J. Austral. Math. Soc. (*Series A*) 40 (1986), 407–413

PERTURBATIONS OF TYPE I AW*-ALGEBRAS

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(Received 14 May 1984; revised 2 October 1984)

Communicated by J. H. Chabrowski

Abstract

The distance between two operator algebras acting on a Hilbert space H is defined to be the Hausdorff distance between their unit balls. We investigate the structural similarities between two close AW*-algebras A and B acting on a Hilbert space H. In particular, we prove that if A is of type I and separable, then A and B are $*$-isomorphic.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 46 L 05.

Introduction

Our main result states that if A and B are AW*-algebras acting on a Hilbert space H and $\|A - B\|$ is sufficiently small (see Section 1 for the definition), then under certain conditions (for example, if A is type I and separable) A and B are $*$-isomorphic. First we show that if A and B are close AW*-algebras, then their central projections corresponding to various portion of type are also close. This is known for von-Neumann algebras [5] and the proof for AW*-algebras given here is similar. However, Lemma 1.8, which corresponds to Lemma 15 of [5], is done completely differently. We also show that close AW*-algebras have close centers. We prove our main Theorem 2.3 by using these and Kaplansky's Theorem 1 of [7].

These results are part of the author's Ph.D. dissertation done at Dalhousie University under the supervision of Professor John Phillips.

1. Stability of type

We recall the the *distance* between C*-algebras A and B acting on a Hilbert space H is defined by

$$\|A - B\| = \sup\left\{\inf_a \|a - b\|, \inf_b \|a - b\| : a \in A_1, b \in B_1\right\}$$

where A_1 and B_1 are the unit balls of A and B respectively.

1.1. NOTATION. As usual for an AW*-algebra A, P_{I}, P_{II}, P_{III} denote the unique maximal central projections in A such that $P_{\mathrm{I}}A$, $P_{\mathrm{II}}A$ and $P_{\mathrm{III}}A$ are of type I, II and III respectively. In the case that A is of type I or II we denote by P_{I_1}, P_{I_∞}, P_{II_1}, P_{II_∞} the central projections in A corresponding to the finite and properly infinite portions of A. By I_A we denote the identity of A. If e is a projection in A, then $c(e)$ denotes the central cover of e in A. Our reference on AW*-algebras is [1].

1.2. REMARK. Let A and B be C*-algebras acting on H with $\|A - B\| < \gamma < 1/2$. By [2, Lemma 2.1], if $p \in A$ is a projection, we can choose a projection $q \in B$ such that $\|p - q\| < 2\gamma$. Moreover, if p is central and $\gamma \leqslant 1/6$, p is abelian and $\gamma \leqslant 1/30$ or p is finite and $\gamma \leqslant 1/40$, then q is central, abelian or finite respectively (cf. [5] and [8, Lemma 2.3]).

1.3. LEMMA. *Let A and B be AW*-algebras acting on H such that $\|A - B\| < \gamma < 1/200$. Let h_i, $i = 1, 2, 3, 4$ be the unique central projections in A such that*
(i) *$h_1 A$ is finite and $(\mathrm{I}_A - h_1)A$ is properly infinite,*
(ii) *$h_2 A$ is abelian and $(\mathrm{I}_A - h_2)A$ is properly non-abelian,*
(iii) *$h_3 A$ is semifinite and $(\mathrm{I}_A - h_3)A$ is purely infinite,*
(iv) *$h_4 A$ is discrete and $(\mathrm{I}_A - h_4)A$ is continous.*
If g_i, $i = 1, 2, 3, 4$ are the corresponding projections in B, then $\|h_i - g_i\| < 2\gamma$, $i = 1, 2, 3, 4$.

PROOF. Let $\{h_\alpha\}$ be a maximal orthogonal family of non-zero finite central projections in A. By [1, §15, Theorem 1], $h_1 = \sup h_\alpha$. Now by 1.2 we can choose for each α a finite central projection $k_\alpha \in B$ such that $\|h_\alpha - k_\alpha\| < 2\gamma$. If $\alpha \neq \beta$, then $h_\alpha h_\beta = 0$ and we have

$$\|k_\alpha k_\beta\| \leqslant \|k_\alpha k_\beta - h_\alpha k_\beta\| + \|h_\alpha k_\beta - h_\alpha h_\beta\| < 4\gamma < 1.$$

https://doi.org/10.1017/S1446788700027579 Published online by Cambridge University Press

Thus $k_\alpha k_\beta = 0$ and $\{k_\alpha\}$ is an orthogonal family of projections in B. Suppose for some finite central projection $k \in B$, $kk_\alpha = 0$ for every α. Choose a finite central projection $h \in A$ such that $\|k - h\| < 2\gamma$. Then it follows that $hh_\alpha = 0$ for every α, which is in contradiction with the maximality of $\{h_\alpha\}$. This shows that $\{k_\alpha\}$ is a maximal family of finite central projections and [1, §15, Theorem 1] implies that $k_1 = \sup k_\alpha$. Let $\hat{k} \in B$ be a finite projection such that $\|h - h_1\| < 2\gamma$. We show that $\hat{k} = k_1$. Now $\hat{k} \leqslant k_1$ and

$$\begin{aligned}\|k_\alpha - \hat{k}k_\alpha\| &\leqslant \|k_\alpha - h_\alpha\| + \|h_1 h_\alpha - h_1 k_\alpha\| + \|h_1 k_\alpha - \hat{k}k_\alpha\| \\ &\leqslant \|k_\alpha - h_\alpha\| + \|h_\alpha - k_\alpha\| + \|h_1 - \hat{k}\| < 6\gamma < 1.\end{aligned}$$

Hence $k_\alpha = \hat{k}k_\alpha$, i.e. $\hat{k} \geqslant k_\alpha$ for every α. Therefore $\hat{k} \geqslant k_1$. This together with $\hat{k} \leqslant k_1$ implies that $\hat{k} = k_1$.

Next we show that $\|h_3 - k_3\| < 2\gamma$. By [1, §15, Theorem 1], $h_3 = \sup h_\alpha$ for a maximal orthogonal family $\{h_\alpha\}$ of semifinite central projections. For each α choose a projection $k_\alpha \in B$ such that $\|k_\alpha - h_\alpha\| < 2\gamma$. We show that k_α is semifinite. Since h_α is semifinite $h_\alpha = c(e_\alpha)$ for some finite projection e_α. Now it follows from $\|h_\alpha - k_\alpha\| < 2\gamma$ and $\|A - B\| < \gamma$ that $\|h_\alpha A - k_\alpha B\| < 5\gamma \leqslant 1/40$. Hence, as mentioned in 1.2, we can choose a finite projection $f_\alpha \in k_\alpha B$ such that $\|f_\alpha - e_\alpha\| < 10\gamma$. Then it follows from [5, Lemma 7] that $\|c(e_\alpha) - c(f_\alpha)\| < 20\gamma$. Now

$$\|k_\alpha - c(f_\alpha)\| \leqslant \|k_\alpha - h_\alpha\| + \|c(e_\alpha) - c(f_\alpha)\| < 22\gamma < 1,$$

so that $c(f_\alpha) = k_\alpha$. Hence k_α is a semifinite projection. Moreover $k_3 = \sup k_\alpha$ and the rest of the proof goes as in the first paragraph. Similar arguments can be used in order to show that $\|h_2 - k_2\| < 2\gamma$ and $\|h_4 - k_4\| < 2\gamma$, and we omit the details.

1.4. LEMMA. *Let A and B be AW*-algebras acting on H with $\|A - B\| < \gamma \leqslant 1/200$. Let P_{I}, P_{II}, P_{III}, P_{I_1}, P_{I_∞}, P_{II_1}, P_{II_∞} be the unique maximal central projections described in* 1.1 *and let Q_{I}, Q_{II}, Q_{III}, Q_{I_1}, Q_{I_∞}, Q_{II_1}, Q_{II_∞} be the corresponding projections in B. Then*

$$\|P_x - Q_x\| < 2\gamma \quad \text{for } x \in \Gamma = \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{I}_1, \mathrm{I}_\infty, \mathrm{II}_1, \mathrm{II}_\infty\}.$$

PROOF. By [1, Section 15, Theorems 2 and 3], we have $P_{\mathrm{I}} = h_4$, $P_{\mathrm{II}} = h_3(\mathrm{I}_A - h_4)$, $P_{\mathrm{III}} = \mathrm{I}_A - h_3$, $P_{\mathrm{I}_1} = P_{\mathrm{I}} h_1$, $P_{\mathrm{I}_\infty} = P_{\mathrm{I}}(\mathrm{I}_A - h_1)$, $P_{\mathrm{II}_1} = P_{\mathrm{II}} h_1$ and $P_{\mathrm{II}_\infty} = P_{\mathrm{II}}(\mathrm{I}_A - h_1)$. Now from 1.4 and the fact that $\|\mathrm{I}_A - \mathrm{I}_B\| < 2\gamma$ one can easily verify that $\|P_x - Q_x\| < 6\gamma$ for every $x \in \Gamma$. Now by 1.2, for each $x \in \Gamma$, we can choose a central projection $Q'_x \in B$ such that $\|P_x - Q'_x\| < 2\gamma$. Hence

$$\|Q_x - Q'_x\| \leqslant \|Q_x - P_x\| + \|P_x - Q'_x\| < 8\gamma \leqslant 1.$$

This implies that $Q_x = Q'_x$ and hence $\|Q_x - P_x\| < 2\gamma$, as desired.

1.5. REMARK. We recall that an AW*-algebra A is said to be $\aleph$- *homogeneous* if there exists an orthogonal family $\{e_\alpha\}_{\alpha\in\Omega}$ of pairwise equivalent abelian projections in A such that $I_A = \sup e_\alpha$, where $\operatorname{card}\Omega = \aleph$. In this case we say that A is of type $I_\aleph$. We note that a homogeneous AW*-algebra is necessarily of type I.

1.6. REMARK. Let A be a C*-algebra and $e, f \in A$ be projections. We write $e \sim f$ if e and f are Murray von-Neumann equivalent. The equivalence class of e under $\sim$ is denoted by $[e]$. The set of these equivalence classes, denoted by $S(A)$, is equipped with a partial addition as follows: $[e]$, $[f] \in S(A)$ can be added if there exist projections e', $f' \in A$ such that $e' \sim e$, $f' \sim f$ and $e'f' = 0$. Then we set $[e] + [f] = [e' + f']$. If A and B are C*-algebras acting on H and $\|A - B\| < \gamma \leqslant 1/8$, then by [8, Theorem 2.6], there exists an isomorphism ρ: $S(A) \to S(B)$ defined by closeness, i.e. by $\rho([e]) = [f]$ if $\|e - f\| < 2\gamma$. We use these notations in the following lemma without further comments.

1.7. LEMMA. *Let A and B be* AW*-*algebras acting on H and $\|A - B\| < \gamma \leqslant 1/60$. Suppose A is of type $I_{\aleph_0}$ and $\{e_n\}$ is an orthogonal sequence of equivalent abelian projections in A such that $I_A = \sup e_n$. Then there exist sequences $\{f_n\}$ and $\{f_n'\}$ of projections in B such that*

(i) $f_n \sim f_n'$ *for every* n,

(ii) $\|f_n' - e_n\| < 2\gamma$ *for every* n,

(iii) $f_n f_m = 0$, *if* $n \neq m$,

(iv) *the* f_n *are abelian and* $f_n \sim f_m$ *for every* n *and* m.

PROOF. We use induction in order to construct $\{f_n\}$ and $\{f_n'\}$. Suppose $f_1, \ldots, f_N$ and $f_1', \ldots, f_N'$ satisfy the conditions of the lemma. Let $f = f_1 + \cdots + f_N$ and choose a projection $e \in A$ such that $\|e - f\| < 2\gamma$ (see 1.2). Then

$$\begin{aligned}[e] = \rho[f] &= \rho[f_1 + \cdots + f_N] \\ &= \rho([f_1]) + \cdots + \rho([f_N]) \\ &= \rho\big([f_1']\big) + \cdots + \rho\big([f_N']\big) \\ &= [e_1] + \cdots + [e_n] = [e_1 + \cdots + e_N],\end{aligned}$$

(see 1.6 for notation). Hence $e \sim e_1 + \cdots + e_N$. By [1, §17, Theorem 2], $e_1 + \cdots + e_n$ is a finite projection and [1, §17, Proposition 5] implies that

$$I_A - e \sim I_A - (e_1 + \cdots + e_N) = \sup_{n > N} e_n.$$

Let $V \in A$ be a partial isometry such that $VV^* = I_A - e$ and $V^*V = I_A - (e_1 + \cdots + e_N)$. Then $(Ve_{N+1}V^*)(I_A - e) = Ve_{N+1}V^*$ and we conclude that

$Ve_{N+1}V^*$ is orthogonal to e. Now since $\|e - f\| < 2\gamma < 1/6$, by [8. Lemma 2.4] we can choose a projection $\hat{f} \in B$ such that $\hat{f}f = 0$ and $\|Ve_{N+1}V^* - \hat{f}\| < 6\gamma$. Let $f_{n+1} = \hat{f}$ and choose $f'(=f'_{N+1})$ in B such that $\|f'_{N+1} - e_{N+1}\| < 2\gamma$. Now $f_1, \ldots, f_{N+1}$ and $f'_1, \ldots, f'_{N+1}$ satisfy conditions (ii) and (iii). Conditions (i) and (iv) follow from [8, Lemma 2.3] and [5, Corollary D]. We note that we need $\gamma \leqslant 1/60$ in order to be able to use Corollary D of [5].

1.8. LEMMA. *Let A and B be* AW*-*algebras acting on H and suppose that $\|A - B\| < \gamma \leqslant 1/300$. If A is of type* $\mathrm{I}_{\aleph}$, *with $\aleph \leqslant \aleph_0$, then B is also of type* $\mathrm{I}_{\aleph}$.

PROOF. We consider the case that A is of type $\mathrm{I}_{\aleph_0}$. The case that A is finite can be dealt with in the same way. Suppose $\mathrm{I}_A = \sup e_n$, where $\{e_n\}$ is an orthogonal sequence of abelian equivalent projections in A. Let $\{f_n\}$ and $\{f'_n\}$ be as constructed in 1.7. Let $F = \sup f_n$ and choose a projection $G \in A$ such that $\|F - G\| < 2\gamma$. Then one verifies that $\|FBF - GAG\| < 5\gamma \leqslant 1/60$. Now 1.7 can be applied to the AW*-algebras FBF and GAG in order to get sequences $\{g_n\}$ and $\{g'_n\}$ of projections in GAG for which the conditions of Lemma 1.7 are fulfilled. Then $\|e_n - f'_n\| < 2\gamma$ and $\|g'_n - f_n\| < 2\gamma$, and [8, Lemma 2.3] implies that $g_n \sim g'_n \sim e_n$. Now $\{e_n\}$ and $\{g_n\}$ are sequences of pairwise orthogonal projections, and $e_n \sim g_n$ for every n. By [6, Theorem 5.5], we have $\sup e_n \sim \sup g_n$, i.e. $\mathrm{I}_A \sim \sup g_n \leqslant G$. Therefore $\mathrm{I}_A \lesssim G$ and since $G \leqslant \mathrm{I}_A$ we must have $G \sim \mathrm{I}_A$. Now standard arguments imply that $\|\mathrm{I}_A - \mathrm{I}_B\| < 2\gamma$, and we have $\|F - G\| < 2\gamma$. Hence it follows from [8, Lemma 2.3] that $F \sim \mathrm{I}_B$. If $w \in B$ is a partial isometry such that $w^*w = F$ and $ww^* = \mathrm{I}_B$, then $\mathrm{I}_B = \sup\{wf_nw^*\}$ and this shows that B is of type $\mathrm{I}_{\aleph_0}$.

2. Main result

2.1. PROPOSITION. *Let A and B be* AW*-*algebras acting on a Hilbert space H and suppose that $\|A - B\| < \gamma$. Then $\|Z(A) - Z(B)\| < 6\gamma$, where $Z(A)$ and $Z(B)$ are the centers of A and B respectively.*

PROOF. Let $a \in Z(A)$ and $\|a\| < 1$. We must show that there exists an element $b \in Z(B)$, $\|b\| \leqslant 1$, such that; $\|a - b\| < 6\gamma$. Choose $c \in B_1$ such that $\|a - c\| < \gamma$. Now let $y \in B_1$ and choose $x \in A_1$ such that $\|x - y\| < \gamma$. Then

$$\|ad_c(y)\| = \|cy - yc\| \leqslant \|cy - cx\| + \|cx - ax\| + \|xa - ya\| + \|ya - yc\| < 4\gamma.$$

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Hence $\|ad_c\| < 4\gamma$. By [4, Corollary 4.8] there exists an element $b' \in Z(B)$ such that $\|ad_c\| = 2\|c - b'\|$. Therefore

$$\|a - b'\| \leqslant \|a - c\| + \|c - b'\| < \gamma + 1/2\|ad_c\| < 3\gamma.$$

Let $b = b'/\|b'\|$. Then $\|b' - b\| < 3\gamma$ and we get

$$\|a - b\| \leqslant \|a - b\| + \|b - b'\| < 6\gamma.$$

By reversing the argument we can show that for any element $b \in Z(B)$, $\|b\| \leqslant 1$, there exists an element a in the unit ball of $Z(A)$ such that $\|a - b\| < 6\gamma$. Hence $\|Z(A) - Z(B)\| < 6\gamma$ as desired.

2.2. PROPOSITION. *Let A and B be* AW*-*algebras acting on a Hilbert space H with $\|A - B\| < \gamma \leqslant 1/300$. If A is of type* $\mathrm{I}_{\aleph}$ $(\aleph \leqslant \aleph_0)$, *then A and B are *-isomorphic.*

PROOF. By 2.1, $\|Z(A) - Z(B)\| < 6\gamma < 1/10$, and [3, Theorem 5.3] implies that $Z(A) = UZ(B)U^*$ for some unitary operator U. Also Lemma 1.8 implies that B is of type $\mathrm{I}_{\aleph}$. Now it follows from [7, Theorem 1] that A and B are *-isomorphic.

2.3. THEOREM. *Let A and B be* AW*-*algebras acting on a Hilbert space H such that $\|A - B\| < \gamma \leqslant 1/6300$. If A is of type* I *and its properly infinite portion is of type* $\mathrm{I}_{\aleph_0}$, *then A and B are *-isomorphic.*

PROOF. Let $h_1 \in A$ and $k_1 \in B$ be the unique central projections as described in the statement of Lemma 1.3. Then $\|h_1 - k_1\| < 2\gamma$ and one easily verifies that $\|h_1A - k_1B\| < 5\gamma$ and $\|(\mathrm{I}_A - h_1)A - (\mathrm{I}_B - k_1)B\| < 9\gamma$. Now, since $(\mathrm{I}_A - h_1)A$ is of type $\mathrm{I}_{\aleph_0}$ by hypothesis, 2.2 implies that $(\mathrm{I}_A - h_1)A \cong (\mathrm{I}_B - k_1)B$. Also, by [1, §18, Theorem 4], there exists an orthogonal sequence $\{\hat{h}_n\}$ of central projections in h_1A such that h_1A is the C*-sum of $\hat{h}_nA$ and each $\hat{h}_nA$ is either 0 or of type I_n. Now for each n, we can choose by 1.2 a central projection $\hat{k}_n \in k_1B$ such that $\|\hat{k}_n - \hat{h}_n\| < 10\gamma$. Then $\|\hat{k}_nB - \hat{h}_nA\| < 21\gamma \leqslant 1/300$ and Lemma 1.8 implies that $\hat{k}_nB$ is of type I_n. Moreover, one can easily verify that $\hat{k}_1B$ is the C*-sum of $\hat{k}_nB$. Hence we conclude from 2.2 that k_1B is *-isomorphic to h_1A. This ends the proof of the theorem.

2.4. COROLLARY. *Let A and B be* AW*-*algebras acting on a separable Hilbert space H with $\|A - B\|l < 1/6300$. If A is of type* I, *then A and B are *-isomorphic.*

https://doi.org/10.1017/S1446788700027579 Published online by Cambridge University Press

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https://doi.org/10.1017/S1446788700027579 Published online by Cambridge University Press