CYCLES OF EACH LENGTH IN REGULAR TOURNAMENTS

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(received November 17, 1966)

1. Introduction. It is known that a strong tournament of order n contains a cycle of each length k, $k=3, \dots, n$, ([1], Thm. 7). Moon [2] observed that each vertex in a strong tournament of order n is contained in a cycle of each length k, $k = 3, \dots, n$. In this paper we obtain a similar result for each arc of a regular tournament, that is, a tournament in which all vertices have the same score.

The property that each arc of a tournament of order n is contained in a cycle of each length k, $k = 3, \dots, n$, is subsequently referred to as property A. If there is an arc from a vertex u to a vertex v in a tournament T, we use the terminology "u defeats v" or "v is defeated by u" and the notation $(u, v) \in T$. $I(v) = \{u \in T : (u, v) \in T\}$ and $0(u) = \{v \in T : (u, v) \in T\}$.

2. Main result.

THEOREM. *A regular tournament of order* $2n + 1$ *satisfies property* A.

Proof. A 3-cycle and a regular tournament of order 5 obviously satisfy property A. In the following we assume $n \geq 3$.

Let $(v, v_o) \in T$ be an arbitrary arc of T. The theorem will follow if for each k, $k = 1, \dots, 2n-1$, there exists a k-path from v_o to some vertex of $I(v)$ such that v is not a vertex in the path.

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[1] This research forms part of the doctoral dissertation submitted by the author to the University of California, Santa Barbara.

Canad. Math. Bull. vol. 10, no. 2, 1967

Since v and v_o are in neither of the sets $0(v_o)$ and $I(v)$, we have $0(v_o) \cap I(v) \neq \phi$. Letting $v_1 \in 0(v_o) \cap I(v)$ we obtain a 1-path (v_o, v_1) of the desired form.

Assume there is an r-path $(v_o, v_1, \ldots, v_r)$, $1 \leq r \leq 2n-2$, such that $v_r \in I(v)$ and v is not a vertex of the r-path. Let $U = \{u_1, \ldots, u_p\}$ be the vertices of $I(v)$ that are not in the r-path.

CASE 1. $U \neq \phi$ and $(v_r, u_j) \in T$ for some $u_j \in U$. Then $(v_o, v_1, \ldots, v_r, u_j)$ is an $(r+1)$-path of the desired form.

CASE 2. $U \neq \phi$ and $(v_r, u_j) \notin T$ for all $u_j \in U$. Assume $r \geq n-1$. If $(u_1, v_i) \in T$ for $i = 0, 1, \ldots, r$, then u_1 would have score greater than n contradicting the regularity of T. Hence, since $(u_1, v_r) \in T$, there is a vertex v_i of the r-path, $i < r$, such that $(v_i, u_1) \in T$ and $(u_1, v_j) \in T$ for $j = i+1, i+2, \ldots, r$. Then $(v_o, v_1, \ldots, v_i, u_1, v_{i+1}, v_{i+2}, \ldots, v_r)$ is an $(r+1)$-path of the desired type. Notice that we have replaced an arc (v_i, v_{i+1}) of the r-path by a 2-path from v_i to v_{i+1} which does not pass through v. Henceforth, this method of obtaining a path of length one greater will be referred to as replacement.

Assume $1 \leq r \leq n-2$. Since $v_o \notin I(v)$, then $p \geq n-r \geq 2$. Since $(u_j, v_r) \in T$ for all $u_j \in U$, then if $(v_i, u_j) \in T$ for some v_i, $i = 0, 1, \ldots, r-1$, and some $u_j \in U$, an $(r+1)$-path of the desired form can be obtained by replacement. Thus we assume $(u_j, v_i) \in T$ for $0 \leq i \leq r$ and all $u_j \in U$. Thus v_o defeats at most r members of the set $S = \{v, v_o, \ldots, v_r, u_1, \ldots, u_p\}$. Hence, there are at least $n-r \geq 2$ members of $0(v_o)$ in the complement of S. Let $w_1, \ldots, w_{n-r}$ denote $n-r$ members of $0(v_o)$ in the complement of S. Since u_1 defeats $v_o, \ldots, v_r$ and has score n, it cannot defeat each w_j. Let $(w_t, u_1) \in T$. If $r = 1$, then (v_o, w_t, u_1) is a 2-path of the desired form. If

$r \neq 1$, then $(v_o, w_t, u_1, v_2, v_3, \ldots, v_r)$ is an $(r+1)$-path of the desired type.

CASE 3. $U = \phi$ which implies $I(v) \subseteq \{v_1, v_2, \ldots, v_r\}$ and $r \geq n$. Let $W = \{w_1, \ldots, w_{2n-r-1}\}$ be the vertices of T in the complement of $\{v, v_o, v_1, \ldots, v_r\}$. If $(w_j, v_r) \in T$ for some $w_j \in W$, or if $(v_o, w_j) \in T$ for some $w_j \in W$, then an $(r+1)$-path of the desired type can be obtained by replacement since $r \geq n$. Consequently, assume $(v_r, w_j) \in T$ and $(w_j, v_o) \in T$ for all $w_j \in W$.

If $r = n$, then W contains $n-1$ vertices. Thus there is a vertex w_j of W that can be defeated by at most $n-1$ vertices not in W . If v_1 defeats w_j , then we can obtain an $(r+1)$-path of the desired type by replacement. On the other hand, if $r > n$, then v_1 can defeat no vertex of W or else replacement yields an $(r+1)$-path of the desired form. In either case, we can assume there is a vertex $w_t \in W$ such that $(w_t, v_1) \in T$.

Consider the $n-1$ vertices among $\{v_2, v_3, \ldots, v_r\}$ that are r-path successors of the vertices of $I(v)$ in the r-path. Since v_o is defeated by v and every member of W , there exists a vertex v_s , $2 \leq s \leq r$, of the r-path such that $(v_o, v_s) \in T$ and $v_{s-1} \in I(v)$. The desired $(r+1)$-path is $(v_o, v_s, v_{s+1}, \ldots, v_r, w_t, v_1, v_2, \ldots, v_{s-1})$.

The theorem follows by induction on the length of the path.

3. Conclusion. Let S be a tournament with vertices $v_1, \ldots, v_n$ which satisfies (i) property A or (ii) each arc of S is contained in a 3-cycle and each vertex is the initial vertex in a path of length k , $k = 1, \ldots, n-1$. Adjoin two vertices v_{n+1} and v_{n+2} to S and let v_{n+1} defeat v_{n+2} , let v_{n+1} be defeated by all the vertices of S , and let v_{n+2} defeat all the vertices of S . The resulting tournament, call it S' , of order $n+2$ satisfies property A . If the score sequence of S is $(s_1, \ldots, s_n)$, then

https://doi.org/10.4153/CMB-1967-028-6 Published online by Cambridge University Press

the score sequence of S' is $(1, s_1+1, s_2+1, \ldots, s_n+1, n)$. Hence, the tournaments satisfying property A form a class lying between strong tournaments and regular tournaments.

In general, an almost regular tournament, that is, a tournament of order $2n$ having n vertices with score n and the remaining vertices with score $n-1$, does not satisfy property A . To see this we construct the following tournament with vertices $v_o, v_1, \ldots, v_{2n-1}$. Let v_i defeat $v_{i+1}, \ldots, v_{i+n}$ for $i = 0, 1, \ldots, n-1$ and let v_i defeat $v_{i+1 (\mathrm{mod}\ 2n)}, \ldots, v_{i+n-1 (\mathrm{mod}\ 2n)}$ for $i = n, \ldots, 2n-1$. The resulting tournament is almost regular and it is easy to see that the arc (v_{n-1}, v_n) is not contained in a 3-cycle.

The author wishes to thank Professor Paul Kelly for his many valuable suggestions during the course of the author's research.

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