mathbf{R}^n \to \mathbf{R}$ and $f\colon \mathbf{R}^n \to \mathbf{R}$, over the feasible region of (FP), are differentiable functions, with $f \geqslant 0$ and $h > 0$. One of the fractional duals to (FP) is the following:

(FD):
$$\text{Maximize } f(u)/h(u) - y^T(Au - b)$$
$$\text{subject to } \nabla[(f/h)(u)] - A^T y \geqslant 0,$$
$$u^T[\nabla(f/h)(u) - A^T y] \leqslant 0,$$
$$y \geqslant 0.$$

Let us now define the $(n + m + 1) \times (n + m + 1)$ skew symmetric matrix $B_3(x)$ as follows:

$$B_3(x) = \begin{bmatrix} 0 & A^T & -\nabla(f/h)(x) \\ -A & 0 & b \\ [\nabla(f/h)(x)]^T & -b^T & 0 \end{bmatrix}.$$

Then we have the following theorems similar to Theorems 3 and 4:

THEOREM 7. *Let $\bar{x}$ and $(\bar{x}, \bar{y})$ be feasible solutions to* (FP) *and* (FD) *respectively such that $\bar{y}^T(A\bar{x} - b) = 0$. Let $z^* = 1/(1 + \Sigma_j \bar{x}_j + \Sigma_i \bar{y}_i)$, $x^* = z^*\bar{x}$ and $y^* = z^*\bar{y}$. Then (x^*, y^*, z^*) solves the matrix game $B_3(\bar{x})$.*

THEOREM 8. *Let (x^*, y^*, z^*) with $z^* > 0$, solve the matrix game $B_3(\bar{x})$, where $\bar{x} = x^*/z^*$. Let $\bar{y} = y^*/z^*$. Then $\bar{x}$ and $(\bar{x}, \bar{y})$ are feasible solutions to* (FP) *and* (FD) *respectively with objective functions being equal. In addition, if there is weak duality between* (FP) *and* (FD) *then $\bar{x}$ and $(\bar{x}, \bar{y})$ are optimal to the respective problems.*

For fractional programs with nonlinear constraints, similar comments and conclusions hold as in Section 3.

Acknowledgement

One of the authors (S. Chandra) acknowledges the finanical support and excellent research facilities provided by the authorities of the University of Melbourne during his term of a visiting research fellowship.

References

[1] C. R. Bector and S. K. Bhat, "Pseudo-monotonic interval programming", *Naval Res. Logist. Quart.* 25 (1978), 309–314.

[2] S. K. Bhat, "Linearization techniques for linear fractional and pseudomonotonic programming revisited", *Cahiers Centre Études Rech. Opér.* 23 (1981), 53–56.

[3] S. Chandra, B. D. Craven and B. Mond, "Generalized concavity and duality with a square root term", *Optimization* (1985), to appear.

[4] B. D. Craven, *Mathematical programming and control theory* (Chapman and Hall, London, 1978).

[5] G. B. Dantzig, "A proof of the equivalence of the programming problem and the game problem", in *Activity analysis of production and allocation* (ed. T. C. Koopmans), *Cowles Commission Monograph No.* 13, (John Wiley and Sons, 1951), 330–335.

[6] M. Frank and P. Wolfe, "An algorithm for quadratic programming", *Naval Res. Logist. Quart.* 3 (1956), 95–110.

[7] F. Forgó, "The relationship between continuous zero-sum two person games and linear programming", *Res. Rep.* 1969-1, Dept. of Mathematics, Karl Marx University of Economics, Budapest.

[8] S. Gass, *Linear programming* (McGraw-Hill and Kogakusha, 1964).

[9] S. Karlin, *Mathematical methods and theory in games, programming and mathematical economics* (Addison-Wesley, Reading, Mass., 1959).

[10] K. O. Kortanek and J. P. Evans, "Pseudo-concave programming and convex Lagrange regularity", *Oper. Res.* 15 (1967), 882–891.

[11] O. L. Mangasarian, "Nonlinear fractional programming", *J. Oper. Res. Soc. Japan* 12 (1969), 1–10.

[12] B. Mond and T. Weir, "Generalized concavity and duality", in *Generalized concavity in optimization and economics* (eds. S. Schaible and W. T. Ziemba), (Academic Press, 1981), 263–279.

[13] S. H. Tijs, "Semi-infinite linear programs and semi-infinite matrix games," *Nieuw Arch. Wisk.* 27 (1969), 197–214.