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SHARPNESS OF SOME PROPERTIES OF WIENER AMALGAM AND MODULATION SPACES

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Abstract

We prove sharp estimates for the dilation operator $f(x) \longmapsto f(\lambda x)$, when acting on Wiener amalgam spaces $W(L^p, L^q)$. Scaling arguments are also used to prove the sharpness of the known convolution and pointwise relations for modulation spaces $M^{p,q}$, as well as the optimality of an estimate for the Schrödinger propagator on modulation spaces.

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1. Introduction

Modulation and Wiener amalgam spaces have been introduced and used to measure the time-frequency concentration of functions and tempered distributions in the framework of time-frequency analysis [9–15, 20]. Recently, these spaces have been employed to study boundedness properties of pseudodifferential operators (see, for example, [4, 18, 19]), Fourier integral operators (in particular, Fourier multipliers) [1, 7, 8] and well-posedness of solutions to partial differential equations (see, for example [2, 3, 5, 6, 21–23] and references therein).

In this paper we present new dilation properties for Wiener amalgam spaces and their optimality. Moreover, we prove the sharpness of the known convolution and pointwise estimates for modulation spaces.

To recall the definition of these spaces, we first introduce the translation and modulation operators, defined by $T_x f(t) = f(t - x)$ and $M_\xi f(t) = e^{2\pi i \xi t} f(t)$, $t, x, \xi \in \mathbb{R}^d$.

Wiener amalgam spaces [10, 12, 15]. Let $g \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ be a test function. We will refer to g as a window function. Let B be either the Banach space $L^p(\mathbb{R}^d)$ or $\mathcal{F}L^p(\mathbb{R}^d)$, $1 \le p \le \infty$. For any given function f which is locally in B (that is, $gf \in B$, for all $g \in \mathcal{C}_0^\infty(\mathbb{R}^d)$), we set $f_B(x) = \|f T_x g\|_B$. The Wiener amalgam space $W(B, L^q)(\mathbb{R}^d)$ with local component B and global component $L^q(\mathbb{R}^d)$, $1 \le q \le \infty$, is defined as the space of all functions f locally in B such that $f_B \in L^q(\mathbb{R}^d)$. Endowed with the norm $\|f\|_{W(B,L^q)} = \|f_B\|_{L^q}$, $W(B, L^q)(\mathbb{R}^d)$ is a Banach space. Moreover, different choices of $g \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ generate the same space and yield equivalent norms. In fact, the space of admissible windows for the Wiener amalgam spaces $W(B, L^q)(\mathbb{R}^d)$ can be enlarged to the so-called Feichtinger algebra $W(\mathcal{F}L^1, L^1)(\mathbb{R}^d)$. Recall that the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ is dense in $W(\mathcal{F}L^1, L^1)(\mathbb{R}^d)$.

Modulation spaces [10, 14]. For a fixed nonzero $g \in \mathcal{S}(\mathbb{R}^d)$, the short-time Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the window g is given by $V_g f(x, \xi) = \langle f, M_\xi T_x g\rangle$, $x, \xi \in \mathbb{R}^d$. Given a nonzero window $g \in \mathcal{S}(\mathbb{R}^d)$, $1 \le p, q \le \infty$, the *modulation space* $M^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_g f \in L^{p,q}(\mathbb{R}^{2d})$ (mixed-norm spaces). The norm on $M^{p,q}$ is given by

$$\|f\|_{M^{p,q}} := \|V_g f\|_{L^{p,q}} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_g f(x, \xi)|^p \, dx\right)^{q/p} d\xi\right)^{1/q},$$

with obvious changes if $p = \infty$ or $q = \infty$. If $p = q$, we write M^p instead of $M^{p,p}$. The space $M^{p,q}(\mathbb{R}^d)$ is a Banach space, whose definition is independent of the choice of the window g. Different nonzero windows $g \in M^1$ yield equivalent norms on $M^{p,q}$. This property will be crucial in the sequel, because we will choose a suitable window g in estimates of the $M^{p,q}$-norm. Within the class of modulation spaces, one finds several standard function spaces, for instance $M^2 = L^2$, $M^1 = W(\mathcal{F}L^1, L^1)$ and, using weighted versions, one also finds certain Sobolev spaces and Shubin–Sobolev spaces [4, 14]. The relationship between modulation and Wiener amalgam spaces is expressed by the following result: *The Fourier transform establishes an isomorphism* $\mathcal{F} : M^{p,q} \to W(\mathcal{F}L^p, L^q)$. Consequently, convolution properties of modulation spaces can be translated into pointwise multiplication properties of Wiener amalgam spaces.

Let us now turn to the topic of the present paper. The importance of the dilation operator

$$f(x) \longmapsto f(\lambda x), \quad \lambda > 0,$$

in classical analysis is well known. For example, in most estimates arising in classical harmonic analysis (for example the Hölder, Young and Hausdorff–Young inequalities) as well as in partial differential equations (for example, Sobolev embeddings, Strichartz estimates) scaling arguments yield the constraints that the Lebesgue exponents must satisfy for the corresponding inequalities to hold.

When dealing with modulation or Wiener amalgam spaces, the situation becomes more subtle. In fact, the corresponding norms are not 'homogeneous' with respect to the scaling. Basically, this is due to the fact that, for example, in $W(L^p, L^q)$ the two spaces L^p, L^q display different scaling if $p \neq q$. Obtaining sharp estimates (in terms of λ) for the dilation operator norm, when acting on such spaces, is therefore a nontrivial problem. This study was carried out in depth in [17] (see also [5, 19]) in the case of modulation spaces $M^{p,q}$. The estimates obtained in [17] turned out to be a fundamental tool for embedding problems of modulation spaces into Besov spaces (see also [22]), and for boundedness of pseudodifferential operators of type (ρ, δ) on modulation spaces [18]. Arguments of this type allowed us to prove the sharpness of some Strichartz estimates in the Wiener amalgam spaces $W(\mathcal{F}L^p, L^q)$ in [6]. Finally, they were also used in [8] to prove sharp boundedness properties of Hörmander's type Fourier integral operators on $\mathcal{F}L^p$ and modulation spaces.

We point out that an investigation of the dilation operator on $W(C, L^1)$ (C being the space of continuous functions) had already appeared in [13].

The first result of this note (Proposition 2.2 below) provides *sharp* upper and lower bounds for the operator norm of the dilation operator on the Wiener amalgam spaces $W(L^p, L^q)$.

Contrary to what one might expect, it does not happen that the exponent p alone has its influence when $\lambda \to +\infty$, and the exponent q when $\lambda \to 0$.

Then, as for the classical function spaces, scaling arguments are employed to prove the sharpness of the known convolution, inclusion and pointwise multiplication relations for modulation spaces. This is precisely the topic studied in Section 3. In pursuit of this goal, we do not use the bounds obtained in [17], which would give constraints that are weaker than optimal. Instead, the sharp results come from explicit computations involving dilation properties of Gaussian functions.

Finally, we observe that these techniques can be applied to prove the sharpness of estimates arising in partial differential equations. As an example, in Section 4 we prove the optimality of an estimate for the Schrödinger propagator, recently obtained in [22] (see also [1]).

2. Dilation properties of Wiener amalgam spaces

In this section we study the dilation properties of Wiener amalgam spaces $W(L^p, L^q)$, $1 \le p, q \le \infty$. First, recall the following complex interpolation result [9].

LEMMA 2.1. *Let B_0, B_1, be local components of Wiener amalgam spaces, as in the Introduction. Then, for $1 \le q_0, q_1 \le \infty$, with $q_0 < \infty$ or $q_1 < \infty$, and $0 < \theta < 1$, we have*

$$[W(B_0, L^{q_0}), W(B_1, L^{q_1})]_{[\theta]} = W([B_0, B_1]_{[\theta]}, L^{q_\theta}),$$

with $1/q_\theta = (1-\theta)/q_0 + \theta/q_1$.

For $\lambda > 0$, we set $f_\lambda(x) = f(\lambda x)$.

PROPOSITION 2.2. *For $1 \le p, q \le \infty$,*

$$\|f_\lambda\|_{W(L^p,L^q)} \lesssim \lambda^{-d\max\{1/p,1/q\}} \|f\|_{W(L^p,L^q)}, \quad \forall\, 0 < \lambda \le 1, \tag{2.1}$$

and

$$\|f_\lambda\|_{W(L^p,L^q)} \lesssim \lambda^{-d\min\{1/p,1/q\}} \|f\|_{W(L^p,L^q)}, \quad \forall \lambda \ge 1. \tag{2.2}$$

Also,

$$\|f_\lambda\|_{W(L^p,L^q)} \gtrsim \lambda^{-d\min\{1/p,1/q\}} \|f\|_{W(L^p,L^q)}, \quad \forall\, 0 < \lambda \le 1, \tag{2.3}$$

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and

$$\|f_\lambda\|_{W(L^p,L^q)} \gtrsim \lambda^{-d\max\{1/p,1/q\}}\|f\|_{W(L^p,L^q)}, \quad \forall\lambda \geq 1. \tag{2.4}$$

We first prove the following weaker estimates.

LEMMA 2.3. *For* $1 \leq p, q \leq \infty$,

$$\|f_\lambda\|_{W(L^p,L^q)} \lesssim \lambda^{-d(1/p+1/q)}\|f\|_{W(L^p,L^q)}, \quad \forall\, 0 < \lambda \leq 1, \tag{2.5}$$

and

$$\|f_\lambda\|_{W(L^p,L^q)} \lesssim \lambda^{d(1-1/p-1/q)}\|f\|_{W(L^p,L^q)}, \quad \forall\lambda \geq 1. \tag{2.6}$$

PROOF. To compute the Wiener norm, we choose the window function $g = \chi_{[0,1]^d}$, the characteristic function of the box $[0, 1]^d$. Then

$$\begin{aligned}\|f_\lambda\|_{W(L^p,L^q)} &\asymp \|\|f(\lambda t)g(t-x)\|_{L^p_t}\|_{L^q_x}\\ &= \lambda^{-d/p}\|\|f(t)g_{1/\lambda}(t-\lambda x)\|_{L^p_t}\|_{L^q_x}\\ &= \lambda^{-d(1/p+1/q)}\|\|f(t)g_{1/\lambda}(t-x)\|_{L^p_t}\|_{L^q_x}.\end{aligned} \tag{2.7}$$

If $0 < \lambda \leq 1$, the window function g fulfills $g_{1/\lambda}(y) \leq g(y)$, and (2.5) follows.

Now let $\lambda \geq 1$. To prove (2.6), we observe that

$$g_{1/\lambda}(t) \leq \sum_{j\in\mathbb{Z}^d\cap[0,\lambda]^d} g(t-j).$$

Notice that the above sum contains $N_\lambda = O(\lambda^d)$ terms. Using this inequality we dominate the expression in (2.7) by

$$\lambda^{-d(1/p+1/q)} \sum_{j\in\mathbb{Z}^d\cap[0,\lambda]^d} \|\|f(t)g(t-x-j)\|_{L^p_t}\|_{L^q_x} = \lambda^{-d(1/p+1/q)}N_\lambda\|f\|_{W(L^p,L^q)},$$

where we have also performed a change of variable in the integral with respect to x. This concludes the proof. □

PROOF OF PROPOSITION 2.2. We first prove (2.1) and (2.2) when $p = \infty$. We see at once that (2.1) coincides with (2.5) when $p = \infty$. On the other hand, (2.2) for $p = \infty$ follows by complex interpolation (Lemma 2.1) from (2.6) with $(p, q) = (\infty, 1)$, that is,

$$\|f_\lambda\|_{W(L^\infty,L^1)} \lesssim \|f\|_{W(L^\infty,L^1)}$$

and the trivial estimate

$$\|f_\lambda\|_{W(L^\infty,L^\infty)} \asymp \|f_\lambda\|_{L^\infty} = \|f\|_{L^\infty} \asymp \|f\|_{W(L^\infty,L^\infty)}.$$

Since the estimates (2.1) and (2.2) also hold for $p = q$ (because $W(L^p, L^p) = L^p$ with equivalent norms), by interpolation with the case $p = \infty$, $1 \leq q \leq \infty$, we see that they hold for any pair (p, q), with $1 \leq q \leq p \leq \infty$.

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When $1 < p < q \leq \infty$ they follow by duality arguments as follows. Suppose, for example, that $\lambda \geq 1$. Then relation (2.1), applied to the pair (p', q'), yields

$$
\begin{aligned}
\|f_\lambda\|_{W(L^p,L^q)} &= \sup_{\|g\|_{W(L^{p'},L^{q'})}=1} |\langle f_\lambda, g\rangle| \\
&= \sup_{\|g\|_{W(L^{p'},L^{q'})}=1} \lambda^{-d}|\langle f, g_{1/\lambda}\rangle| \\
&\leq \lambda^{-d} \sup_{\|g\|_{W(L^{p'},L^{q'})}=1} \|f\|_{W(L^p,L^q)}\|g_{1/\lambda}\|_{W(L^{p'},L^{q'})} \\
&\lesssim \lambda^{-d}\lambda^{d\max\{1/p',1/q'\}}\|f\|_{W(L^p,L^q)},
\end{aligned}
$$

which is (2.2). The proof of (2.1) is similar in this case.

It remains to prove the estimates (2.1) and (2.2) for $p = 1$, $1 < q \leq \infty$. They follow from interpolation from the case $(p, q) = (1, 1)$ and the case $(p, q) = (1, \infty)$, where (2.1) and (2.2) coincide with (2.5) and (2.6) respectively.

Finally, (2.3) and (2.4) follow at once from (2.2) and (2.1), respectively, applied to the function $f_{1/\lambda}$. □

We now show that the result above is sharp.

PROPOSITION 2.4 (Sharpness of (2.1) and (2.2)).

(i) *Suppose that, for some* $\alpha \in \mathbb{R}$,

$$\|f_\lambda\|_{W(L^p,L^q)} \lesssim \lambda^\alpha \|f\|_{W(L^p,L^q)}, \quad \forall\, 0 < \lambda \leq 1. \tag{2.8}$$

Then

$$\alpha \leq -d \max\left\{\frac{1}{p}, \frac{1}{q}\right\}. \tag{2.9}$$

(ii) *Suppose that, for some* $\alpha \in \mathbb{R}$,

$$\|f_\lambda\|_{W(L^p,L^q)} \lesssim \lambda^\alpha \|f\|_{W(L^p,L^q)}, \quad \forall \lambda \geq 1. \tag{2.10}$$

Then

$$\alpha \geq -d \min\left\{\frac{1}{p}, \frac{1}{q}\right\}. \tag{2.11}$$

This also shows the sharpness of the estimates (2.3) and (2.4), since they are equivalent to (2.2) and (2.1), respectively.

PROOF. (i) First, consider the case $p \geq q$. We have $W(L^p, L^q) \hookrightarrow W(L^q, L^q) = L^q$. Hence

$$\lambda^{-d/q}\|f\|_{L^q} = \|f_\lambda\|_{L^q} \lesssim \|f_\lambda\|_{W(L^p,L^q)}.$$

Combining this estimate with (2.8) and letting $\lambda \to 0^+$, we obtain $\alpha \leq -d/q$.

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Now assume that $p<q$. It suffices to verify that, for every $\varepsilon>0$, there exists $f\in W(L^p,L^q)$ such that

$$\|f_\lambda\|_{W(L^p,L^q)}\geq C\lambda^{-d/p+\varepsilon}. \tag{2.12}$$

We study the case of dimension $d=1$. The general case follows by tensor products of functions of one variable. To this end, we choose

$$f(t)=\begin{cases}|t|^{-1/p+\varepsilon} & \text{for } |t|\leq 1,\\ 0 & \text{for } |t|>1.\end{cases}$$

Observe that $f\in W(L^p,L^1)\hookrightarrow W(L^p,L^q)$, for every $1\leq q\leq\infty$, and

$$f(\lambda t)=\lambda^{-1/p+\varepsilon}f(t)\quad \text{for } |t|\leq\frac{1}{\lambda}. \tag{2.13}$$

Now take $g=\chi_{B(0,1)}$ as window function. Of course,

$$\|f_\lambda\|_{W(L^p,L^q)}=\bigg(\int\|f_\lambda T_y g\|_{L^p}^q\,dy\bigg)^{1/q}\geq\bigg(\int_{B(0,1)}\|f_\lambda T_y g\|_{L^p}^q\,dy\bigg)^{1/q}.$$

By using (2.13) and the choice $g=\chi_{B(0,1)}$, for $\lambda\leq 1/2$ the last expression is estimated from below by

$$\geq\lambda^{-1/p+\varepsilon}\bigg(\int_{B(0,1)}\|fT_y g\|_{L^p}^q\,dy\bigg)^{1/q},$$

that is, (2.12).

(ii) Again, we first consider the case $p\geq q$. Then $W(L^p,L^q)\hookrightarrow W(L^p,L^p)=L^p$. Hence,

$$\|f_\lambda\|_{W(L^p,L^q)}\gtrsim\|f_\lambda\|_{L^p}=\lambda^{-d/p}\|f\|_{L^p}.$$

Combining this estimate with (2.10) and letting $\lambda\to+\infty$, we obtain $\alpha\geq -d/q$.

Now suppose that $p<q$. As before, it suffices to prove, in dimension $d=1$, that for every $\varepsilon>0$ there exists a function $f\in W(L^p,L^q)$ such that

$$\|f_\lambda\|_{W(L^p,L^q)}\geq C\lambda^{-1/q-\varepsilon}.$$

Therefore, choose

$$f(t)=\begin{cases}|t|^{-1/q-\varepsilon} & \text{for } |t|\geq 1,\\ 0 & \text{for } |t|<1.\end{cases}$$

Then $f\in W(L^\infty,L^q)\hookrightarrow W(L^p,L^q)$, for every $1\leq p\leq\infty$, and

$$f(\lambda t)=\lambda^{-1/q-\varepsilon}f(t)\quad \text{for } |t|\geq\frac{1}{\lambda}. \tag{2.14}$$

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Again choose $g = \chi_{B(0,1)}$ as window function. Then

$$\|f_\lambda\|_{W(L^p,L^q)} \geq \left(\int_{B(0,2)} \|f_\lambda T_y g\|_{L^p}^q \, dy\right)^{1/q}.$$

By using (2.14) and the choice $g = \chi_{B(0,1)}$, for $\lambda \geq 1$ the last expression is

$$\geq \lambda^{-1/q-\varepsilon} \left(\int_{B(0,2)} \|f T_y g\|_{L^p}^q \, dy\right)^{1/q},$$

which concludes the proof of (ii). □

3. Convolution, inclusion and multiplication relations for modulation spaces

In this section we study the optimality of the convolution, inclusion and pointwise multiplication relations for modulation spaces. We need some preliminary results.

If one chooses the Gaussian $e^{-\pi|x|^2}$ as window function to compute Wiener amalgam norms, then an easy computation (see, for example, [6, Lemma 5.3]) yields the result below.

LEMMA 3.1. *For $a, b \in \mathbb{R}$, $a > 0$, set $\mathcal{G}_{(a+ib)}(x) = (a+ib)^{-d/2} e^{-\pi|x|^2/(a+ib)}$. Then, for every $1 \leq p, q \leq \infty$,*

$$\|\mathcal{G}_{(a+ib)}\|_{W(\mathcal{F}L^p,L^q)} = \frac{((a+1)^2+b^2)^{d(1/p-1/2)/2}}{p^{d/2p}(aq)^{d/2q}(a(a+1)+b^2)^{d(1/p-1/q)/2}}. \tag{3.1}$$

For tempered distributions compactly supported either in time or in frequency, the $M^{p,q}$-norm is equivalent to the $\mathcal{F}L^q$-norm or L^p-norm, respectively. This result is well known [11, 12, 16]. For the sake of completeness we provide an outline of the proof.

LEMMA 3.2. *Let $1 \leq p, q \leq \infty$.*

(i) *For every $u \in \mathcal{S}'(\mathbb{R}^d)$, supported in a compact set $K \subset \mathbb{R}^d$, $u \in M^{p,q}$* if and only if *$u \in \mathcal{F}L^q$, and*

$$C_K^{-1}\|u\|_{M^{p,q}} \leq \|u\|_{\mathcal{F}L^q} \leq C_K\|u\|_{M^{p,q}}, \tag{3.2}$$

where $C_K > 0$ depends only on K.

(ii) *For every $u \in \mathcal{S}'(\mathbb{R}^d)$, whose Fourier transform is supported in a compact set $K \subset \mathbb{R}^d$, $u \in M^{p,q}$* if and only if *$u \in L^p$, and*

$$C_K^{-1}\|u\|_{M^{p,q}} \leq \|u\|_{L^p} \leq C_K\|u\|_{M^{p,q}}, \tag{3.3}$$

where $C_K > 0$ depends only on K.

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PROOF. (i) This is detailed in [16, Lemma 1].

(ii) It is well known (see, for example, [20]) that

$$\|u\|_{M^{p,q}} \asymp \left(\sum_{k\in\mathbb{Z}^d} \|\nu(D-k)u\|_{L^p}^q \right)^{1/q},$$

where ν is a test function satisfying $\sum_{k\in\mathbb{Z}^d} \nu(\xi-k)\equiv 1$. Now, if $\widehat{u}$ has compact support, the above sum is finite and one deduces at once the first estimate in (3.3), since the multipliers $\nu(D-k)$ are (uniformly) bounded on L^p. To obtain the second estimate in (3.3), we write $u=\sum_{k\in\mathbb{Z}^d} \nu(D-k)u$, then apply the triangle inequality and the finiteness of the sum over k again. □

We now turn our attention to the sharpness of the convolution properties for modulation spaces.

PROPOSITION 3.3. *Let* $1\le p,q,p_1,p_2,q_1,q_2\le\infty$. *Then*

$$\|f*g\|_{M^{p,q}} \lesssim \|f\|_{M^{p_1,q_1}}\|g\|_{M^{p_2,q_2}} \tag{3.4}$$

if and only if the following relations hold:

$$\frac{1}{p}+1\le\frac{1}{p_1}+\frac{1}{p_2}, \tag{3.5}$$

and

$$\frac{1}{q}\le\frac{1}{q_1}+\frac{1}{q_2}. \tag{3.6}$$

PROOF. *Sufficiency.* The inclusion relations (3.4) were proved in [4, 19]. There the relations (3.5) and (3.6) were shown with equality. The inequalities follow by the inclusion relations $M^{p_1,q_1}\hookrightarrow M^{p_2,q_2}$ for $p_1\le p_2$ and $q_1\le q_2$ [10, 14].

Necessity. We consider the family of Gaussians $\varphi^{(\lambda)}(x):=e^{-\pi\lambda|x|^2}$, for $\lambda>0$. Obviously, $\varphi^{(\lambda)}\in\mathcal{S}(\mathbb{R}^d)\subset M^{p,q}(\mathbb{R}^d)$, for every $1\le p,q\le\infty$. Since

$$\|f\|_{M^{p,q}}\asymp\|\widehat{f}\|_{W(\mathcal{F}L^p,L^q)} \quad \text{and} \quad \widehat{\varphi^{(\lambda)}}=\lambda^{-d/2}\varphi^{(1/\lambda)},$$

Lemma 3.1 yields

$$\begin{aligned}\|\varphi^{(\lambda)}\|_{M^{p,q}} &\asymp \lambda^{-d/2}\|\varphi^{(1/\lambda)}\|_{W(\mathcal{F}L^p,L^q)}\asymp\|\mathcal{G}_{(\lambda)}\|_{W(\mathcal{F}L^p,L^q)}\\ &\asymp \frac{(\lambda+1)^{d(1/p-1/2)}}{\lambda^{d/2q}(\lambda^2+\lambda)^{(1/p-1/q)d/2}}.\end{aligned} \tag{3.7}$$

A straightforward calculation shows that $(\varphi^{(\lambda)}*\varphi^{(\lambda)})(x)=(2\lambda)^{-d/2}\varphi^{(\lambda/2)}(x)$. Hence, using (3.7), we obtain

$$\|\varphi^{(\lambda)}*\varphi^{(\lambda)}\|_{M^{p,q}}\asymp\lambda^{-(1+1/p)d/2} \quad \text{for } \lambda\to 0^+. \tag{3.8}$$

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Using (3.7) again, we also obtain

$$\|\varphi^{(\lambda)}\|_{M^{p_i,q_i}} \asymp \lambda^{-d/2p_i}, \quad i=1,2, \text{ for } \lambda \to 0^+. \tag{3.9}$$

Substituting in (3.4), we obtain (3.5). The relation (3.6) can be obtained similarly. Indeed, the estimate (3.7) gives, for $\lambda \to +\infty$,

$$\|\varphi^{(\lambda)} * \varphi^{(\lambda)}\|_{M^{p,q}} \asymp \lambda^{-d(1-1/2q)}, \quad \|\varphi^{(\lambda)}\|_{M^{p_i,q_i}} \asymp \lambda^{-d(1-1/q_i)/2}, \quad i=1,2,$$

and, using (3.4) again, the relation (3.6) follows.

An alternative proof of the necessary conditions (3.5) and (3.6) is provided by Lemma 3.2. Specifically, to prove (3.6), consider two compactly supported smooth functions f, g and their scaling $f_\lambda(x)=f(\lambda x)$, $g_\lambda(x)=g(\lambda x)$, with $\lambda \geq 1$. Since $\lambda \geq 1$, f_λ and g_λ (and therefore $f_\lambda * g_\lambda$) are all supported in a compact subset K, independent of λ. By Lemma 3.2, (i), the bilinear estimate (3.4) for f_λ and g_λ becomes

$$\|f_\lambda * g_\lambda\|_{\mathcal{F}L^q} \lesssim \|f_\lambda\|_{\mathcal{F}L^{q_1}} \|g_\lambda\|_{\mathcal{F}L^{q_2}}.$$

Using $f_\lambda * g_\lambda = \lambda^{-d}(f*g)_\lambda$, the dilation property for $\mathcal{F}L^q$ spaces given by $\|h(\lambda\cdot)\|_{\mathcal{F}L^q} = \lambda^{-d/q'}\|h\|_{\mathcal{F}L^q}$, and letting $\lambda \to +\infty$, we obtain (3.6).

In order to prove (3.5), one argues similarly. Here the functions f, g have Fourier transforms $\widehat{f}, \widehat{g}$ compactly supported and the scale λ satisfies $0<\lambda\leq 1$. By Lemma 3.2, (ii), the estimate (3.4) becomes

$$\|f_\lambda * g_\lambda\|_{L^p} \lesssim \|f_\lambda\|_{L^{p_1}} \|g_\lambda\|_{L^{p_2}}.$$

Using $f_\lambda * g_\lambda = \lambda^{-d}(f*g)_\lambda$, the dilation property $\|h(\lambda\cdot)\|_{L^p} = \lambda^{-d/p}\|h\|_{L^p}$, and letting $\lambda \to 0^+$, we prove (3.5). □

The family of Gaussians $\varphi^{(\lambda)}$ provides an alternative proof for the sharpness of the inclusion relation for modulation spaces, already obtained by the inclusion relations for the sequence spaces $\ell^{p,q}$, via the norm equivalence $\|f\|_{M^{p,q}} \asymp \|\langle f, T_m M_n g\rangle\|_{\ell^{p,q}}$, with $\{T_m M_n g\}$ being a Gabor frame (see, for example, [14, Theorem 13.6.1]).

Proposition 3.4. *Let* $1 \leq p_1, p_2, q_1, q_2 \leq \infty$. *Then*

$$\|f\|_{M^{p_2,q_2}} \lesssim \|f\|_{M^{p_1,q_1}} \tag{3.10}$$

if and only if

$$p_1 \leq p_2 \quad \text{and} \quad q_1 \leq q_2. \tag{3.11}$$

Proof. We show the necessity of (3.11). Let $\varphi^{(\lambda)}(x)=e^{-\pi\lambda|x|^2}$, $\lambda>0$. From the proof of Proposition 3.3,

$$\|\varphi^{(\lambda)}\|_{M^{p_i,q_i}} \asymp \lambda^{-d/2p_i} \quad \text{for } \lambda \to 0^+ \quad \text{and} \quad \|\varphi^{(\lambda)}\|_{M^{p_i,q_i}} \asymp \lambda^{-d(1-1/q_i)/2} \quad \text{for } \lambda \to +\infty. \tag{3.12}$$

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Hence, for (3.10) to be satisfied it must be

$$\lambda^{-d/2p_2} \lesssim \lambda^{-d/2p_1} \quad \text{for } \lambda \to 0^+ \quad \text{and} \quad \lambda^{-d(1-1/q_2)/2} \lesssim \lambda^{-d(1-1/q_1)/2}$$
$$\text{for } \lambda \to +\infty,$$

that give the relations in (3.11). □

In what follows we study the pointwise multiplication operator in modulation spaces (which is equivalent to studing the convolution operator for the Wiener amalgam spaces $W(\mathcal{F}L^p, L^q)$).

PROPOSITION 3.5. *Let* $1 \le p, q, p_1, p_2, q_1, q_2 \le \infty$. *Then*

$$\|fg\|_{M^{p,q}} \lesssim \|f\|_{M^{p_1,q_1}} \|g\|_{M^{p_2,q_2}} \tag{3.13}$$

if and only if the following relations hold:

$$\frac{1}{p} \le \frac{1}{p_1} + \frac{1}{p_2}, \tag{3.14}$$

and

$$\frac{1}{q} + 1 \le \frac{1}{q_1} + \frac{1}{q_2}. \tag{3.15}$$

PROOF. The sufficiency can be found in [10] (see also [23]). For the necessity of the conditions (3.14) and (3.15) we test the estimate (3.13) on the Gaussians $f(x) = g(x) = \varphi^{(\lambda)}(x) = e^{-\lambda\pi|x|^2}$. We observe that $\varphi^{(\lambda)}\varphi^{(\lambda)} = \varphi^{(2\lambda)}$. Hence by applying (3.12) and substituting in (3.13), relation (3.15) follows by letting $\lambda \to 0^+$, while (3.14) follows by letting $\lambda \to +\infty$. □

4. An estimate for the Schrödinger propagator

Consider the Fourier multiplier $e^{it\Delta}$, with symbol $e^{-it|2\pi\xi|^2}$, that is,

$$(e^{it\Delta}u_0)(x) = \frac{1}{(4\pi i t)^{d/2}} \int e^{i(|x-y|^2)/4t} u_0(y)\, dy.$$

It is shown in [22, Proposition 4.1] that, given $2 \le p < \infty$, $1/p + 1/p' = 1$, $1 \le q \le \infty$,

$$\|e^{it\Delta}u_0\|_{M^{p,q}} \lesssim (1+|t|)^{-d(1/2-1/p)} \|u_0\|_{M^{p',q}}. \tag{4.1}$$

(Similar estimates were obtained in [1].) We now show that the condition $p \ge 2$ is necessary in (4.1), and the decay at infinity is optimal.

PROPOSITION 4.1 (Sharpness of (4.1)). *Suppose that, for some fixed* $t_0 \in \mathbb{R}$, $1 \le p$, $q \le \infty$, $C > 0$, *the following estimate holds:*

$$\|e^{it_0\Delta}u_0\|_{M^{p,q}} \le C\|u_0\|_{M^{p',q}}, \qquad \forall u_0 \in \mathcal{S}(\mathbb{R}^d). \tag{4.2}$$

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Then $p \geq 2$.

Assume now that, for some $\alpha \in \mathbb{R}$, $C > 0$, $M > 0$, $1 \leq \gamma, \delta \leq \infty$, $1 \leq p, q \leq \infty$, *the estimate*

$$\|e^{it\Delta} u_0\|_{M^{p,q}} \leq C t^{\alpha} \|u_0\|_{M^{\gamma,\delta}}, \quad \forall u_0 \in \mathcal{S}(\mathbb{R}^d), \tag{4.3}$$

holds for every $t > M$. *Then*

$$\alpha \geq -d\left(\frac{1}{2} - \frac{1}{p}\right). \tag{4.4}$$

PROOF. We consider the family of initial data $u_0(\lambda x) = e^{-\pi\lambda^2|x|^2}$, $\lambda > 0$. A direct computation shows that the corresponding solutions are

$$\begin{aligned} u(\lambda^2 t, \lambda x) &= (1 + 4\pi i t\lambda^2)^{-d/2} \exp\left[-\frac{\pi\lambda^2|x|^2}{1 + 4\pi i t\lambda^2}\right] \\ &= \lambda^{-d} \mathcal{G}_{(\lambda^{-2}+4\pi i t)}(x), \end{aligned} \tag{4.5}$$

where we have used the notation in Lemma 3.1. It follows from (3.1) that

$$\begin{aligned} \|u_0(\lambda \cdot)\|_{M^{p',q}} &\asymp \lambda^{-d} \|\widehat{u}_0(\lambda^{-1} \cdot)\|_{W(\mathcal{F}L^{p'},L^q)} \\ &= \|\mathcal{G}_{(\lambda^2)}\|_{W(\mathcal{F}L^{p'},L^q)} \asymp \lambda^{-d/p'}, \quad \text{as } \lambda \to 0^+. \end{aligned} \tag{4.6}$$

On the other hand, by (4.5),

$$\begin{aligned} \|u(\lambda^2 t, \lambda \cdot)\|_{M^{p,q}} &\asymp \|\mathcal{F}(u(\lambda^2 t, \lambda \cdot))\|_{W(\mathcal{F}L^p,L^q)} \\ &\asymp \lambda^{-d}(a^2 + b^2)^{d/4} \|\mathcal{G}_{(a+ib)}\|_{W(\mathcal{F}L^p,L^q)}, \end{aligned} \tag{4.7}$$

where

$$a = \frac{\lambda^{-2}}{\lambda^{-4} + (4\pi t)^2}, \quad b = -\frac{4\pi t}{\lambda^{-4} + (4\pi t)^2}.$$

Hence, for fixed $t = t_0$, (3.1) gives

$$\|u(\lambda^2 t_0, \lambda \cdot)\|_{M^{p,q}} \asymp \lambda^{-d/p} \quad \text{as } \lambda \to 0^+. \tag{4.8}$$

Estimates (4.6), (4.8) and (4.2) yield $-d/p \geq -d/p'$, namely $p \geq 2$.

Choosing $\lambda = 1$ in (4.7) and using (3.1), we obtain

$$\|u(t, \cdot)\|_{M^{p,q}} \asymp t^{-d(1/2-1/p)} \quad \text{as } t \to +\infty.$$

This shows that (4.4) is necessary for (4.3) to hold. □

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