BULL. AUSTRAL. MATH. SOC. 14B05, 14E15, 14H50, 14M25, 14Q05, 12D05,
VOL. 69 (2004) [403–414] 26C10

A FAST ALGORITHM FOR CURVE SINGULARITIES

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We demonstrate a fast algorithm to resolve local singularities of algebraic curves. The algorithm is based on the monomial transform and is independent of any other coordinate change. Two new invariants are introduced to gauge the singularities and sharply control the number of algorithmic steps. Our algorithm is applicable to both real and complex domains.

1. INTRODUCTION

Resolution of an algebraic curve's singularity is one of the incipient topics for algebraic geometry. There are two routine algorithms to resolve curve singularities in textbooks. Puiseux series ([**2, 10**]) is an algorithm conceived by Newton along with his Newton polygon whose geometric property fully characterises the singularity of its associated curve. This is a fast and powerful algorithm except that the fractional exponents of the series inevitably lead to excessiveness of multiple root branches and it is not adapted to algebraic variety of higher dimensions.

The birational transform ([**2, 4**]) is a modern method introduced by Noether [**6**] and is applicable to high-dimensional algebraic varieties. However the algorithm is slow and has no direct relationship to the curve's Newton polygon. As a result it is difficult to discern singularity attributes of the curve from the resolution process.

As a substitute for the birational transform, the monomial transform emerged in the 1970s ([**5**]) and was employed by Varchenko on oscillatory integrals [**1, 9**]. The first resolution algorithm based on it appeared in 1990s by Oka [**7**]. Nevertheless his algorithm relies on ambiguous coordinate changes, abstract algebraic attributes and contradiction by assuming infinite algorithmic steps, which are also the inconveniences of previous resolution algorithms based on birational transform.

Resorting to two decreasing singularity invariants, we rely on the power of monomial transform itself without additional coordinate change to eliminate ambiguity and enhance efficiency. In addition, computational implementation of the resolution algorithms ([**3, 8**]) constitutes another motivation for the author.

Received 4th November, 2003

All our polynomials are in two variables unless stated explicitly. Further, we are interested in local singularity of a curve at the origin of the (x, y)-plane. In part 2 the *perfect* Newton polygon consolidates the traditional one so that the induced monomial transforms are bijective regardless of the axes. The *partial resolution* and *partial reduction* in part 3 constitute a resolution step that factorises a polynomial and then reduces its singularity to a *branch point* of the origin. In part 4 and part 5 we introduce two singularity invariants, *singularity height* and *singularity index*, and prove that their alternate decreases terminate the algorithm in finite steps whose number can be sharply estimated by these invariants. We exemplify their usage in Example 4.4 and Example 5.4.

2. Perfect Newton polygon

A *positive quadrant* with vertex (a,b) is defined as $\{(x,y) \in \mathbb{R}^2 \mid x \geqslant a, y \geqslant b\}$. Given a polynomial, consider the union of the positive quadrants whose vertices correspond to the exponents of its monomials.

Definition 2.1: The *Newton polygon* of a polynomial is defined as the convex hull of the above union of positive quadrants.

The Newton polygon of a monomial $x^a y^b$ is simply the positive quadrant with vertex (a,b); while the Newton polygon of polynomial $x^3y + xy^3 - 2y^4$ is $\{(x,y) \in \mathbb{R}^2 \mid x \geqslant 0, y \geqslant 1, x + y \geqslant 4\}$.

The boundary of a Newton polygon always contains two noncompact faces that are either part of or parallel to the two axes. Denote a compact or noncompact face of a Newton polygon satisfying equation $mx + ny = p$ as $[mx + ny = p]$ with $m, n \in \mathbb{N} \cup \{0\}$ and $(m,n) = 1$ if $mn \neq 0$.

Suppose $\{(a,b)\} = [mx + ny = p] \cap [\widetilde{m}x + \widetilde{n}y = \widetilde{p}]$ is a vertex of the Newton polygon. Then we have the following lemma:

Lemma 2.2. *If*

$$\det \begin{pmatrix} m & \widetilde{m} \\ n & \widetilde{n} \end{pmatrix} > 1, \tag{2.1}$$

then there are a sequence of straight lines $r_j x + s_j y = q_{j\ (1 \leqslant j \leqslant J)}$ *passing through* (a,b) *such that for* $1 \leqslant j < J$,

$$\det \begin{pmatrix} m & r_1 \\ n & s_1 \end{pmatrix} = \det \begin{pmatrix} r_j & r_{j+1} \\ s_j & s_{j+1} \end{pmatrix} = \det \begin{pmatrix} r_J & \widetilde{m} \\ s_J & \widetilde{n} \end{pmatrix} = 1\,. \tag{2.2}$$

Proof: (2.1) implies that the integer vectors $\begin{pmatrix} m \\ n \end{pmatrix}$ and $\begin{pmatrix} \widetilde{m} \\ \widetilde{n} \end{pmatrix}$ are not a basis of the integer lattice in $\mathbb{R}^2$. Hence $\exists\ \lambda_1, \lambda_2 \in (0,1)$ and an integer vector $\begin{pmatrix} r \\ s \end{pmatrix}$ such that $\begin{pmatrix} r \\ s \end{pmatrix} = \lambda_1 \begin{pmatrix} m \\ n \end{pmatrix} + \lambda_2 \begin{pmatrix} \widetilde{m} \\ \widetilde{n} \end{pmatrix}$. We have:

$$\det \begin{pmatrix} m & r \\ n & s \end{pmatrix} = \lambda_2 \det \begin{pmatrix} m & \widetilde{m} \\ n & \widetilde{n} \end{pmatrix} < \det \begin{pmatrix} m & \widetilde{m} \\ n & \widetilde{n} \end{pmatrix},$$
$$\det \begin{pmatrix} r & \widetilde{m} \\ s & \widetilde{n} \end{pmatrix} = \lambda_1 \det \begin{pmatrix} m & \widetilde{m} \\ n & \widetilde{n} \end{pmatrix} < \det \begin{pmatrix} m & \widetilde{m} \\ n & \widetilde{n} \end{pmatrix}.$$

The conclusion of the lemma follows from a decreasing induction on the integer values of the determinants. ∎

We name the sequence of straight lines $r_j x + s_j y = q_j$ $(1 \leqslant j \leqslant J)$ satisfying (2.2) as a sequence of *auxiliary lines* at vertex (a, b).

DEFINITION 2.3: A Newton polygon with auxiliary lines added to each vertex is a *perfect Newton polygon.*

Lemma 2.2 indicates that we can always refine a Newton polygon into a perfect Newton polygon. Referring to each compact or noncompact face and each auxiliary line of a perfect Newton polygon simply as a face hereafter, we sort and enumerate all faces of a perfect Newton polygon in increasing order of their slopes and denote them as $L_\kappa = [m_\kappa x + n_\kappa y = p_\kappa]$ respectively $(1 \leqslant \kappa \leqslant \rho)$. Lemma 2.2 indicates that two adjacent faces L_κ and $L_{\kappa+1}$ of the perfect Newton polygon satisfy:

$$\det \begin{pmatrix} m_\kappa & m_{\kappa+1} \\ n_\kappa & n_{\kappa+1} \end{pmatrix} = 1 \tag{2.3}$$

for $1 \leqslant \kappa < \rho$. In particular we have $L_1 = [x = p_1]$ and $L_\rho = [y = p_\rho]$.

Consider the perfect Newton polygon of a generic polynomial $P(x,y)$ with $\rho > 2$. Let $\{(a_\kappa, b_\kappa)\} = L_\kappa \cap L_{\kappa+1}$ be one of its vertices. We reorganise $P(x,y)$ by separating the monomials of L_κ and $L_{\kappa+1}$ from other ones as follows:

$$P(x,y) = c_\kappa x^{a_\kappa} y^{b_\kappa} + \sum_{(\alpha,\beta)\in L_\kappa \cup L_{\kappa+1}} c_{\alpha\beta} x^\alpha y^\beta + \sum_{(\delta,\gamma)} c_{\delta\gamma} x^\delta y^\gamma \tag{2.4}$$

with $(\alpha, \beta) \neq (a_\kappa, b_\kappa)$ and $(\delta, \gamma) \notin L_\kappa \cup L_{\kappa+1}$.

It is apparent that $\forall (\alpha,\beta) \in L_\kappa$, $\exists\, l_{\alpha\beta} \in \mathbb{N} \cup \{0\}$ such that $\alpha = a_\kappa - n_\kappa l_{\alpha\beta}$ and $\beta = b_\kappa + m_\kappa l_{\alpha\beta}$. By (2.3) one instantly has:

$$m_{\kappa+1}\alpha + n_{\kappa+1}\beta - p_{\kappa+1} = l_{\alpha\beta}, \qquad (\alpha,\beta) \in L_\kappa. \tag{2.5}$$

Similarly for $\forall (\alpha,\beta) \in L_{\kappa+1}$, $\exists\, \bar{l}_{\alpha\beta} \in \mathbb{N} \cup \{0\}$, such that $\alpha = a_\kappa + n_{\kappa+1}\bar{l}_{\alpha\beta}$, $\beta = b_\kappa - m_{\kappa+1}\bar{l}_{\alpha\beta}$, and

$$m_\kappa \alpha + n_\kappa \beta - p_\kappa = \bar{l}_{\alpha\beta}, \qquad (\alpha,\beta) \in L_{\kappa+1}. \tag{2.6}$$

3. Partial Resolution and Partial Reduction

Based on two adjacent faces L_κ and $L_{\kappa+1}$ of a perfect Newton polygon as above, consider a monomial transform $\mathcal{T}_\kappa$ from the (X_κ, Y_κ)-plane to the (x,y)-plane:

$$(3.1) \qquad \mathcal{T}_\kappa : \begin{cases} x &= X_\kappa^{m_\kappa} Y_\kappa^{m_{\kappa+1}} \\ y &= X_\kappa^{n_\kappa} Y_\kappa^{n_{\kappa+1}} \end{cases}, \qquad \mathcal{T}_\kappa^{-1} : \begin{cases} X_\kappa &= x^{n_{\kappa+1}}/y^{m_{\kappa+1}} \\ Y_\kappa &= y^{m_\kappa}/x^{n_\kappa} \end{cases}.$$

To obtain $\mathcal{T}_\kappa^{-1}$, we assume $xy \neq 0$ and resort to (2.3). Thus if we exclude all the axes, $\mathcal{T}_\kappa$ is bijective.

Each $\mathcal{T}_\kappa$ factorises $P(x,y)$ in (2.4) into $P(x,y) = X_\kappa^{p_\kappa} Y_\kappa^{p_{\kappa+1}} P_1(X_\kappa, Y_\kappa)$ with:

$$(3.2) \quad P_1(X_\kappa, Y_\kappa) = c_\kappa + \sum_{(\alpha,\beta)\in L_\kappa} c_{\alpha\beta} Y_\kappa^{l_{\alpha\beta}} + \sum_{(\alpha,\beta)\in L_{\kappa+1}} c_{\alpha\beta} X_\kappa^{\bar{l}_{\alpha\beta}} + \sum_{(\delta,\gamma)} c_{\delta\gamma} X_\kappa^{m_\kappa\delta+n_\kappa\gamma-p_\kappa} Y_\kappa^{m_{\kappa+1}\delta+n_{\kappa+1}\gamma-p_{\kappa+1}},$$

here $l_{\alpha\beta}$ and $\bar{l}_{\alpha\beta}$ are as in (2.5) and (2.6) respectively. Due to the convexity of Newton polygon, $m_\kappa\delta + n_\kappa\gamma - p_\kappa, m_{\kappa+1}\delta + n_{\kappa+1}\gamma - p_{\kappa+1} > 0$ since $(\delta,\gamma) \notin L_\kappa \cup L_{\kappa+1}$.

Definition 3.1: Define $P_1(X_\kappa, Y_\kappa)$ as a *partial transform* of $P(x,y)$ and the above factorisation via a monomial transform as a *partial resolution* of singularity.

Each pair of adjacent faces of a perfect Newton polygon induces a monomial transform and thus a partial resolution.

Definition 3.2: A *proper polynomial* is a univariate polynomial that consists of all the univariate terms in a partial transform including the constant term.

In the partial transform (3.2), the proper polynomial in Y is factorised as:

$$(3.3) \qquad c + \sum_{(\alpha,\beta)\in L_\kappa} c_{\alpha\beta} Y^{l_{\alpha\beta}} = Q(Y) \prod_j (Y - r_j)^{h_j},$$

where for simplicity we disposed of the subscript κ. $Q(Y)$ is a nonzero coefficient on the complex domain, or a product of indecomposable quadratic polynomials and a nonzero coefficient on the real domain; r_j is a root of the proper polynomial. The proper polynomial in X is $c + \sum_{(\alpha,\beta)\in L_{\kappa+1}} c_{\alpha\beta} X^{\bar{l}_{\alpha\beta}} = \widetilde{Q}(X) \prod_i (X - s_i)^{h'_i}$.

Hereafter by an *integral point* we refer to a point corresponding to a monomial in the polynomial $P(x,y)$. Denote the number of integral points on L_κ as N_κ that is also the number of terms in the proper polynomial (3.3). Then

$$(3.4) \qquad h_j + 1 \leqslant N_\kappa.$$

DEFINITION 3.3: Each root r_j of the proper polynomial in Y corresponds to a point $(0, r_j)$ on the (X,Y)-plane at which the partial transform is singular. Define $(0, r_j)$ as a *branch point* of the origin $(0,0)$ on the (x,y)-plane. The same for $(s_i, 0)$.

To address the singularity of the partial transform $P_1(X,Y)$ in (3.2) at the branch point $(0, r_j)$, we substitute the factorisation (3.3) into $P_1(X,Y)$ and transform it into $\widetilde{P}_1(X, Y - r_j) = P_1(X, (Y - r_j) + r_j)$.

DEFINITION 3.4: Define $\widetilde{P}_1(X, Y - r_j)$ as a *reduced transform* at branch point $(0, r_j)$ and its derivation as a *partial reduction* of singularity to branch point $(0, r_j)$.

We build the perfect Newton polygon of $\widetilde{P}_1(X, Y - r_j)$ at the origin $(0,0)$ of the $(X, Y - r_j)$-plane in the same way as for $P(x,y)$ in part 2, based on which the partial resolutions in part 3 can be repeated. We still parameterise these partial resolutions by the same subscript κ.

4. SINGULARITY HEIGHT

With partial resolution and partial reduction repeated at each new branch point, the branch points form a tree whose root is the origin of the (x,y)-plane. The subscript κ as in part 3 parameterises different tree branches that are on the same level and from a common branch point. The difference between this tree and a regular one is that each value of κ further branches out to different branch points.

Following the above procedure we choose a branch on each level of the tree to obtain a path from its root to one of its last branch points. Henceforth we use a subscript t to parameterise different levels of the tree on a path. And the composition of monomial transforms on a path factorises the polynomial $P(x,y)$ into a product $\prod_{i=1}^{t} X_i^{p_i} Y_i^{p_i'} P_t(X_t, Y_t)$ with P_t bearing the same form as P_1 in (3.2). We prove later that for any path in the tree, $\exists N \in \mathbb{N}$ such that $P_N(X,Y) = [Y - r(X)]^h E(X, Y - r)$ with $E(X, Y - r)$ nonsingular satisfying $E(0,0) \neq 0$. $r(X)$ is either a convergent series with $r(0) \neq 0$ or a constant $r \neq 0$. $h \geqslant 0$.

The procedure of moving from one branch point to the next on a path is defined as a *resolution step* that includes a partial resolution and a partial reduction.

We disregard the roots of a proper polynomial in Y that corresponds to the first face L_1 of the Newton polygon with $(\alpha, \beta) \in L_1$ in (3.3). In fact, (3.1) would imply that $\mathcal{T}_1(0, r_j) = (0, r_j) \neq (0,0)$. Similarly we disregard the proper polynomial in X corresponding to the last face L_ρ of the Newton polygon.

A partial transform is *nonsingular* if its proper polynomials either have no roots as $Q(Y)$ in (3.3) or can be disregarded as above.

DEFINITION 4.1: Define the *singularity height* of a branch point as the multiplicity of the corresponding root of proper polynomial. It is also an attribute of the reduced transform at the branch point.

DEFINITION 4.2: A *degenerate* transform is defined as a reduced transform with a single compact face of perfect power whose exponent equals the singularity height.

The paradigm of a degenerate transform with $n, h \in \mathbb{N}$ and $\delta, \gamma \in \mathbb{N} \cup \{0\}$ is:

$$(4.1) \qquad (Y - r - r_1 X^n)^h + \sum_{\delta + n\gamma > nh} c_{\delta\gamma} X^\delta (Y - r)^\gamma .$$

We obtain another paradigm of a degenerate transform by exchanging Y and X and substituting s for r in (4.1).

If a reduced transform has singularity height h, then $(0, h)$ is the initial vertex of its Newton polygon and each compact face with N integral points satisfies:

$$(4.2) \qquad N \leqslant h + 1 .$$

Similar to (3.4), a partial resolution to the above reduced transform involving the above compact face with N integral points induces an inequality:

$$(4.3) \qquad h' + 1 \leqslant N ,$$

where h' represents the singularity height of each branch point derived from the ensuing partial transform.

(4.2) and (4.3) imply that the singularity height is decreasing: $h' \leqslant h$.

LEMMA 4.3. *If a reduced transform is neither degenerate nor nonsingular, then the singularity height strictly decreases after a resolution step. After finite resolution steps we shall obtain a transform that is either degenerate or nonsingular.*

PROOF: A reduced transform with singularity height $h = 1$ is degenerate. Hence to prove the lemma it suffices to prove the first conclusion.

For a reduced transform of singularity height h whose Newton polygon has either multiple compact faces or a single compact face $L = [mX + n(Y - r) = p]$ with $m > 1$, (4.2) is a strict inequality and hence the singularity height strictly decreases.

When $m = 1$, the single compact face $L = [X + n(Y - r) = nh]$ is a perfect power if and only if the proper polynomial of its partial transform is a perfect power under the monomial transform $X = X_1$ and $Y - r = X_1^n Y_1$. And the proper polynomial is a perfect power if and only if (3.4) or (4.3) is an equality. ∎

EXAMPLE 4.4. The perfect Newton polygon of $P(x, y) = xy^3 - 2x^2y^2 + x^3y - x^6$ has five faces enumerated in increasing order of their slopes: two noncompact faces $L_1 = [x = 1]$

https://doi.org/10.1017/S0004972700036182 Published online by Cambridge University Press

and $L_5 = [y = 0]$, two compact faces $L_2 = [x + y = 4]$ and $L_4 = [x + 3y = 6]$, and one auxiliary line $L_3 = [x + 2y = 5]$.

Each two adjacent pair L_κ and $L_{\kappa+1}$ induces a monomial transform $\mathcal{T}_\kappa$ as in (3.1) with $1 \leqslant \kappa \leqslant 4$. Consider $\mathcal{T}_2$ with $x = XY$ and $y = XY^2$, under which $P(x, y) = X^4Y^5P_1(X, Y)$ and the partial transform $P_1 = Y^2 - 2Y + 1 - X^2Y$. The proper polynomial $Y^2 - 2Y + 1 = (Y - 1)^2$ leads to a branch point $(0, 1)$ at which the reduced transform $\widetilde{P}_1(X, Y - 1) = (Y - 1)^2 - X^2 - X^2(Y - 1)$.

The Newton polygon of $\widetilde{P}_1$ is perfect with singularity height $h = 2$. It has three faces $L_1 = [x = 0]$, $L_2 = [x + y = 2]$ and $L_3 = [y = 0]$. Pick a partial resolution $X = X_1Y_1$ and $Y - 1 = Y_1$ based on L_1 and L_2. Then $\widetilde{P}_1(X, Y - 1) = -Y_1^2P_2(X_1, Y_1)$ with $P_2 = X_1^2 - 1 + X_1^2Y_1$. A partial reduction at branch point $(1, 0)$ yields a reduced transform $\widetilde{P}_2(X_1 - 1, Y_1) = 2(X_1 - 1) + Y_1 + 2(X_1 - 1)Y_1 + (X_1 - 1)^2(1 + Y_1)$ whose singularity height equals $h = 1$. $\widetilde{P}_2$ is a degenerate transform.

After another partial resolution $X_1 - 1 = X_2Y_2$ and $Y_1 = Y_2$, $\widetilde{P}_2(X_1 - 1, Y_1) = 2Y_2P_3(X_2, Y_2)$ with $P_3 = X_2 + 1/2 + X_2Y_2 + X_2^2Y_2(1 + Y_2)/2$. Invoking Weierstrass Preparation Theorem, $P_3 = \left(X_2 + 1/2 + \sum_{\alpha\in\mathbb{N}} c_\alpha Y_2^\alpha\right)E(X_2 + 1/2, Y_2)$ with $E(0, 0) \neq 0$.

5. Singularity Index

Consider the degenerate transform (4.1). After a partial resolution $X = X_1$ and $Y - r = X_1^nY_1$, its partial transform

$$(Y_1 - r_1)^h + \sum_{\delta+n\gamma>nh} c_{\delta\gamma}X_1^{\delta+n\gamma-nh}Y_1^\gamma \tag{5.1}$$

has a proper polynomial of perfect power.

When $h = 1$, the exponents $\delta + n\gamma - nh > 0$ imply the factorisation of (5.1) as:

$$\left(Y_1 - r_1 + \sum_{\alpha\in\mathbb{N}} c_\alpha X_1^\alpha\right)E(X_1, Y_1 - r_1) \tag{5.2}$$

according to Weierstrass Preparation Theorem. Here $E(0, 0) \neq 0$ and the series $\sum_{\alpha\in\mathbb{N}} c_\alpha X_1^\alpha$ is convergent in a nonempty convergence interval. This also explains the last partial resolution of the degenerate transform $\widetilde{P}_2$ in Example 4.4.

Suppose the singularity height $h > 1$ hereafter. We organise the partial transform (5.1) into a polynomial in $Y_1 - r_1$:

$$(Y_1 - r_1)^h[1 + R_0(X_1, Y_1)] + \sum_{j=1}^{h}(Y_1 - r_1)^{h-j}R_j(X_1, Y_1) \tag{5.3}$$

such that each monomial pair in $R_{j\,(1\leqslant j\leqslant h)}$ has no factor $Y_1 - r_1$ and we say that R_j is *clean* of factor $Y_1 - r_1$.

From $\delta + n\gamma - nh > 0$ in (5.1), it is easily seen that $X_1 \mid R_j$, that is, $R_j(0,Y_1) = 0_{\,(0\leqslant j\leqslant h)}$. Thus we have the following definition:

DEFINITION 5.1: Define $R_{j\,(0\leqslant j\leqslant h)}$ as a *remnant* and the *index* of $R_j \neq 0_{\,(1\leqslant j\leqslant h)}$ as $d_j \equiv \min\{(\alpha/j) \mid cX_1^\alpha Y_1^\beta \in R_j; \alpha \in \mathbb{N}, \beta \in \{0\} \cup \mathbb{N}\} \in \mathbb{Q}$; $d_j \equiv \infty$ when $R_j = 0$. We say that the partial transform (5.3) is *ready* if $\exists\, i \neq j_{\,(1\leqslant i,j\leqslant h)}$ such that $d_i \neq d_j$.

If (5.3) is ready, it is easily seen that the singularity height strictly decreases after another resolution step. If (5.3) is not ready, let $d_j = n \in \mathbb{N}_{\,(1\leqslant j\leqslant h)}$. We complete the perfect power of $(Y_1 - r_1)^h$ to obtain:

$$(5.4)\qquad \left(Y_1 - r_1 - r_2X_1^nY_1^\beta\right)^h + \sum_{j=0}^{h}(Y_1 - r_1)^{h-j}R'_j(X_1,Y_1)$$

with $R'_0 = R_0$ and it is evident that $X_1 \mid R'_{j\,(1\leqslant j\leqslant h)}$.

Now let $\beta = 1$ in (5.4) with discussion of other cases postponed.

Expand each factor $(Y_1 - r_1)^k{}_{\,(1\leqslant k\leqslant h)}$ in (5.4) as $\left[(Y_1 - r_1 - r_2X_1^nY_1) + r_2X_1^nY_1\right]^k$ so as to have a polynomial in $Y_1 - r_1 - r_2X_1^nY_1$:

$$(5.5)\qquad \sum_{j=0}^{h}(Y_1 - r_1 - r_2X_1^nY_1)^{h-j}\widetilde{R}_j(X_1,Y_1)$$

such that $\widetilde{R}_{j\,(1\leqslant j\leqslant h)}$ is clean of factor $Y_1 - r_1$ and $\widetilde{R}_0(0,Y_1) = 1$. The factor $Y_1 - r_1 - r_2X_1^nY_1$ will be referred to as a *series factor* henceforth.

A possible simplification of the above expansion occurs when $\exists\,\alpha_{\,(1\leqslant\alpha\leqslant h-j)}$ such that $(1 - r_2X_1^n)^\alpha \mid R'_{j\,(1\leqslant j<h)}$ in (5.4). In this case we begin our expansion with the identity $(Y_1 - r_1)^\alpha(1 - r_2X_1^n)^\alpha = \left[(Y_1 - r_1 - r_2X_1^nY_1) + r_1r_2X_1^n\right]^\alpha$.

Notice from the above expansion that $X_1 \mid \widetilde{R}_{j\,(1\leqslant j\leqslant h)}$. Thus Definition 5.1 still applies to (5.5). Momentarily suppose the partial transform (5.5) is ready.

DEFINITION 5.2: With d_j being the index of the remnant $\widetilde{R}_{j(1\leqslant j\leqslant h)}$, define $d \equiv \min_{1\leqslant j\leqslant h}\{d_j\}$ as the *singularity index* of the partial transform (5.5). If $d_j = d$, define the monomial in $\widetilde{R}_j$ with exponent dj in X_1 as an *index term*.

If $d \leqslant n$, then the reduced transform of (5.5) is not a degenerate transform and thus the singularity height strictly decreases after another resolution step. In fact, we can write (5.5) in a recursive form P_h defined as follows:

$$(5.6)\qquad P_j = (Y_1 - r_1 - r_2X_1^nY_1)P_{j-1} + \widetilde{R}_{j\,(1\leqslant j\leqslant h)}, \qquad P_0 = \widetilde{R}_0(X_1,Y_1).$$

When $d_j = d \leqslant n$, the single compact face of the reduced transform of P_j is determined by $(Y_1 - r_1 - r_2 X_1^n Y_1)^j + \widetilde{R}_j$ and is not a perfect power. Thus the reduced transform of P_j is not degenerate.

The following discussions are under the assumption that $d > n$.

After another t resolution steps with monomial transforms $X_s = X_{s+1}$ and $Y_s - r_1 r_2^{s-1} = X_{s+1}^n Y_{s+1}\ {}_{(1 \leqslant s \leqslant t)}$, we define the *index* of a generic monomial $c(Y_{t+1} - r_1 r_2^t)^\gamma X_{t+1}^\alpha Y_{t+1}^\beta$ in the ensuing partial transform as $\alpha + n\gamma$. It is easily seen that along with resolution steps the series factor has constant index n. The definition can be explained by the fact that the index of $Y_s - r_1 r_2^{s-1} = X_{s+1}^n Y_{s+1}$ is exactly the exponent of $X_{s+1\ (1 \leqslant s \leqslant t)}$ in the next resolution step. Hence a natural cancellation rule is that two monomials should not cancel with each other if they have different indices. Geometrically the index $\alpha + n\gamma$ of the monomial $c(Y_{t+1} - r_1 r_2^t)^\gamma X_{t+1}^\alpha$ in the reduced transform indicates that the monomial lies on the line $x + ny = \alpha + n\gamma$ parallel to the single compact face of the Newton polygon.

PROPOSITION 5.3. *Along with resolution steps, the singularity index d of a partial transform* (5.5) *strictly decreases to the index n of its series factor such that $d \leqslant n$.*

PROOF: Suppose $cX_1^{dj} Y_1^\beta \in \widetilde{R}_{j\ (1 \leqslant j \leqslant h)}$ is an index term. During partial reduction to branch point $(0, r_1)$, it generates a univariate monomial $cX_1^{dj} r_1^\beta$ as the new index term that does not cancel with any other monomial in the reduced transform of $\widetilde{R}_j$ since $\widetilde{R}_j$ is clean of factor $Y_1 - r_1$.

For $1 \leqslant k < j \leqslant h$, the new index term $cX_1^{dj} r_1^\beta$ cannot be nullified by $-cX_1^{dj} r_1^\beta$ in the reduced transform of $(Y_1 - r_1 - r_2 X_1^n Y_1)^{j-k} \widetilde{R}_k$. The reason is that the two monomials will have different indices after another resolution step since the series factor has constant index n along with resolution steps. Hence according to the above cancellation rule the new index term should not be nullified.

For $1 \leqslant j < k \leqslant h$, $cX_1^{dj} r_1^\beta (Y_1 - r_1)^{k-j}$ cannot be nullified by monomials in the reduced transform of $\widetilde{R}_k$. Otherwise the reduced transform of $\widetilde{R}_k$ would have index at most $\big(dj + n(k-j)\big)/k < d$, contradicting the singularity index being d.

After the monomial transform $X_1 = X_2$ and $Y_1 - r_1 = X_2^n Y_2$, a monomial $c' X_1^\alpha (Y_1 - r_1)^\gamma$ in the reduced transform of $\widetilde{R}_j$ is transformed to $c' X_2^{\alpha + n\gamma - nj} Y_2^\gamma$ with a decrease of $n(j - \gamma)$ in the exponent of $X_1 = X_2$; whereas the index term $cX_1^{dj} r_1^\beta$ is transformed to $cX_2^{(d-n)j} r_1^\beta$ with a maximum decrease of $nj \geqslant n(j - \gamma)$ in the exponent. Hence there is no cancellation for $cX_2^{(d-n)j} r_1^\beta$ after the above monomial transform. Further, the maximum decrease nj itself is proportional to j, which ensures that $cX_2^{(d-n)j} r_1^\beta$ is still an index term.

We conclude that a resolution step strictly reduces singularity index from d to

$d-n$. For $\forall x \in \mathbb{Q}$, denote $\lceil x \rceil \equiv \{m \in \mathbb{Z} \mid 0 \leqslant m-x < 1\}$. After $\lceil d/n \rceil - 1$ resolution steps the singularity index, denoted as $\widetilde{d}$, satisfies $\widetilde{d} \leqslant n$. ∎

When $d \leqslant n$, with essentially the same argument as (5.6), we can show that the singularity height strictly decreases after another resolution step.

If the partial transform (5.5) is not ready, we continue to complete perfect power like (5.4) and expand like (5.5) to either make it ready, or proceed to obtain a perfect power in the end with a generic form:

$$\Big(Y_1 - r_1 + \sum_{\alpha \geqslant n} c_{\alpha\beta} X_1^\alpha Y_1^\beta\Big)^h \widehat{R}_0, \tag{5.7}$$

where $\widehat{R}_0(0, Y_1) = 1$ and $\beta \in \mathbb{N} \cup \{0\}$.

We can also regard the polynomial inside the perfect power of (5.7) as a generic form of the series factor in (5.5) that accounts for the cases $\beta = 0$ or $\beta > 1$ in (5.4). It is not difficult to verify that this generic form makes no essential difference from the above discussions on (5.5).

EXAMPLE 5.4. Consider a degenerate transform $(Y-1-X)^2 + X(Y-1-X)[2(Y-1) + X^3(1+X)] + X^2(Y-1)^2$ with singularity height $h = 2$ at branch point $(0,1)$.

After the partial resolution $X = X_1$ and $Y - 1 = X_1 Y_1$, we organise the ensuing partial transform into a polynomial in $Y_1 - 1$ to obtain $(Y_1 - 1)^2 + (Y_1 - 1) R_1(X_1, Y_1) + R_2(X_1, Y_1)$ with $R_1 = 2X_1Y_1 + X_1^3(1+X_1)$ and $R_2 = X_1^2 Y_1^2$. The indices of $R_{j\ (j=1,2)}$ are $d_1 = d_2 = 1$ and the partial transform is not ready.

By completing perfect square we obtain $(Y_1 - 1 + X_1Y_1)^2 + (Y_1 - 1)X_1^3(1+X_1) = (Y_1 - 1 + X_1Y_1)^2 + (Y_1 - 1 + X_1Y_1)\widetilde{R}_1 + \widetilde{R}_2$ with the series factor $Y_1 - 1 + X_1Y_1$ and the remnants $\widetilde{R}_1 = X_1^3$ and $\widetilde{R}_2 = -X_1^4$. Here we used the identity $(Y_1 - 1)(1 + X_1) = (Y_1 - 1 + X_1Y_1) - X_1$. The remnants have indices $d_1 = 3$ and $d_2 = 2$. The singularity index is $d = d_2 = 2$ with the index term being $-X_1^4$, which should not cancel with the X_1^4 in the reduced transform of $(Y_1 - 1 + X_1Y_1)\widetilde{R}_1$.

The partial resolution with monomial transform $X_1 = X_2$ and $Y_1 - 1 = X_2Y_2$ yields the partial transform $(Y_2 + 1 + X_2Y_2)^2 + (Y_2 + 1 + X_2Y_2)\widehat{R}_1 + \widehat{R}_2$ with remnants $\widehat{R}_1 = X_2^2$ and $\widehat{R}_2 = -X_2^2$. The singularity index is reduced to $d = d_2 = 1$ that is equal to the index of the series factor. Thus its reduced transform $(Y_2+1)^2 - 2X_2(Y_2+1) - X_2^3 + X_2(Y_2+1)[2(Y_2+1) - X_2 + X_2(Y_2 + 1 + X_2)]$ at branch point $(0,-1)$ has two compact faces $L_2 = [x + y = 2]$ and $L_3 = [x + 2y = 3]$ and is not degenerate. Its singularity height equals $h = 2$.

By Lemma 4.3, the singularity height strictly decreases to $h = 1$ after another resolution step. Consider, for example, the monomial transform $X_2 = X_3Y_3$ and Y_2

$+1 = X_3Y_3^2$ based on L_2 and L_3. In this case we have partial transform $Y_3 - 2 - X_3 + \sum_{\delta>1,\gamma>1} X_3^\delta Y_3^\gamma$ and Weierstrass Preparation Theorem can be applied as in (5.2).

6. Conclusion

Theorem 6.1. *After finite resolution steps whose number can be sharply estimated by singularity height and singularity index, the partial transform of a polynomial becomes either a nonsingular transform or perfect power of a partial transform whose proper polynomial is linear as in* (5.7).

It is easily seen that the theorem follows from an alternate decreasing induction on singularity height and singularity index. The conclusion also holds for birational transforms since a monomial transform is a composition of birational transforms.

The partial transform inside the perfect power in (5.7) can be factorised as (5.2).

We conclude by posing the obvious question of generalising this algorithm to higher dimensions, which might elucidate Hironaka's resolution theorem. As for the invariants introduced here, the singularity height and the singularity index, their counterparts in higher dimensions and positive characteristic fields and connections to the classical ones like Puiseux pair and Milnor number will be interesting.

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