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ON STOCHASTIC OPTIMAL CONTROL LAWS

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§ 1. **Introduction.** Let us begin by recalling the existence of optimal controls for a class of stochastic differential equations

$$(1.1) \qquad dX(t) = \beta(t, X(s), s \le t, U(t))dB(t) + \alpha(t, X(s), s \le t, U(t))dt, t \ge 0 ,$$

with given initial condition $X(0) = x$, where B is an n-dimensional Brownian motion and the control U is a stochastic process. As admissible controls, let us allow all non-anticipative process $U(t) = (U_1(t), \cdots U_m(t)) \in \Gamma$ where Γ is a compact subset of $\boldsymbol{R}^m$. We call Γ a control region. Assume that the matrix valued functional β and the n-vector valued α satisfy a Lipscitz condition in X and some growth condition. Then we have a unique solution X^U for an admissible control U.

We shall consider the minimization problem for the expectation of cost functional $\Phi(X^U, U)$. If β does not depend on U and α is linear in U, i.e

$$\alpha_i(t f u) = \sum_{j=1}^{m} \alpha_{ij}(t f) u_j .$$

Fleming and Nisio [4] consider the existence of an optimal control U_0, (open loop control), in the case where $\Phi(X, U)$ is non-negative and lower semi-continuous on X and $V(t) = \int^t U(s)ds$. But in many problems of controls, we would like to minimize $E\Phi(X^U, U)$ subject the condition that the control $U(t)$, selected at time t, should depend only on the observed data up to time t. Let us suppose that the system X of (1.1) is completely observable. Thus an admissible control will be a function $u; [0 \infty) \times C_n \to \Gamma$, which satisfies the non-anticipative condition. If $\beta \equiv 1$ and $\alpha(t f \Gamma)$ is convex (Roxin's condition), then Beneš [2] proved the existence of an optimal control u_0, (control based on a complete

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observation), in the case where the cost functional Φ is given by an integral form.

We shall remark, in § 5, that an optimal control based on a complete observation turns out to an optimal open loop control under some conditions. This means that the synthesis problem is decided, i.e. an optimal open loop control will be determined as a function of data of the system X. In §§ 3∼5, we assume that β does not depend on U. The existence of an optimal open loop control will be proved under the Roxin's condition, (A 6) and (A 7), in § 3. Moreover when β is uniformly positive definite, we shall consider the correspondence between the laws of response of open loop controls and controls based on a complete observation. Namely, let $\mathscr{P}$ and Q be set of all laws of response of open loop controls and controls based on a complete observation respectively. Then we show that $\mathscr{P} = Q$, in § 5. When β depends on U, the existence of optimal controls and the synthesis problem will be discussed in § 6. In § 7, we discuss controls of diffusion type processes. We shall sketch the Krylov's work [9], i.e. when the cost functional Φ is given by an integral form, an optimal control is attained by a Markovian policy, under some conditions. But, when Φ is not an integral form, we have a little counter example.

Let us now introduce some preliminary definitions and notations.

Given a stochastic process $X(t), t \geq 0$, $\mathfrak{B}_t(X)$ denotes the least σ-algebra generated by $\{X(s), s \leq t\}$.

The n-dimensional Brownian motion is denoted by $B(t) = (B_1(t), \cdots B_n(t)), t \geq 0$, and we normalize it by $B(0) = 0$. $\mathfrak{B}_{tv}(dB)$ denotes the least σ-algebra generated by $\{B(s) - B(\tau), t \leq \tau < s < v\}$.

C_n denotes the space of all $\boldsymbol{R}^n$-valued continuous functions defined on $[0\ \infty)$, with the usual metric ρ,

$$\rho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup_{t \leq n} |f(t) - g(t)|}{1 + \sup_{t \leq n} |f(t) - g(t)|}, \qquad f, g \in C_n$$

where $|\ \ |$ means the Euclid norm of $\boldsymbol{R}^n$. Let S_t be the σ-algebra generated by $\{f(s), s \leq t\}$. According to Beneš [1], we define the σ-algebra G_n on $[0\ \infty) \times C_n$ as follows, a Borel subset E of $[0\ \infty) \times C_n$ is in G_n, if and only if

(i) every t-section of E is S_t-measurable, for $t \in [0\ \infty)$, and

(ii) every f-section of E is a Borel set of $[0\ \infty)$, for $f \in C_n$.

Let $X(t), t \geq 0$, be a $\boldsymbol{R}^n$-valued stochastic process with continuous simple paths. Define $\pi; [0\ \infty) \times \Omega \to [0\ \infty) \times C_n$, by $\pi(t\omega) = (t\ X(\omega))$. We denote $\pi^{-1}(G_n)$ by $\mathfrak{G}_n$.

Let the control region Γ be a compact subset of $\boldsymbol{R}^m$. A process $U(t), t \geq 0$, is called an admissible control if, with probability 1,

(i) $U(t) \in \Gamma\ ,\quad 0 \leq t\ ,$ and if

(ii) $\mathfrak{B}_t(U, B)$ is independent of $\mathfrak{B}_{t\infty}(dB)$ for every $t \geq 0$.

To be more precisely (B, U) is called an admissible system. We denote, by $\mathfrak{A}$, the set of all admissible systems.

Let $\alpha(tfu)$ and $\beta(tfu)$ be an n-vector and an $n \times n$ matrix valued $G_n \times \mathfrak{B}_m(\Gamma)$-measurable function, defined on $[0\ \infty) \times C_n \times \Gamma$. Then the equation (1.1) can be understood as

$$dX(t) = \beta(t\ X\ U(t))dB(t) + \alpha(t\ X\ U(t))dt\ .$$

By a solution of (1.1), we mean a stochastic process $(\tilde{X}(t)\tilde{B}(t)\tilde{U}(t)), t \geq 0$, defined on a suitable probability space (we may assume the Lebesgue space [0 1]), such that

(i) $\tilde{X}$ has continuous paths,

(ii) $(\tilde{B}\tilde{U})$ has the same law as the given admissible system (BU),

(iii) $\mathfrak{B}_t(\tilde{X}\tilde{B}\tilde{U})$ is independent of $\mathfrak{B}_{t\infty}(d\tilde{B})$ for any $t \geq 0$, and, with probability 1,

(iv) $$\tilde{X}(t) = x + \int_0^t \beta(s\ \tilde{X}\ \tilde{U}(s))d\tilde{B}(s) + \int_0^t \alpha(s\ \tilde{X}\ \tilde{U}(s))ds\ , \qquad \text{for any } t \geq 0.$$

For simplicity, we call X a solution of (1.1), or a response to the control U.

§ 2. Existence and uniqueness of solution. Let us impose following assumptions

(A.1) $\alpha(tfu)$ and $\beta(tfu)$ are $G_n \times \mathfrak{B}_m(\Gamma)$-measurable,

(A.2) there exists a bounded measure dM on $(-\infty\ 0]$, such that

$$\sum_i |\alpha_i(tfu) - \alpha_i(tgu)|^2 + \sum_{ij} |\beta_{ij}(tfu) - \beta_{ij}(tgu)|^2$$
$$\leq \int_{-t}^0 |f(s+t) - g(s+t)|^2\, dM(s)$$

and

(A.3) there exists a increasing function $L(t)$, such that

$$\sum_i |\alpha_i(t\ 0\ u)|^2 + \sum_{ij} |\beta_{ij}(t\ 0\ u)|^2 \leq L(t)\ , \qquad \forall u \in \Gamma\ .$$

THEOREM 1. *Under assumptions* (A 1) (A 2) *and* (A 3) *there exists a solution of* (1.1) *uniquely. Moreover, this solution* $X(t)$ *is* $\mathfrak{B}_t(UB)$-*measurable and has the following moments,*

$$E|X(t)|^k \leq K(k,T), \quad \text{for } t \leq T, \quad k = 1,2 \cdots. \tag{2.1}$$

$$E|X(t) - X(s)|^4 \leq K(T)((t-s)^2 + (t-s)^4), \quad \text{for } t,s \leq T, \tag{2.2}$$

where $K(k,T)$ *and* $K(T)$ *are independent of an admissible control* U *and increasing in* T.

Proof. The method of proof is just a repeat of Sect. 2 of [4]. To show the existence of a solution, we shall use the well-known successive approximation. Let us define a sequence of approximate solutions $X_n, n = 0, 1, \cdots,$ as follows,

$$X_0(t) = x$$

$$X_{n+1}(t) = x + \int_0^t \beta(s\, X_n\, U(s))dB(s) + \int_0^s \alpha(s\, X_n\, U(s))ds .$$

Then the following inequality will be proved by induction;

$$\sum_i E|X_{m\,i}(t) - X_{m-1\,i}(t)|^2 \leq \frac{\gamma_1(t)\gamma_2^m t^m (t+1)^m}{m\,!}, \quad m = 1,2 \cdots, \tag{2.3}$$

where γ_2 is a constant determined by $\|M\|(=M(-\infty,0])$ and $\gamma_1(t)$ is independent of m and increasing in t.

Therefore, we have

$$\begin{aligned} &P\Big(\sup_{0\leq s\leq t} |X_{m\,i}(s) - X_{m-1\,i}(s)| > 2\varepsilon\Big) \\ &\leq P\Big(\int_0^t |\alpha_i(\tau X_{m-1} U(\tau)) - \alpha_i(\tau X_{m-2} U(\tau)|\, d\tau > \varepsilon\Big) \\ &+ P\Big(\sup_{0\leq s\leq t} \Big|\sum_j \int_0^s [\beta_{ij}(\tau X_{m-1} U(\tau)) - \beta_{ij}(\tau X_{m-2} U(\tau))] dB_j(\tau)\Big| > \varepsilon\Big) \\ &\leq \varepsilon^{-2} t \int_0^t E\,|\alpha_i(\tau X_{m-1} U(\tau)) - \alpha_i(\tau X_{m-2} U(\tau))|^2 d\tau \\ &+ \varepsilon^{-2} n \sum_j \int_0^t E\,|\beta_{ij}(\tau X_{m-1} U(\tau)) - \beta_{ij}(\tau X_{m-2} U(\tau))|^2 d\tau \\ &\leq \varepsilon^{-2} n \frac{\gamma_1(t)\gamma_2^{m-1} t^m (t+1)^m}{m\,!} \|M\| . \end{aligned}$$

Setting $\varepsilon = 2^{-m}$, we get

$$\sum P\Big(\sup_{0\leq s\leq t} |X_{m\,i}(s) - X_{m-1\,i}(s)| > 2^{-m+1}\Big) < \infty .$$

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Therefore, by Borel-Cantelli's lemma, we see that, with probability 1,

$$X_m(s) = x + \sum_{\ell=1}^{m} [X_\ell(s) - X_{\ell+1}(s)]$$

converges uniformly on $[0\ t]$, and so, on every bounded subinterval of $[0, \infty)$. Hence the limit process X has continuous paths. Applying an usual method, it is easy to see that X is a solution. In order to prove (2.1) we may assume that k is even.

$$\begin{aligned} X^k_{m\ i}(t) &\leq 3^{k-1}\Big[x^k_i + \Big(\int_0^t \alpha_i(\tau X_{m-1}U(\tau))d\tau\Big)^k \\ &\qquad + \Big(\sum_j \int_0^t \beta_{ij}(\tau X_{m-1}U(\tau))dB_j(\tau)\Big)^k\Big] \qquad (2.4) \\ &\leq 3^{k-1}\Big[x^k_i + t^{k-1}\int_0^t \alpha^k_i d\tau + n^{k-1}\sum_j \Big(\int_0^t \beta_{ij} dB_j\Big)^k\Big] \end{aligned}$$

Putting $\xi(t) = \int_0^t \beta_{ij} dB_j$, we shall evaluate its k-th moment. Let σ_A be the first passage time of ξ to $(-A, A)^c$. From a formula on stochastic differentials [6], we get

$$E\xi^k_A(t) = \frac{k(k-1)}{2}\int_0^t E\xi^{k-2}_A(s)ds \qquad (2.5)$$

where $\xi_A(t) = \xi(t \wedge \sigma_A)$ and $\beta_A = \chi_{[0\sigma_A]}\beta_{ij}$. Hence by Hölder's inequality, we get

$$E\xi^k_A(t) \leq \frac{k(k-1)}{2}\int_0^t (E\xi^k_A(s))^{(k-2)/k}(E\beta^k_A(s))^{2/k} ds$$

Since (2.5) implies that $E\xi^k_A(t)$ is increasing in t, we have

$$E\xi^k_A(t) \leq \frac{k(k-1)}{2}(E\xi^k_A(t))^{(k-2)/k}\int_0^t (E\beta^k_A)^{2/k} ds$$

namely,

$$\begin{aligned} E\xi^k_A(t) &\leq \Big(\frac{k(k-1)}{2}\int_0^t (E\beta^k_A(s))^{2/k} dt\Big)^{k/2} \\ &\leq K_1(k)t^{k/2-1}\int_0^t E\beta^k_A(s)ds\ . \end{aligned}$$

From (A 2) and (A 3),

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$$|\beta_{ij}(tfu)|^k \leq \left(L(t) + \int_{-t}^{0} |f(s+t)|^2 dM(s)\right)^{k/2}$$
$$\leq K_2(t, k)\left(1 + \int_{-t}^{0} |f(s+t)|^k dM(s)\right).$$

Hence, setting $d_m(t) = \sup_{\substack{0 \leq s \leq t \\ i=1\cdots n}} EX_i^k(s)$, we have

$$E\xi^k(t) \leq \varliminf_{A\to\infty} E\xi_A^k(t) \leq K_3(k, t)\left(1 + \int_0^t d_{m-1}(s)ds\right).$$

Therefore, by virtue of (2.4), we can easily see

$$EX_{m\ i}^k(t) \leq K_4(k, t)\left(1 + \int_0^t d_{m-1}(s)ds\right) \qquad i = 1, \cdots n, m = 1, 2 \cdots .$$

where $K_4(k, T)$ is independent of U and increasing in t. So we have

$$d_m(t) \leq K_4(k, t)\left(1 + \int_0^t d_{m-1}(s)ds\right). \tag{2.6}$$

On account of $d_0 < \infty$, this (2.6) implies (2.1).

In order to prove (2.2) we can apply a similar calculation. Since

$$X_i(t) - X_i(s) = \int_s^t \alpha_i(\tau\, X\, U(\tau))d\tau + \sum_j \int_s^t \beta_{ij}(\tau\, X\, U(\tau))dB_j(\tau),$$

we have

$$|X_i(t) - X_i(s)|^4 \leq 3^3\left[(t-s)^3 \int_s^t \alpha_i^4 d\tau + n^3 \sum_j \left(\int_s^t \beta_{ij} dB_j\right)^4\right].$$

From (A 2) and (2.1) we see

$$E\int_s^t \alpha_i^4 d\tau \leq K_5(T)(t-s), \qquad \text{for } t\, s \leq T,$$

where $K_5(T)$ is independent of U. Putting $\xi(t) = \int_s^t \beta_{ij} dB_j$ and using same notation as above, we have

$$E\xi_A^4(t) = 6\int_s^t E\xi_A^2(\tau)\beta_A^2(\tau)d\tau$$
$$\leq \frac{1}{2(t-s)}\int_s^t E\xi_A^4(\tau)d\tau + 18(t-s)\int_s^t E\beta_A^4(\tau)d\tau.$$

Since $E\xi_A^4(t)$ is increasing in t, we get

$$E\xi_A^4(t) \leq 36(t-s)\int_s^t E\beta_A^4 d\tau.$$

Hence from (A 2) and (2.1), we have

$$E\xi^4(t) \leq \varliminf_{A\to\infty} E\xi_A^4(t) \leq K_6(T)(t-s)^2 .$$

Therefore (2.2) holds.

Let Y be a solution with bounded second moment. Recalling that $X(t)$ is $\mathfrak{B}_t(U, B)$-measurable, we shall evaluate $Y(t) - X(t)$. In a routine, we can show

$$E\,|X_i(t) - Y_i(t)|^2 = 0 , \qquad i = 1 \cdots n .$$

This completes the proof of Theorem 1.

§ 3. **Existence of optimal controls for** $\beta(tf)$. Let us introduce following assumptions

(A 4) $\beta(tfu) = \beta(tf)$

(A 5) $\beta(tfu)$ is continuous in (tfu)

(A 6) $\alpha(tfu)$ is continuous in (tfu)

(A 7) $\alpha(tf\Gamma)$ is convex, for each (tf).

For an admissible system (BU), we denote a solution of (1.1) by X^U.

THEOREM 2. *Let Φ be lower semi-continuous on C_n, with $0 \leq \Phi(f) \leq \infty$. Then, under the assumptions* (A 1)~(A 7), *there exists an admissible system* (U_0B_0) *such that*

$$E\Phi(X^{U_0}) \leq E\Phi(X^U) , \qquad \forall (BU) \in \mathfrak{A}$$

Setting $\theta^U(t) = \int_0^t \alpha(s\, X^U U(s))ds$ and $\mathfrak{M} = \{(X^U, B, \theta^U),\ (B, U) \in \mathfrak{A}\}$, we can see, from (2.2)

LEMMA 1. *Under assumptions* (A 1) (A 2) *and* (A 3), *$\mathfrak{M}$ is L-totally bounded.*

Hereafter we suppose that (A 1)~(A 7) hold.

LEMMA 2. *Let X be the solution for $(B\ U)$ of $\mathfrak{A}$. Then there exists a G_{2n}-measurable function v; $[0\ \infty) \times C_{2n} \to \Gamma$, such that with probability* 1,

$$X(t) = x + \int_0^t \beta(s\, X)dB(s) + \int_0^t \alpha(s\, X\, v(s\, X\, B))ds , \qquad \forall t \geq 0 \tag{3.1}$$

Proof. Let $(\Omega\mathfrak{B}P)$ be a probability space on which (XBU) is defined. For simplicity, we may suppose that $X(t\omega)$ and $B(t\omega)$ are continuous in t, for all $\omega \in \Omega$. Let us define π; $[0\ \infty) \times \Omega \to [0\ \infty) \times C_{2n}$ by $\pi(t\omega) = (t\, X(\omega)\, B(\omega))$ and endow the σ-algebra $\mathfrak{G} = \pi^{-1}(G_{2n})$ on $[0\ \infty) \times \Omega$. $\mathfrak{G}$

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is the completion of $\mathfrak{G}$ by the product measure, Lebesque measure $\times P$. Since

$$\theta^{U}(t) = X(t) - x - \int_0^t \beta(s\,X)dB(s)\ , \tag{3.2}$$

θ^U is $\bar{\mathfrak{G}}$-measurable. Moreover $\alpha(t\,X\,U(t))$ is a (Radon-Nikodym) derivative of the right side of (3.2). So, $\alpha(t\,X\,U(t))$ is $\bar{\mathfrak{G}}$-measurable. Put $\tilde{\alpha}(t\,\omega\,u) = \alpha(t\,X(\omega)\,u)$. Then $\tilde{\alpha}$ is continuous in u and $\alpha(t\,X(\omega)\,U(t\omega)) \in \tilde{\alpha}(t\omega\Gamma)$. Hence an implicit function theorem [1] guarantees the existence of a $\bar{\mathfrak{G}}$-measurable $\bar{V}$: $[0\ \infty) \times \Omega \to \Gamma$, such that

$$\alpha(t\,X(\omega)U\,(t\omega)) = \tilde{\alpha}(t\,\omega\,\bar{V}(t\omega))\ .$$

Since there exists a $\mathfrak{G}$-measurable modification V of $\bar{V}$, i.e.

$$\bar{V}(t\omega) = V(t\omega)\ , \qquad \tilde{\forall}(t\omega)\ ,$$

we have, with probability 1,

$$\theta^{U}(t\omega) = \int_0^t \alpha(s\,X(\omega)\,V(s\omega))ds\ , \qquad \forall t \geq 0\ .$$

From the definition of $\mathfrak{G}$, $V(t\omega)$ turns out to $v(t\,X(\omega)\,B(\omega))$ with a G_{2n}-measurable v. This completes the proof of Lemma 2.

This lemma 2 means that we may change $U(t)$ to $v(t\,X\,B)$, if we are concerned with an event of (X, B).

LEMMA 3. *Suppose that $(X_\ell B_\ell \theta_\ell) \in \mathfrak{M}$ converges to $(XB\theta)$ in L-metric. Then $(XB\theta)$ is in $\mathfrak{M}$, i.e. there exists a control U such that $(B\ U)$ is in $\mathfrak{A}$ and $X = X^U, \theta = \theta^U$.*

Proof. By Lemma 2, we may assume

$$X_\ell(t) = x + \int_0^t \beta(s\,X_\ell)dB_\ell(s) + \int_0^t \alpha(s\,X_\ell v_\ell(sX_\ell B_\ell))ds$$

and

$$\theta_\ell(t) = \int_0^t \alpha(s\,X_\ell v_\ell(sX_\ell B_\ell))ds\ .$$

Using Skorohod's theorem, we can construct $(\tilde{X}_\ell \tilde{B}_\ell \tilde{\theta}_\ell)$ and $(XB\theta)$ on the Lebesgue space (again we denote by Ω), such that

$$\tilde{X}_\ell(t) = x + \int_0^t \beta(s\,\tilde{X}_\ell)d\tilde{B}_\ell + \tilde{\theta}_\ell(t)\ ,$$

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$$\tilde{\theta}_\ell(t) = \int_0^t \alpha(s\,\tilde{X}_\ell v_\ell(s\tilde{X}_\ell\tilde{B}_\ell))ds$$

and, with probability 1, $(\tilde{X}_\ell(t)\tilde{B}_\ell(t)\tilde{\theta}_\ell(t))$ tends to $(X(t)B(t)\theta(t))$ uniformly on any bounded subinterval of $[0\ \infty)$. Hence B is a Brownian process adapted to $\mathfrak{B}_t(XB\theta)$ and, by (2.1),

$$E\,|X(t)|^4 \leq K(t)\ . \tag{3.3}$$

From the continuity of β, we have

$$X(t) = x + \int_0^t \beta(s\,X)dB(s) + \theta(t) \qquad t \geq 0\ . \tag{3.4}$$

On the other hand, by virtue of (A 2) and (A 3), we see

$$|\theta(t) - \theta(s)| = \lim_\ell |\tilde{\theta}_\ell(t) - \tilde{\theta}_\ell(s)| = \lim_\ell \left|\int_s^t \alpha(\tau\tilde{X}_\ell v_\ell(\tau\tilde{X}_\ell\tilde{B}_\ell))d\tau\right| \leq K_1(T\omega)\,|t-s|$$
$$\text{for } t, s \leq T\ .$$

Moreover, setting $\alpha_\ell(t\omega) \equiv \alpha(t\,X(\omega)\,v_\ell(s\tilde{X}_\ell(\omega)\tilde{B}_\ell(\omega))$, we have

$$\begin{aligned} &\left|\theta(t) - \int_0^t \alpha_\ell(s)ds\right| \\ &\leq |\theta(t) - \tilde{\theta}_\ell(t)| + \left|\int_0^t \alpha(s\,\tilde{X}_\ell v_\ell(s\tilde{X}_\ell\tilde{B}_\ell)) - \alpha(s\,Xv_\ell(s\,\tilde{X}_\ell\tilde{B}_\ell))\,ds\right| \\ &\leq |\theta(t) - \tilde{\theta}_\ell(t)| + K_2(T\omega)\sup_{s\leq t}|\tilde{X}_\ell(s) - X(s)|\ , \qquad t \leq T \end{aligned} \tag{3.5}$$

Hence, with probability 1,

$$\int_0^t \alpha_\ell(s)ds \to \theta(t)\ , \text{ uniformly on any bounded interval}\ . \tag{3.6}$$

For simplicity, we may assume that $(X(t)B(t)\theta(t))$ is continuous in all ω and (3.4) holds for all ω. Define $\pi\,;\,[0\ \infty) \times \Omega \to [0\ \infty) \times C_{2n}$ by $\pi(t\omega) = (t\,X(\omega)\,B(\omega))$ and put $\mathfrak{G} = \pi^{-1}(G_{2n})$. From (3.4), we can take a $\mathfrak{G}$-measurable derivative γ of θ, i.e, with probability 1,

$$\theta(t) = \int_0^t \gamma(s)ds\ . \tag{3.7}$$

Therefore, by (3.6), we have, for any n-vector $\eta \in L_2([0\ T] \times \Omega)$,

$$\int_0^T (\eta(s), \alpha_\ell(s))ds \to \int_0^T (\eta(s), \gamma(s))ds\ , \qquad \tilde{\forall}\omega\ .$$

Recalling (3.3), we get the following estimate,

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$$E\left|\int_0^T (\eta(s), \alpha_\ell(s))ds\right|^{4/3} \leq T^{1/3} E\int_0^T |\eta(s)|^{4/3} |\alpha_\ell(s)|^{4/3} ds$$
$$\leq T^{1/3}\left(E\int_0^T |\eta(s)|^2 ds\right)^{2/3}\left(E\int_0^T |\alpha_\ell(s)|^4 ds\right)^{1/3} \leq K_3(T)\|\eta\|^{4/3}, \qquad \ell = 1, 2 \cdots .$$

Therefore, by virtue of uniform integrability, $E\int_0^T (\eta(s), \alpha_\ell(s))ds$ tends to $E\int_0^T (\eta(s), \gamma(s))ds$. Consequently α_ℓ tends to γ weakly in $L_2([0\ T] \times \Omega)$. Hence a convex combination of α_ℓ can converge to γ strongly. So, we have a subsequence which converges almost everywhere. Since $\alpha(s\ X(\omega)\ \Gamma)$ is convex and closed, for almost all $(t\omega)$,

$$\gamma(t\omega) \in \alpha(s\ X(\omega)\ \Gamma) . \tag{3.8}$$

We can modify γ, so that (3.8) holds for all $(t\omega)$, i.e. there exists a $\overline{\mathfrak{G}}$-measurable $\bar\gamma$ such that

$$\bar\gamma(t\omega) = \gamma(t\omega) \qquad \text{for almost all } (t\omega) ,$$

and

$$\bar\gamma(t\omega) \in \alpha(t\ X(\omega)\ \Gamma) \qquad \forall(t\omega) .$$

Again, by an implicit function theorem, we have a G_{2n}-measurable $v; [0\ \infty) \times C_{2n} \to \Gamma$, such that

$$\gamma(t\omega) = \alpha(t\ X(\omega)\ v(tX(\omega)B(\omega))) \qquad \forall(t\omega) .$$

Hence, $(B\ v(sXB))$ is an admissible system and by (3.7), with probability 1,

$$\theta(t) = \int_0^t \alpha(s\ X\ v(sXB))ds , \qquad \forall t \geq 0 .$$

Recalling (3.4), we conclude that $(XB\theta)$ is in $\mathfrak{M}$.

Proof of Theorem 2. Let X_m be approximate optimal, i.e.

$$\lim E\Phi(X_m) = \inf_{\mathfrak{A}} E\Phi(X^U) .$$

Let X_m be a response for $(B_m U_m)$. By Lemmas 2 and 3, $\mathfrak{M}$ is sequentially compact. Hence it is enough to verify that $E\Phi(X)$ is lower semi-continuous under L-convergence. If X_m tends to X in L-metric, then Skorohod's theorem tells us that we may assume that, with probability 1, $X_m(t)$ converges to $X(t)$ uniformly on any bounded interval. Hence, with probability 1,

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$$\Phi(X) \leq \underline{\lim}\, \Phi(X_m) .$$

By Fatou's lemma, we have

$$E\Phi(X) \leq \underline{\lim}\, E\Phi(X_m)$$

which proves Theorem 2.

§ 4. **Transformation of measure.** Consider a stochastic differential equation

(4.1) $$dX(t) = \beta(t\ X)dB(t) + \gamma(t\ X)dt , \qquad X(0) = x .$$

We assume the following conditions,

(C 1) β is G_n-measurable

(C 2) $\beta(tf)$ is locally square integrable in t, for any $f \in C_n$.

(C 3) there exists a bounded G_n-measurable n-vector function ϕ, such that

$$\gamma(tf) = \beta(tf)\phi(tf) .$$

Under these assumptions, we can apply the method of the so-called transformation of measure [2]. We have the following

THEOREM 3. *Suppose that a stochastic differential equation*

(4.2) $$d\xi(t) = \beta(t\xi)dB(t) , \qquad \xi(0) = x$$

has a solution and the explosion does not occur. Then (4.1) *has a solution. Moreover if the law of the joint process* (ξB) *is unique for any solution* ξ *of* (4.2), *then the law of* $(X\ B)$ *is unique for any solution* X *of* (4.1).

Proof. Put $F_t = \mathfrak{B}_t(B\xi)$ and

$$D(t) = \exp \sum_{k=1}^{n} \left(\int_0^t \phi_k(s\xi)dB_k(s) - \frac{1}{2}\int_0^t \phi_k^2(s\xi)\, ds \right) .$$

Then it is well-known that D is an F_t-martingale. Define the probability measure Q_T, on (Ω, F_T), by

(4.3) $$dQ_T = D(t)dP , \qquad T \geq 0 .$$

Appealing to the extension theorem of measure, we have the probability measure Q on (ΩF), where $F = \bigvee_T F_T$. The following lemma 1 is easy.

LEMMA 1. *Let* ζ *be a bounded and* F_t *-measurable random variable. Then*

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$$E(\zeta D(t)/F_s) = \bar{E}(\zeta/F_s)D(s) \qquad t > s ,$$

where $\bar{E}$ means the expection with respect to Q. Using Lemma 1, we shall show

LEMMA 2. $W(t) \equiv B(t) - \int_0^t \phi(s\xi)ds$ *is a Brownian motion adapted to F_t, on $(\Omega F Q)$.*

Proof. Put $Z(t) \equiv W(t)D(t)$. Then, using a formula on stochastic differentials, we have

$$dZ(t) = D(t)dB(t) + D(t)W(t)\sum_{k=1}^{n} \phi_k(t\xi)dB_k(t) .$$

So, Z is an F_t-martingale on $(\Omega F P)$. Therefore, by virtue of Lemma 1, W is an F_t-martingale on $(\Omega F P)$. Since $\bar{E}|W(t)|^2 < \infty$, we now seek the variation process $\langle W_i W_j \rangle(t)$ on $(\Omega F Q)$, [10]. Put $Z(t) \equiv W_i(t)W_j(t)D(t)$. Then again by a formula on stochastic differentials, we have

$$\begin{aligned} dZ(t) = D(t)\delta_{ij}dt + L(t)(W_i(t)dB_j(t) + W_j(t)dB_i(t)) \\ + L(t)W_i(t)W_j(t) \sum \phi_k(t\xi)dB_k(t) . \end{aligned}$$

Hence, from Lemma 1, we get on $(\Omega F Q)$

$$\langle W_i W_j \rangle(t) = \delta_{ij} t .$$

This implies that W is an F_t-Brownian motion, on $(\Omega F Q)$, [10].

Let us show that the process ξ is a solution of (4.1) on $(\Omega F Q)$. According to McKean [11], we define β' and β'' by

$$\beta'_\ell(tf) = 2^\ell \int_{t-2^{-\ell}}^{t} \beta(sf)ds \quad \text{and} \quad \beta''_{\ell m}(tf) = \beta'_\ell(2^{-m}[2^m t], f) ,$$

where $[c]$ is the largest integer less than c. Then β'' is G_n-measurable and simple in t. Moreover, by (C 2), we see that for any $f \in C_n$,

$$\int_0^T |\beta(sf) - \beta''_{\ell m}(sf)|^2 ds \to 0 , \qquad \text{if } m \to \infty \quad \text{and} \quad \ell \to \infty .$$

Therefore, we can take a G_n-measurable function β_k, which is simple in t, so that

$$P\left(\int_0^T |\beta(s\xi) - \beta_k(s\xi)|^2 ds > 2^{-k}\right) < 2^{-k} \tag{4.4}$$

and

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(4.5) $$Q\left(\int_0^T |\beta(s\xi) - \beta_k(s\xi)|^2 ds > 2^{-k}\right) < 2^{-k} .$$

Hence there exist sets $N_i(\in F)$, such that $P(N_1) = 0$ and $Q(N_2) = 0$. Moreover, for $\omega \notin N_1$,

(4.6) $$\int_0^t \beta_k(s\xi)dB(s) \to \int_0^t \beta(s\xi)dB(s) , \qquad \text{uniformly on } [0\ T] ,$$

and, for $\omega \notin N_2$

(4.7) $$\int_0^t \beta_k(s\xi)dW(s) \to \int_0^t \beta(s\xi)dW(s) , \qquad \text{uniformly on } [0\ T] .$$

Since Q is absolutely continuous to P, putting $N = N_1 \cup N_2$, we set that $Q(N) = 0$ and (4.6) and (4.7) hold for $\omega \notin N$. Because β_k is simple in t, we have

$$\int_0^t \beta_k(s\xi)dB(s) = \int_0^t \beta_k(s\xi)dW(s) + \int_0^t \beta_k(s\xi)\phi(s\xi)ds .$$

Furthermore,

$$\left|\int_0^t \beta(sf)\phi(sf)ds - \int_0^t \beta_k(sf)\phi(sf)ds\right| \le \|\phi\|_\infty \int_0^t |\beta(sf) - \beta_k(sf)|\, ds .$$

Therefore, recalling (4.5), we see that, with Q-probability 1,

$$\int_0^t \beta_k(s\xi)\phi(s\xi)ds \to \int_0^t \beta(s\xi)\phi(s\xi)ds , \qquad \text{uniformly on } [0\ T] .$$

Consequently, with Q-probability 1,

(4.8) $$\xi(t) = x + \int_0^t \beta(s\xi)dW(s) + \int_0^t \gamma(s\xi)ds , \qquad \forall t \ge 0 .$$

Let X be a solution of (4.1) and μ the probability law of (XB). For convenience, we take the coordinate representation of (XB), i.e. we endow the probability measure μ on $\Omega = C_{2n}$, setting $X_i(t\omega) = \omega_i(t), i = 1 \cdots n$, and $B_i(t\omega) = \omega_{n+i}(t), i = 1 \cdots n$. Put

$$D(t) = \exp\left(-\sum_k \int_0^t \phi_k(sX)dB_k(s) - \tfrac{1}{2}\sum_k \int_0^t \phi_k^2(sX)ds\right.$$

and $d\nu_T = D(T)d\mu$ on $F_T(\equiv \mathfrak{B}_T(XB))$. Then ν_T can be extended to the probability measure ν on $F(\equiv \bigvee_T F_T)$ uniquely. Repeating the same calculations as (4.8), we see that, on $(\Omega F \nu)$, $W(t) \equiv B(t) + \int_0^t \phi(sX)ds$ is an

F_t-Brownian motion and

$$X(t) = x + \int_0^t \beta(sX)dW(s) .$$

Therefore the law of (XW) is unique. Since $B(t) = W(t) - \int_0^t \phi(sX)ds$, the law of (XB) is unique. This turns out that the law of coordinate is unique in $(\Omega F\nu)$. Hence ν_T is unique on F_T. This means that μ is unique on F_T, since $D(T)$ is positive. Consequently, μ is unique on F. This completes the proof of Theorem 3.

COROLLARY. *Suppose that* (A 8) *is satisfied, besides* (A 1)~(A 4), (A 8) *there exists a bounded* $G_n \times \mathfrak{B}_m(\Gamma)$*-measurable n-vector function* ϕ, *such that*

$$\alpha(tfu) = \beta(tf)\phi(tfu) .$$

Then, for any G_n*-measurable function* $v; [0\ \infty) \times C_n \to \Gamma$, *the following stochastic differential equation*

$$dX(t) = \beta(t\, X)dB(t) + \alpha(t\, X\, v(tX))dt , \qquad X(0) = x ,$$

has a law unique solution X.

This means that $(B\, v(tX))$ is an admissible system and X is the response.

5. Laws of solutions. Let A be the set of all G_n-measurable functions $v; [0\ \infty) \times C_n \to \Gamma$. We introduce two sets of probability measures on C_n, namely

$$\mathscr{P} = \{\text{law of } X^U ; (BU) \in \mathfrak{A}\}$$

and

$$Q = \{\text{law of } X^v ; v \in A\} .$$

THEOREM 4. *Suppose that* (A 9) *is satisfied, besides* (A 1)~(A 4) *and* (A 6)~(A 8),

(A 9) $\beta(tfu)$ *is uniformly positive definite, i.e.*

$$\sum \beta_{ij}(tfu)c_i c_j \geq K\,|c|^2 \qquad \textit{with } K > 0 .$$

Then $\mathscr{P} = Q$.

This theorem means that the response of an admissible control may be regarded as the response of control based on a complete observation.

From Theorems 2 and 4, we can easily see

COROLLARY. *Under the conditions* (A 1)~(A 9), *there exists a function* v *of* A *such that* X^v *is optimal, i.e.*

$$X(t) = x + \int_0^t \beta(s\,X)dB(s) + \int_0^t \alpha(s\,X\,v(sX))ds$$

and

$$E\Phi(X) = \inf_{\mathfrak{A}} E\Phi(X^U)\ .$$

Let X be the response for (BU). Then we have

LEMMA 1. *There exists a function* v *of* A, *such that*

$$E(\alpha(tXU))/\mathfrak{B}_t(X)) = \alpha(t\,X\,v(tX)) \qquad \tilde{\forall}(t\omega)\ .$$

Proof. Let $\phi(t\omega)$ be a measurable and $\mathfrak{B}_t(X)$-adapted version of the conditional expectation, $E(\alpha(tXU(t))/\mathfrak{B}_t(X))$. Hence ϕ is $\mathfrak{S}_n$-measurable and, by (A 7),

$$\phi(t\omega) \in \alpha(t\,X(\omega)\,\Gamma) \qquad \tilde{\forall}(t\omega)\ . \tag{5.1}$$

Therefore we may modify ϕ, so that (5.1) holds for any $(t\omega)$. This means that there exists a $\bar{\phi}_n$-measurable $\bar{\phi}$ such that $\bar{\phi}(t\omega) = \phi(t\omega)$ for almost all $(t\omega)$, and $\bar{\mathfrak{S}}(t\omega) \in \alpha(t\,X(\omega)\,\Gamma)$ for all $(t\omega)$. Hence an implicit function theorem guarantees the existence of a $\bar{\mathfrak{S}}_n$-measurable $\bar{V}; [0\ \infty) \times \Omega \to \Gamma$, such that

$$\bar{\phi}(t\omega) = \alpha(t\,X(\omega)\,\bar{V}(t\omega))\ .$$

Taking a $\mathfrak{S}_n$-measurable modification of $\bar{V}$, we have a function $v \in A$, such that

$$\phi(t\omega) = \alpha(t\,X(\omega)\,v(tX(\omega))) \qquad \tilde{\forall}(t\omega)\ .$$

This completes the proof of Lemma 1.

Put $Z(t) = X(t) - x - \int_0^t \alpha(s\,X\,v(sX))ds$. Then we see

LEMMA 2. Z *is an* L_2-*martingale adapted to* $\mathfrak{B}_t(X)$ *and its variation process* $\langle Z_i Z_j \rangle$ *is given by*

$$\langle Z_i Z_j \rangle(t) = \sum_{\ell=1}^{n} \beta_{i\ell}(sX)\beta_{\ell j}(sX)ds\ . \tag{5.2}$$

Proof. From the definition of $Z, Z(t)$ is $\mathfrak{B}_t(X)$-measurable. On the other hand

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$$Z(t) = \int_0^t \beta(sX)ds + \int_0^t (\alpha(sXU(s)) - E(\alpha(sXU(s)/\mathfrak{B}_s(X)))ds$$

Hence Z is an L_2-martingale adapted to $\mathfrak{B}_t(X)$. Using a formula on stochastic differentials, we have

$$d(Z_i(t)Z_j(t)) = Z_i(t)dZ_j(t) + Z_j(t)dZ_i(t) + \sum_\ell \beta_{i\ell}(tX)\beta_{j\ell}(tX)dt .$$

Therefore $Z_i(t)Z_j(t) - \sum_\ell \int_0^t \beta_{i\ell}(sX)\beta_{j\ell}(sX)ds$ is a $\mathfrak{B}_t(X)$-martingale. This means (5.2).

Proof of Theorem 4. In order to show $\mathscr{P} \subset Q$, we shall apply the method of the so-called innovation, [5]. Set $\theta(sf) = \beta(sf)^{-1}$. Then, by (A 9), θ is bounded, symmetric and G_n-measurable. Hence, by Lemma 2, we can define the stochastic integral $\zeta(t) = (\zeta_1(t), \cdots, \zeta_n(t))$,

$$\zeta(t) = \int_0^t \theta(sX)dX(s) ,$$

as an L_2-martingale adapted to $\mathfrak{B}_t(X)$, with

$$\langle \zeta_i \eta \rangle(t) = \sum_k \int_0^t \theta_{ik}(sX)d\langle Z_k \eta \rangle(s) \qquad i = 1 \cdots n ,$$

for any L_2-martingale η adapted to $\mathfrak{B}_t(X)$, [10]. Hence, by (5.2),

$$\begin{aligned}\langle \zeta_j Z_k \rangle(t) &= \sum_\ell \int_0^t \theta_{j\ell}(sX)d\langle Z_\ell Z_k \rangle(s) \\ &= \sum_{\ell p} \int_0^t \theta_{j\ell}(sX)\beta_{\ell p}(sX)\beta_{pk}(sX)ds .\end{aligned}$$

Therefore,

$$\begin{aligned}\langle \zeta_i \zeta_j \rangle(t) &= \sum_k \int_0^t \theta_{ik}(sX)d\langle Z_k \zeta_j \rangle(s) \\ &= \sum_{k\ell p} \int_0^t \theta_{ik}(sX)\theta_{j\ell}(sX)\beta_{\ell p}(sX)\beta_{pk}(sX)ds = \delta_{ij}t .\end{aligned}$$

This means that ζ is a Brownian process adapted to $\mathfrak{B}_t(X)$.

Consider the stochastic integral $\xi(t) = \int_0^t \beta(sX)d\zeta(s)$. Then

$$\langle \xi_i \eta \rangle(t) = \sum_j \int_0^t \beta_{ij}(sX)d\langle \zeta_j \eta \rangle(s) .$$

Since

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$$\langle\zeta_j\eta\rangle(t) = \sum_{\ell}\int_0^t \theta_{j\ell}(sX)d\langle Z_\ell\eta\rangle(s) ,$$

we have

$$\langle\zeta_i\eta\rangle(t) = \sum_{j\ell}\int_0^t \beta_{ij}(sX)\theta_{j\ell}(sX)d\langle Z_\ell\eta\rangle(s) = \langle Z_i\eta\rangle(t) .$$

This implies that, with probability 1, $Z_i(t) = \xi_i(t)$ for all $t \geq 0$.

Because ζ is a Brownian motion, Cor. of Theorem 3 implies "$\mathscr{P} \subset Q$". Appealing to "$\mathscr{P} \supset Q$", this completes the proof of Theorem 4.

§ **6. Optimal controls for** $\beta(tfu)$. In this section we drop the condition (A 4). An optimal control is obtained by a little different assumption, i.e. (A 9) instead of (A 5). But, since the solvability of stochastic differential equation

$$dX(t) = \beta(t\,X\,v(tX))dB , \qquad v \in A$$

is not yet decided, the synthesis problem is settled only in a weak sense. Hereafter we assume (B 1) and (B 2), besides (A 1)$\sim$(A 3) and (A 9),

(B 1) $\alpha(tfu)$ and $\beta(tfu)$ are continuous in u for any (tf),

(B 2) $\left\{\begin{pmatrix}\beta^2(tfu)\\ \alpha(tfu)\end{pmatrix}; u \in \Gamma\right\}$ is convex,

PROPOSITION 1. *Let $B = (B_1 \cdots B_n)$ be an n-dimensional Brownian process. Suppose that $e, \tilde{e}, \gamma$ and $\tilde{\gamma}$ are real non-anticipative processes, whose 4th moments are locally bounded, say $E\,|\eta(t)|^4 \leq K^4(t)$, $\eta = e, \tilde{e}\ \gamma, \tilde{\gamma}$. We define ξ and $\tilde{\xi}$ by*

$$\xi(t) = \int_0^t e(s)dB_i(s) + \int_0^t \gamma(s)ds$$

and

$$\tilde{\xi}(t) = \int_0^t \tilde{e}(s)dB_j(s) + \int_0^t \tilde{\gamma}(s)ds .$$

Then, putting $\Delta_{nk} = \xi\left(\frac{k}{2^n}\right) - \xi\left(\frac{k-1}{2^n}\right)$ *and* $\tilde{\Delta}_{nk} = \tilde{\xi}\left(\frac{k}{2^n}\right) - \tilde{\xi}\left(\frac{k-1}{2^n}\right)$,

we have

$$(6.1) \qquad P\left(\sup_{t\leq T}\left|\delta_{ij}\int_0^t e(s)\tilde{e}(s)ds - \sum_{k<t2^n}\Delta_{nk}\tilde{\Delta}_{nk}\right| > 2^{n/4}\right) < C2^{-n/4}$$

where a constant C depends only on T and $K(T)$.

Proof. Put $\Delta_{nk}(t) = \Delta(t) = \xi(t) - \xi(c)$ where $c = (k-1)/2^n$ and $\tilde{\Delta}(t)$ similarly. Then, in the same way as (2.2) of Theorem 1, we obtain, for $t < T$,

$$E\,|\Delta(t)|^4 \leq K_1(T)((t-c)^2 + (t-c)^4) \tag{6.2}$$

where $K_1(T)$ depends only on T and $K(T)$. By a formula on stochastic differentials,

$$\begin{aligned}\Delta(t)\tilde{\Delta}(t) = \int_c^t e(s)\tilde{\Delta}(s)dB_i(s) + \int_c^t \tilde{e}(s)\Delta(s)dB_j(s) \\ + \int_c^t \gamma(s)\tilde{\Delta}(s) + \tilde{\gamma}(s)\Delta(s)ds + \delta_{ij}\int_c^t e(s)\tilde{e}(s)ds\,.\end{aligned} \tag{6.3}$$

From (6.2), we have

$$\begin{aligned} E\left|\int_c^t \gamma(s)\tilde{\Delta}(s) + \tilde{\gamma}(s)\Delta(s)ds\right| &\leq K(T)\int_c^t \sqrt{E\tilde{\Delta}(s)^2} + \sqrt{E\Delta(s)^2}\,ds \\ &\leq K_2(T)\int_c^t \sqrt{s-c}\,ds = \tfrac{2}{3}K_2(T)(t-c)^{3/2}\,. \end{aligned} \tag{6.4}$$

$$\begin{aligned} E\left(\int_c^t e(s)\tilde{\Delta}(s)dB_i\right)^2 &= \int_c^t Ee^2(s)\tilde{\Delta}(s^2)ds \leq K^2(T)\int_c^t \sqrt{E\tilde{\Delta}(s)^4}\,ds \\ &\leq K_3(T)\int_c^t (s-c)ds = \tfrac{1}{2}K_3(T)(t-c)^2\,. \end{aligned}$$

Hence, by a martingale inequality, we see

$$\begin{aligned} P\left(\sup_{t\leq T}\left|\sum_{k<t2^n}\int_{(k-1)/2^n}^{k/2^n} e(s)\tilde{\Delta}_{nk}(s)dB_i(s)\right| > 2^{-n/4}\right) \\ \leq \sum_{k<T2^n} E\left(\int_{(k-1)/2^n}^{k/2} e(s)\tilde{\Delta}_{nk}(s)dB_i(s)\right)^2 2^{n/2} = 0(2^{-n/2}) \end{aligned} \tag{6.5}$$

Using (6.3) (6.4) and (6.5) we can obtain (6.1).

By virtue of Borel-Cantelli's lemma, (6.1) implies that, with probability 1,

$$\lim_{n\to\infty}\sum_{k<t2^n} \Delta_{nk}\tilde{\Delta}_{nk} = \int_0^t e(s)\tilde{e}(s)ds \qquad \text{uniformly} \tag{6.6}$$

on any bounded subinterval of $[0\ \infty)$. We denote the left side of (6.6) by $\int_0^t d\xi(s)d\tilde{\xi}(s)$.

COROLLARY. *Let X be a response for an admissible (BU). Then, with probability* 1,

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$$\int_0^t dX_i(s)dX_j(s) = \sum_k \int_0^t \beta_{ik}(sXU(s))\beta_{kj}(sXU(s))ds\,, \qquad \forall t\,.$$

Hence, using a mapping π *with* $\pi(t\omega) = (t\,X(\omega))$, *we see that the process* $Y(t) = \int_0^t \beta^2(sXU(s))ds$ *is* $\overline{\mathfrak{G}}_n$*-measurable.*

Now we shall prove the following existence theorem.

THEOREM 5. *Under the assumptions* (A 1)~(A 3), (A 9) (B 1) *and* (B 2), *we have an optimal control.*

LEMMA 1. *There exists a* G_n*-measurable function* $v\,;(0\ \infty) \times C_n \to \Gamma$, *such that for almost all* $(t\omega)$

$$\beta^2(t\,X(\omega)\,U(t\omega)) = \beta^2(t\,X(\omega)\,v(tX(\omega)))\,. \tag{6.7}$$

Because a symmetric and positive definite root of a symmetric positive definite matrix is unique, (6.7) means that, for almost all $(t\omega)$

$$\beta(t\,X(\omega)\,U(t\omega)) = \beta(t\,X(\omega)\,v(tX(\omega)))\,.$$

LEMMA 2. *There exists a* G_{2n}*-measurable* $w\,;[0\ \infty) \times C_{2n} \to \Gamma$, *such that, for almost all* $(t\omega)$

$$\beta(t\,X(\omega)\,U(t\omega)) = \beta(t\,X(\omega)\,w(tX(\omega)B(\omega)))$$

and

$$\alpha(t\,X(\omega)\,U(t\omega)) = \alpha(t\,X(\omega)\,w(tX(\omega)B(\omega)))$$

Proof. Define $\pi\,;[0\ \infty) \times \Omega \to C_{2n}$ by $\pi(t\omega) = (t\,X(\omega)\,B(\omega))$. Put $Z(t) = \int_0^t \alpha(s\,X\,U(s))ds$. Then $Z(t) = X(t) - x - \int_0^t \beta(s\,X\,U(s))ds$ is also $\overline{\mathfrak{G}}_{2n}$-measurable by Lemma 1. Therefore $\alpha(t\,X\,U(t))$ is also $\overline{\mathfrak{G}}_{2n}$-measurable as a derivative of Z. Put $\gamma(t\omega) = \begin{pmatrix}\beta^2(tX(\omega)U(t\omega))\\ \alpha(tX(\omega)U(t\omega))\end{pmatrix}$ and $\tilde{\gamma}(t\omega u) = \begin{pmatrix}\beta^2(tX(\omega)u)\\ \alpha(tX(\omega)u)\end{pmatrix}$. Then $\gamma(t\omega) \in \tilde{\gamma}(t\,\omega\,\Gamma)$, and $\gamma(\cdot\,\cdot)$ and $\tilde{\gamma}(\cdot\,\cdot\,u)$ are $\overline{\mathfrak{G}}_{2n}$-measurable. So, we have a $\overline{\mathfrak{G}}_{2n}$-measurable $\overline{W}\,;[0\ \infty) \times \Omega \to \Gamma$, such that

$$\gamma(t\omega) = \tilde{\gamma}(t\,\omega\,\overline{W}(t\omega))\,.$$

Again taking a $\mathfrak{G}_{2n}$-modification of $\overline{W}$, we get a G_{2n}-measurable function $w\,;[0\ \infty) \times C_{2n} \to \Gamma$, such that for almost all $(t\omega)$

$$\gamma(t\omega) = \tilde{\gamma}(t\,\omega\,w(tX(\omega)B(\omega)))\,.$$

Recalling the definitions of γ and $\tilde{\gamma}$, we have Lemma 2, since a symmetric

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positive definite root of β^2 is β.

For $(BU)\in\mathfrak{A}$, we define θ^U and Θ^U by

$$\theta^U(t)=\int_0^t \alpha(s\ X\ U(s))ds$$

and

$$\Theta^U(t)=\int_0^t \beta^2(s\ X\ U(s))ds$$

where $X=X^U$ is a response of (BU). Put $\mathfrak{M}=\{(X^U\theta^U\Theta^U),(BU)\in\mathfrak{A}\}$. Then, by (2.2) of Theorem 1, we can easily see the following lemma,

LEMMA 3. *$\mathfrak{M}$ is L-totally bounded.*

LEMMA 4. *Suppose that $(X_\ell\theta_\ell\Theta_\ell)$ comes from an admissible system $(B_\ell U_\ell)$ and $(X_\ell\theta_\ell\Theta_\ell B_\ell)$ converges to $(X\theta\Theta B)$ in L-metric. Then $(X\theta\Theta)$ is in $\mathfrak{M}$.*

Proof. By Lemma 2, we may assume $U_\ell(t)=v_\ell(tX_\ell B_\ell)$, with a G_{2n}-measurable function v_ℓ. Hence, using Skorohod's theorem, we can construct $(\tilde X_\ell\tilde\theta_\ell\tilde\Theta_\ell\tilde B_\ell)$ and $(\tilde X\tilde\theta\tilde\Theta\tilde B)$, so that

$$\tilde X_\ell(t)=x+\int_0^t\beta(s\tilde X_\ell v_\ell(s\tilde X_\ell\tilde B_\ell))d\tilde B_\ell+\int_0^t\alpha(s\tilde X_\ell v_\ell(s\tilde X_\ell\tilde B_\ell))ds$$

$$\tilde\theta_\ell(t)=\int_0^t\alpha(s\tilde X_\ell v_\ell(s\tilde X_\ell\tilde B_\ell))ds$$

$$\tilde\Theta_\ell(t)=\int_0^t\beta^2(s\tilde X_\ell v_\ell(s\tilde X_\ell\tilde B_\ell))ds$$

and, with probability 1, $(\tilde X_\ell(t)\tilde\theta_\ell(t)\tilde\Theta_\ell(t)\tilde B_\ell(t))$ tends to $(\tilde X(t)\tilde\theta(t)\tilde\Theta(t)\tilde B(t))$ uniformly on any bounded interval.

$$\text{Put } \xi_\ell(t)=\tilde X_\ell(t)-x-\tilde\theta_\ell(t)=\int_0^t\beta(s\tilde X_\ell v_\ell(s\tilde X_\ell\tilde B_\ell))d\tilde B_\ell\,.$$

Then ξ_ℓ is an L_2-martingale adapted to $\mathfrak{B}_t(\tilde X_\ell\tilde\theta_\ell)$. Tending ℓ to ∞, we can see that $\xi(t)\equiv\tilde X(t)-x-\tilde\theta(t)$ is an L_2-martingale adapted to $\mathfrak{B}_t(\tilde X\tilde\theta)$. Now we shall show

$$\tilde\Theta_{ij}(t)=\int_0^t d\tilde X_i(s)d\tilde X_j(s) \tag{6.8}$$

Define Δ_{nk} by $\tilde X_i(k/2^n)-\tilde X_i((k-1)/2^n)$ and $\tilde\Delta_{nk}$ similarly for j.

$$\sup_{t\le T}\Big|\tilde\Theta_{ij}(t)-\sum_{k<t2^n}\Delta_{nk}\tilde\Delta_{nk}\Big|$$
$$\le\sup_{t\le T}|\tilde\Theta_{ij}(t)-\tilde\Theta_{\ell ij}(t)|+\sup_{t\le T}\Big|\tilde\Theta_{\ell ij}(t)-\sum_{k<t2^n}\Delta^\ell_{nk}\tilde\Delta^\ell_{nk}\Big|$$

$$+ \sup_{t \leq T} \left| \sum_{k < t2^n} \Delta^\ell_{nk} \tilde{\Delta}^\ell_{nk} - \Delta_{nk} \tilde{\Delta}_{nk} \right| .$$

Recalling (6.1) of Prop. 1, we have

$$P(\text{2nd term} > 2^{-n/4}) \leq C 2^{-n/4} , \qquad \ell = 1, 2 \cdots .$$

For any n, we can take a large $\ell_0(n)$ so that

$$P(\text{3rd term} > 2^{-n/4}) < 2^{-n/4} , \qquad \text{for } \ell > \ell_0(n)$$

and

$$P(\text{1st term} > 2^{-n/4}) < 2^{-n/4} , \qquad \text{for } \ell > \ell_0(n) .$$

Hence we see that, with probability 1, $\sum_{k < t2^n} \Delta_{nk} \tilde{\Delta}_{nk}$ tends to $\tilde{\Theta}_{ij}(t)$ uniformly on any bounded interval, namely we have (6.8). Therefore $\tilde{\Theta}$ is $\overline{\mathfrak{G}}_n$-measurable.

$$\begin{aligned} |\tilde{\Theta}_{ij}(t) - \tilde{\Theta}_{ij}(s)| = \lim_{\ell \to \infty} |\tilde{\Theta}_{\ell ij}(t) - \tilde{\Theta}_{\ell ij}(s)| &= \lim \left| \int_s^t \beta^2_{ij}(\tau \tilde{X}_\ell v_\ell) d\tau \right| \\ &< 2 \lim \int_s^t d\tau \left[L(\tau) + \int_{-\tau}^0 |\tilde{X}_\ell(\tau + \lambda)|^2 dM(\lambda) \right] \\ &= 2 \int_s^t d\tau \left[L(\tau) + \int_{-\tau}^0 |\tilde{X}(\tau + \lambda)|^2 dM(\lambda) \right] \end{aligned}$$

So, there exists a symmetric $\overline{\mathfrak{G}}_n$-measurable H, such that

$$\tilde{\Theta}(t) = \int_0^t H(s) ds$$

Moreover, by (A 9), we see, setting $V_\ell(t) = v_\ell(t \tilde{X}_\ell \tilde{B}_\ell)$,

$$\sum_{ij} (\tilde{\Theta}_{ij}(t) - \tilde{\Theta}_{ij}(s)) c_i c_j = \lim_\ell \int_s^t \sum \beta^2_{ij}(\tau \tilde{X}_\ell V_\ell) c_i c_j d\tau \geq K |c|^2 (t - s)$$

Hence, we may assume that H is uniformly positive definite. If necessary, we may take a $\mathfrak{G}_n$-modification of H, and $\tilde{\Theta}$ may be regarded as $\mathfrak{G}_n$-measurable. From

$$\xi_i(t) \xi_j(t) - \tilde{\Theta}_{ij}(t) = \lim (\xi_{\ell i}(t) \xi_{\ell j}(t) - \tilde{\Theta}_{\ell ij}(t)) ,$$

we see that $\xi_i(t) \xi_j(t) - \tilde{\Theta}_{ij}(t)$ is a $\mathscr{B}_t(\tilde{X} \tilde{\theta})$-martingale. This means

$$\langle \xi_i \xi_j \rangle (t) = \tilde{\Theta}_{ij}(t) .$$

Let $\sqrt{H}$ be the symmetric positive definite root of H. The $\sqrt{H}$ is $\mathfrak{G}_n$-measurable and we can define W by

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$$W(t) = \int_0^t \Lambda(s) d\xi(s)$$

where $\Lambda = \sqrt{H}^{-1}$, i.e. W is a $\mathscr{B}_t(\tilde{X}\tilde{\theta})$-martingale with variation process

$$\begin{aligned}\langle W_i W_j \rangle(t) &= \sum_{k\ell} \int_0^t \Lambda_{ik}(s) \Lambda_{\ell j}(s) d\langle \xi_k \xi_\ell \rangle(s) \\ &= \sum_{k\ell} \int_0^t \Lambda_{ik}(s) \Lambda_{\ell j}(s) H_{k\ell}(s) ds = \delta_{ij} t \,.\end{aligned}$$

Namely, W is a Brownian process. Moreover, in the same way as in § 5, we have

$$\xi(t) = \int_0^t \sqrt{H}(s) dW(s) \,.$$

Consequently

$$\tilde{X}(t) = x + \int_0^t \sqrt{H(s)}\, dW(s) + \tilde{\theta}(t) \,. \tag{6.9}$$

$$\begin{aligned}|\tilde{\theta}_i(t) - \tilde{\theta}_i(s)| &= \lim |\tilde{\theta}_{\ell i}(t) - \tilde{\theta}_{\ell i}(s)| = \lim \left| \int_s^t \alpha_i(\tau \tilde{X}_\ell V_\ell) d\tau \right| \\ &< 2 \lim \int_s^t d\tau \left(\int_{-\tau}^0 |\tilde{X}_\ell(\tau + p)|^2 dM(P) + L(\tau) \right)^{1/2} \\ &= 2 \int_s^t d\tau \left(\int_{-\tau}^0 |X(\tau + p)|^2 dM(p) + L(\tau) \right)^{1/2} .\end{aligned}$$

Hence there exists a Radon-Nykodym derivative of $\tilde{\theta}$. Define π; $[0\ \infty) \times \Omega \to C_{2n}$ by $\pi(t\omega) = (t\, \tilde{X}(\omega)\, W(\omega))$. Then (6.9) tells us that $\tilde{\theta}$ is $\mathfrak{S}_{2n}$-measurable. So,

$$\tilde{\theta} = (t) \int_0^t h(s) ds$$

with a $\mathfrak{S}_{2n}$-measurable h.

Recalling the definition of $\tilde{\theta}$ and $\tilde{\Theta}$, we see that, with probability 1,

$$\tilde{\Theta}_\ell(t) = \int_0^t \beta^2(t \tilde{X}_\ell V_\ell) ds \to \int_0^t H(s) ds \qquad \text{uniformly on } [0\ T]$$

and

$$\tilde{\theta}_\ell(t) = \int_0^t \alpha(t \tilde{X}_\ell V_\ell) ds \to \int_0^t h(s) ds \qquad \text{uniformly on } [0\ T] \,.$$

But, from (A 2), we see

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$$\left|\tilde{\Theta}_{ij}(t) - \int_0^t \beta_{ij}^2(t\tilde{X}V_\ell)ds\right|$$
$$< |\tilde{\Theta}_{ij}(t) - \tilde{\Theta}_{\ell ij}(t)| + \left|\tilde{\Theta}_{\ell ij}(t) - \int_0^t \beta_{ij}^2(t\tilde{X}V_\ell)ds\right|$$
$$< |\tilde{\Theta}_{ij}(t) - \tilde{\Theta}_{\ell ij}(t)| + \int_0^t ds\left(\int_{-s}^0 |\tilde{X}_\ell(\tau + s) - \tilde{X}(\tau + s)|^2 dM(\tau)\right)^{1/2} K_4(T\omega)$$

where $K_4(T\omega)$ depends on T and $\sup_{\substack{t\leq T \\ \ell=1,2\cdots}} |\tilde{X}_\ell(t)|$. Since $\tilde{\Theta}_\ell$ and $\tilde{X}_\ell$ converge uniformly on $[0\ T]$, we have, with probability 1,

$$\int_0^t \beta^2(t\tilde{X}V_\ell)ds \to \int_0^t H(s)ds \qquad \text{uniformly on } [0\ T]$$

and

$$\int_0^t \alpha(t\tilde{X}V_\ell)ds \to \int_0^t h(s)ds \qquad \text{uniformly on } [0\ T]\ .$$

Put $\gamma_\ell(t\omega) = (\beta_{11}^2(t\tilde{X}V_\ell), \cdots \beta_{nn}^2(t\tilde{X}V_\ell), \alpha_1(t\tilde{X}V_\ell) \cdots \alpha_n(t\tilde{X}V_\ell))$ and $\gamma(t\omega) = (H_{11}(s), \cdots H_{nn}(s), h_1(s) \cdots h_n(s))$. Then for any $(n^2 + n)$-vector $\eta \in L_2([0\ T] \times \Omega)$,

$$\int_0^T (\gamma_\ell(s), \eta(s))ds \to \int_0^T (\gamma(s), \eta(s))ds\ , \qquad \text{for } \tilde{\forall}\omega\ .$$

Recalling (2.1), we get the following estimate

$$E\left|\int_0^T (\gamma_\ell(s), \eta(s)ds\right|^{4/3} < T^{1/3}E\int_0^T |\eta(s)|^{4/3}\,|\gamma_\ell(s)|^{4/3}ds$$
$$< T^{1/3}\left(E\int_0^T |\eta(s)|^2 ds\right)^{2/3}\left(E\int_0^T |\gamma_\ell(s)|^4\right)^{1/3} \leq K_5(T)\,\|\eta\|^{4/3}\ , \qquad \ell = 1, 2 \cdots$$

Therefore, by virtue of uniform integrability, we have

$$\lim_\ell E\int_0^T (\gamma_\ell(s), \eta(s))ds = E\int_0^T (\gamma(s), \eta(s))ds\ .$$

Consequently γ_ℓ tends to γ weakly in $L_2([0\ T] \times \Omega)$. Hence a convex combination of γ_ℓ can tend to γ strongly. Therefore we can take a subsequence which converges almost everywhere. From (B 1) and (B 2), we have, for almost all $(t\omega)$,

$$\begin{pmatrix} H(t\omega) \\ h(t\omega) \end{pmatrix} \in \left\{\begin{pmatrix} \beta^2(t\tilde{X}(\omega)u) \\ \alpha(t\tilde{X}(\omega)u) \end{pmatrix} ; u \in \Gamma\right\}\ . \tag{6.10}$$

We can modify H and h on a null set, so that (6.10) holds for all $(t\omega)$, namely there exists a $\mathfrak{G}_{2n}$-measurable $\bar{H}$ and $\bar{h}$ such that, for any $(t\omega)$

$$\begin{pmatrix}\bar{H}(t\omega)\\ \bar{h}(t\omega)\end{pmatrix} \in \left\{\begin{pmatrix}\beta^2(t\tilde{X}(\omega)u)\\ \alpha(t\tilde{X}(\omega)u)\end{pmatrix}; u \in \Gamma\right\} . \tag{6.11}$$

Hence, from an implicit function theorem we can take a $\mathfrak{G}_{2n}$-measurable $\bar{V}$; $[0\ \infty) \times \Omega \to \Gamma$, such that

$$\beta^2(t\tilde{X}(\omega)\bar{V}(t\omega)) = \bar{H}(t\omega)$$

and

$$\alpha(t\tilde{X}(\omega)\bar{V}(t\omega)) = \bar{h}(t\omega) .$$

Taking a $\mathfrak{G}_{2n}$-modification of $\bar{V}$, we have a G_{2n}-measurable function v; $[0\ \infty) \times C_{2n} \to \Gamma$, such that, for almost all $(t\omega)$,

$$H(t\omega) = \beta^2(t\ \tilde{X}(\omega)\ v(t\tilde{X}(\omega)W(\omega)))$$

and

$$h(t\omega) = \alpha(t\ \tilde{X}(\omega)\ v(t\tilde{X}(\omega)W(\omega))) .$$

Since $\beta(t\ \tilde{X}(\omega)\ v(t\tilde{X}(\omega)))$ is the symmetric positive definite root of $H(t\omega)$, we have, from (6.9)

$$\tilde{X}(t) = x + \int_0^t \beta(t\ \tilde{X}\ v(t\tilde{X}W))dW(s) + \int_0^t \alpha(t\ \tilde{X}\ v(t\tilde{X}W))ds \qquad t \geq 0$$

with probability 1. This means that $(\tilde{X}\tilde{\theta}\tilde{\Theta})$ comes from the admissible system $(W\ v(t\tilde{X}W))$, namely $(\tilde{X}\tilde{\theta}\tilde{\Theta})$ is in $\mathfrak{M}$.

Proof of Theorem 5. By Lemmas 3 and 4, we can apply the same method as Theorem 2. Let X_m be approximate optimal. Since $\mathfrak{M}$ is sequential compact, we have a subsequence X_{m_i} of X_m which converges in $\mathfrak{M}$, say $X = \lim X_{m_i}$. The lower semi-continuity of Φ induces

$$E\Phi(X) \leq \varliminf E\Phi(X_{m_i})$$

This completes the proof of Theorem 5.

Concerning synthesis problems, we have

THEOREM 6. *Let* (BU) *be an admissible system and* X *its response i.e.*

$$X(t) = x + \int_0^t \beta(s\ X\ U(s))dB(s) + \int_0^t \alpha(s\ X\ U(s))ds .$$

Under the assumption of Theorem 5, we have a G_n*-measurable function* v; $[0\ \infty) \times C_n \to \Gamma$, *such that*

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$$X(t) = x + \int_0^t \beta(s\ X\ v(sX))d\zeta(s) + \int_0^t \alpha(s\ X\ v(sX))ds$$

with a Brownian process ζ.

Proof. By Lemma 1 of Theorem 5, $\beta(t\ X\ U(t))$ may be regarded as $\mathfrak{G}_n$-measurable ψ. This fact guarantees the possibility of the method in § 5.

Let $\phi(s)$ be a $\mathfrak{G}_n$-measurable version of $E(\alpha(s\ X\ U(s))/\mathfrak{B}_s(X))$ and put $Z(t) = X(t) - x - \int_0^t \phi(s)ds$. Then Z is an L_2-martingale adapted to $\mathfrak{B}_t(X)$ and its variation process is given by

$$\langle Z_i Z_j \rangle(t) = \sum_k \int_0^t \psi_{ik}(s)\psi_{kj}(s)ds\ .$$

Set $\theta(s) = \psi(s)^{-1}$ and define ζ by

$$\zeta(t) = \int_0^t \theta(s)dX(s)\ .$$

Then ζ is an L_2-martingale adapted to $\mathfrak{B}_t(X)$ and

$$\langle \zeta_i \zeta_j \rangle(t) = \sum_{k\ell p} \int_0^t \theta_{i\ell}(s)\theta_{jk}(s)\psi_{\ell p}(s)\psi_{pk}(s)ds = \delta_{ij}t\ .$$

Namely, ζ is a $\mathfrak{B}_t(X)$-Brownian motion and

$$Z(t) = \int_0^t \psi(s)d\zeta(s)\ .$$

Hence we have

$$X(t) = x + \int_0^t \psi(s)d\zeta(s) + \int_0^t \phi(s)ds\ . \tag{6.12}$$

On the other hand, for almost almost all $(t\omega)$

$$E(\beta(t\ X\ U(t))/\mathfrak{B}_t(X)) = \beta(t\ X(\omega)\ U(t\omega)) = \psi(t\omega)$$

and

$$E(\alpha(t\ X\ U(t))/\mathfrak{G}_t(X)) = \phi(t\omega)\ .$$

Therefore, by (B 1) and (B 2),

$$\begin{pmatrix}\psi^2(t\omega)\\ \phi(t\omega)\end{pmatrix} \in \left\{\begin{pmatrix}\beta^2(t\ X(\omega)\ u)\\ \alpha(t\ X(\omega)\ u)\end{pmatrix}; u \in \Gamma\right\} \tag{6.13}$$

for almost all $(t\omega)$. Taking a $\overline{\mathfrak{G}}_n$-modification of ψ^2 and ϕ, so that (6.13)

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holds for any $(t\omega)$, we have a $\overline{\mathfrak{G}}_n$-measurable $\overline{V}; [0\ \infty) \times \Omega \to \Gamma$, such that, for almost all $(t\ \omega)$,

$$\psi^2(t\omega) = \beta^2(t\, X(\omega)\, \overline{V}(t\omega)) \quad \text{and} \quad \phi(t\omega) = \alpha(t\, X(\omega)\, \overline{V}(t\omega))\ .$$

Hence we can take a G_n-measurable $v; [0\ \infty) \times C_n \to \Gamma$ such that, for almost all $(t\omega)$,

$$\psi^2(t\omega) = \beta^2(t\, X(\omega)\, v(tX(\omega))$$

and

$$\phi(t\omega) = \alpha(t\, X(\omega)\, v(tX(\omega))\ .$$

Since ψ is symmetric and positive definite, we see that, for almost all $(t\omega)$,

$$\psi(t\omega) = \beta(t\, X(\omega)\, v(tX(\omega))\ .$$

Recalling (6.11), we complete the proof.

§ 7. **Diffusion type processes.** In this section, we assume that $\beta(tfu) = \sigma(f(t)\, u)$ and $\alpha(tfu) = \gamma(f(t)\, u)$. Namely, we treat a stochastic differential equation of diffusion type,

(7.1) $$dX(t) = \sigma(X(t)\, U(t))dB(t) + \gamma(X(t)\, U(t))dt\ , \qquad X(0) = x\ .$$

Suppose that σ and γ are Lipschitz continuous in x. Then (7.1) has a unique solution X^U. When Φ is given by an integral form, Krylov proved that an optimal control can be given by a Markovian policy, i.e. he showed the following theorem

THEOREM [9]. *Let $\tau(X)$ be the hitting time of X to the boundary of a bounded open set A. Put $\Phi(U) = \int_0^{\tau(X)} F(X(s)\, U(s))ds$, where $X = X^U$. Suppose that is uniformly positive definite and σ, γ and F are bounded and continuous in u. If Bellman equation*

(7.2) $$\sup_{u \in \Gamma} \left[\tfrac{1}{2} \sum a_{ij}(x\, u) \frac{\partial^2 v(x)}{\partial x_i \partial x_j} + \sum \gamma_i(x\, u) \frac{\partial v(x)}{\partial x_i} - F(x\, u) \right] = 0\ , \qquad \text{on } A$$

$$v(x) = 0 \qquad \text{on } \partial A$$

where $a = \sigma^2$, has a continuous solution v of W_2, i.e. (7.2) is satisfied for almost all $x \in A$, then $\inf E\Phi(U)$ can be attained by a Markovian policy.

We shall sketch the outline of his proof. Fix measurable version of derivatives of v arbitrary. Since Γ is compact and coefficients are continuous in u, the supremum is attained. From an implicit function theorem, we take a Borel function $w; A \to \Gamma$, such that, for almost all x of A,

$$\tfrac{1}{2}\sum a_{ij}(x\, w(x))v_{ij}(x) + \sum \gamma_i(x\, w(x))v_i(x) - F(x\, w(x)) = 0$$

where $v_i = \partial v/\partial x_i$ and $v_{ij} = \partial^2 v/\partial x_i \partial x_j$. If it is necessary, we may extend w to the whole R^n. By virtue of uniform positivity of σ, the stochastic differential equation

$$dY(t) = \sigma(Y(t)\, w(Y(t))dB(t) + \gamma(Y(t)\, w(Y(t)))dt\,, \qquad Y(0) = x \tag{7.3}$$

has a solution [7]. Moreover we have, with probability 1,

Lebesgue meas. $\{t; Y(t\omega) \in N\} = 0$, for any null set N of R^n. This means that there exists no trouble about the ambiguity of w on a null set of A.

From a formula on stochastic differentials [7], [8],

$$v(X(t)) - v(x) = \int_0^t \tfrac{1}{2}\sum a_{ij}(X(s)\, U(s))v_{ij}(X(s)) + \sum \gamma_i(X(s)\, U(s))v_i(X(s))ds$$
$$+ \text{ martingale}$$

Hence we have

$$-v(x) = E\int_0^{\sigma(X)} \tfrac{1}{2}\sum a_{ij}(X(s)\, U(s))v_{ij}(X(s)) + \sum \gamma_i(X(s)\, U(s))v_i(X(s))ds$$
$$< \int_0^{\sigma(X)} F(X(s)\, U(s)ds\,.$$

On the other hand,

$$-v(x) = E\int_0^{\sigma(Y)} F(Y(s)w(Y(s))ds\,.$$

Therefore, Y is an optimal trajectory and w an optimal Markovian policy.

Application to Linear Controls. Suppose that $\sigma(x\, u)$ and $F(x\, u)$ are independent of u and

$$\gamma_i(x\, u) = \sum_{j=1}^m \gamma_{ij}(x)u_j\,, \qquad i = 1 \cdots n\,.$$

So, we have the following Bellman equation,

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$$\sup_{u\in\Gamma}\left[\tfrac{1}{2}\sum a_{ij}(x)v_{ij}(x)+\sum v_i(x)\gamma_{ij}(x)u_j-F(x)\right]=0\,,\qquad \text{on } A \tag{7.4}$$
$$v=0 \qquad \text{on } \partial A\,.$$

If A is a connected and bounded open set with a smooth boundary, all coefficients a, γ and F are bounded and smooth and σ is uniformly positive definite, then there is a unique solution v of $C(\bar{A}) \cap C^2(A)$, [3]. We set the inside of the parenthesis of (7.4) by $S(x\,u)$. Since $S(x\,u)$ is linear in u, we have

$$\sup_{u\in\Gamma} S(x\,u)=\sup_{u\in\partial\Gamma} S(x\,u)=0\,,\qquad x\in A$$

namely, S can be regarded as a mapping; $A\times\partial\Gamma\to R^1$, and by an implicit function theorem, a Borel function $w; A\to\partial\Gamma$, such that $S(x\,w(x))=0$, exists. Hence we get a Bang-Bang control which is an optimal Markovian policy.

When Φ is not an integral form, we have a little example, where any optimal control cannot be given by a Markovian policy.

EXAMPLE. Consider the 1-dimensional stochastic differential equation,

$$dX(t)=dB(t)+U(t)dt\,,\qquad X(0)=x\,.$$

Let a control region Γ be $[-1\ 1]$ and $\Phi(f)=f(1)f(2)$. Hence,

$$E\Phi(X)=EX(1)X(2)=EX^2(1)+EX(1)\int_1^2 U(s)ds\,. \tag{7.5}$$

Since any non-anticipative process $U(t)$, such that $U(t)$ is in Γ, is an admissible control, we have

$$X(1)\int_1^2 U(s)ds\geq -|X(1)| \tag{7.6}$$

and, for $U(s)=-\operatorname{sgn}X(1)$, the equality of (7.6) is satisfied. Hence, putting $v=\inf_{\mathfrak{A}} E\Phi(X^U)$, we see

$$v=\inf E(X^2(1)-|X(1)|)=\inf E(|X(1)|-\tfrac{1}{2})^2-\tfrac{1}{4}\,.$$

Because the cost functional of $(|f(1)|-\frac{1}{2})^2$ is non-negative and continuous, an optimal control $\tilde{U}$ exists by Theorem 2. Since $\tilde{U}(t), t>1$, is irrelevant, the control U_0, defined by

$$U_0(t)=\tilde{U}(t)\,,\quad t\leq 1\,,\quad =-\operatorname{sgn}X(1)\quad t>1\,,$$

is optimal, i.e.

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$$v = EX^{U_0}(1)X^{U_0}(2) .$$

Suppose that a Markovian policy w satisfies

$$v = EX^{w}(1)X^{w}(2) .$$

Putting $Y(t) = X^{w}(t)$, we have, from (7.6),

$$Y(1)\int_1^2 w(Y(s))ds \geq -|Y(1)| . \tag{7.7}$$

If the inequality of (7.7) holds with positive probability,

$$v > E(|Y(1)| - \tfrac{1}{2})^2 - \tfrac{1}{4} .$$

On the other hand, "$v \leq E(|Y(1)| - \frac{1}{2})^2 - \frac{1}{4}$" is satisfied by the definition of $\tilde{U}$. Hence, we have, with probability 1,

$$Y(1)\int_1^2 w(Y(s))ds = -|Y(1)| . \tag{7.8}$$

But, Y is a diffusion whose law is mutually absolutely continuous to the law of Brownian process. So, $P(Y(1) = 0) = 0$ and, for any Borel set D with positive Lebesgue measure,

$$p(t\ x\ D) > 0 . \tag{7.9}$$

Hence, (7.8) means that, with probability 1,

$$\int_1^2 w(Y(s))ds = -\text{sgn}\ Y(1) .$$

Since $|w| \leq 1$, we have, for almost all ω

$$w(Y(s)) = -\text{sgn}\ Y(1) , \qquad \text{for almost all } s \text{ of } [1, 2]$$

Hence, for almost all s of [1, 2]

$$w(Y(s)) = -\text{sgn}\ Y(1) , \qquad \text{for almost all } \omega .$$

Appealing to (7.9), we have, $w(\cdot) = 1$ for almost everywhere and $w(\cdot) = -1$ for almost everywhere. This is absurd. Consequently we have not a Markovian policy which can give v.

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