A NOTE ON A RECENT PAPER BY ZAKS, FROSTIG AND LEVIKSON

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ABSTRACT

In the present paper we give a short proof of a result of Zaks, Frostig and Levikson [2006] on the solution of an optimization problem which is related to the problem of optimal pricing of a heterogeneous portfolio.

Following Zaks, Frostig and Levikson [2006], we consider a heterogeneous portfolio which is composed by k risk classes such that for each $j \in \{1, \ldots, k\}$ the risk class j contains n_j risks $X_{j,1}, \ldots, X_{j,n_j}$ which are assumed to be i.i.d. with finite first and second moments and non-zero variance. Then the total risk of risk class j is defined as

$$S_j := \sum_{i=1}^{n_j} X_{j,i}$$

Consider also $r_1, \ldots, r_k \in (0,\infty)$ and $\alpha \in (0,1)$, and let $z_{1-\alpha}$ denote the $1-\alpha$ percentile of the standard normal distribution. The authors prove the following result:

Theorem 1. *The minimization problem*

Minimize

$$\sum_{j=1}^{k} \left(\frac{1}{r_j} E\left[\left(S_j - n_j \pi_j \right)^2 \right] \right)$$

over $\pi_1, \ldots, \pi_k$ *subject to*

$$\sum_{j=1}^{k} n_j \pi_j = E\left[\sum_{j=1}^{k} S_j \right] + z_{1-\alpha} \sqrt{\operatorname{var}\left[\sum_{j=1}^{k} S_j \right]}$$

has a unique solution $\pi_1^*, \ldots, \pi_k^*$ *and the identity*

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$$\pi_j^* = \frac{1}{n_j}\left(E[S_j] + \frac{r_j}{\sum_{i=1}^k r_i} z_{1-\alpha}\sqrt{\mathrm{var}\left[\sum_{i=1}^k S_i\right]}\right)$$

holds for all $j \in \{1, \ldots, k\}$.

Let now $\mathbf{S}$ denote the random vector with coordinates $S_1, \ldots, S_k$ and let $\boldsymbol{v} := E[\mathbf{S}]$. Let also $\mathbf{V}$ denote the diagonal matrix with diagonal elements $r_1, \ldots, r_k$, let $\mathbf{1}$ denote the vector with all coordinates being equal to one, and consider $t \in \mathbb{R}$. Using this notation, Theorem 1 can be stated in the following form, which suggests a simple proof based on the projection theorem in Hilbert spaces (see e.g. De Vylder [1996; Part III] or Swartz [1994; Section 6.6]):

Theorem 1'. *The minimization problem*

Minimize

$$E\left[(\mathbf{S}-\mathbf{p})'\mathbf{V}^{-1}(\mathbf{S}-\mathbf{p})\right]$$

over $\mathbf{p}$ *subject to* $\mathbf{1}'\mathbf{p} = \mathbf{1}'\boldsymbol{v} + t$

has a unique solution $\mathbf{p}^*$ *and the solution satisfies* $\mathbf{p}^* = \boldsymbol{v} + t(\mathbf{1}'\mathbf{V}\mathbf{1})^{-1}\mathbf{V}\mathbf{1}$.

Proof. Since the matrix $\mathbf{V}$ is symmetric and positive definite, the vector space $L^2(\mathbb{R}^k)$ consisting of all k-dimensional random vectors having finite second moments is a Hilbert space under the inner product $\langle .,. \rangle_{\mathbf{V}}$ given by

$$\langle \mathbf{X}, \mathbf{Y} \rangle_{\mathbf{V}} := E[\mathbf{X}'\mathbf{V}^{-1}\mathbf{Y}]$$

and the induced norm $\| . \|_{\mathbf{V}}$ given by

$$\| \mathbf{X} \|_{\mathbf{V}} := \langle \mathbf{X}, \mathbf{X} \rangle_{\mathbf{V}}^{1/2}$$

(Here, as usual, two random vectors $\mathbf{X}, \mathbf{Y}$ are identified if $P[\{\mathbf{X} = \mathbf{Y}\}] = 1$.) Furthermore, the set

$$A := \left\{\mathbf{p} \in \mathbb{R}^k \,\middle|\, \mathbf{1}'\mathbf{p} = \mathbf{1}'\boldsymbol{v} + t\right\}$$

is a nonempty closed subset of $L^2(\mathbb{R}^k)$. Since A is convex, it follows from the projection theorem in Hilbert spaces that the minimization problem

Minimize

$$\| \boldsymbol{S} - \mathbf{p} \|_{\mathbf{V}}$$

over $\mathbf{p} \in A$

has a unique solution $\mathbf{p}^* \in A$. Since A is even affine, $\mathbf{p}^*$ is also the unique solution to the normal equations

$$\langle \mathbf{S} - \mathbf{p}^*, \mathbf{p} - \mathbf{p}^* \rangle_{\mathbf{V}} = 0$$

with $\mathbf{p} \in A$ being arbitrary. Using the definition of the inner product $\langle .,. \rangle_{\mathbf{V}}$, the normal equations can also be written as

$$(\boldsymbol{v} - \mathbf{p}^*)' \mathbf{V}^{-1} (\mathbf{p} - \mathbf{p}^*) = 0$$

We now observe that every vector $\mathbf{q}_\gamma := \boldsymbol{v} + \gamma \mathbf{V1}$ with $\gamma \in \mathbb{R}$ satisfies

$$(\boldsymbol{v} - \mathbf{q}_\gamma)' \mathbf{V}^{-1} (\mathbf{p} - \mathbf{q}_\gamma) = -\gamma (\mathbf{1}'\mathbf{p} - \mathbf{1}'\mathbf{q}_\gamma)$$

and that $\mathbf{q}_\gamma \in A$ if and only if $\gamma = t(\mathbf{1}'\mathbf{V1})^{-1}$. We have thus shown that the vector $\mathbf{q} := \boldsymbol{v} + t(\mathbf{1}'\mathbf{V1})^{-1}\mathbf{V1}$ satisfies $\mathbf{q} \in A$ and

$$(\boldsymbol{v} - \mathbf{q})' \mathbf{V}^{-1} (\mathbf{p} - \mathbf{q}) = 0$$

for all $\mathbf{p} \in A$. Therefore, we have $\mathbf{q} = \mathbf{p}^*$. □

References

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