Mathematical Notes.

A Review of Elementary Mathematics and Science.

PUBLISHED BY

THE EDINBURGH MATHEMATICAL SOCIETY.

EDITED BY P. PINKERTON, M.A., D.SC.

No. 2. **July 1909.**

Notes on the Logic of Equation-Solving.—Consider an equation with one unknown, x. A "solution" or "root" of such an equation is a value which, being substituted for x in the equation, reduces it to an identity.

If the equation be rational and integral with respect to x, and of the n^{th} degree, then there are in general n solutions (some or all of which may be imaginary), corresponding to the n factors of the first degree into which $f(x)$ may be resolved, when the equation is put in the form $f(x)=0$.

But if the equation is not rational and integral, it may have no finite solution, or no solution at all: for example, the equation $\dfrac{1}{x}=0$ has no finite solution, and $\sqrt{x}=-1$ has no solution. (Here $\sqrt{x}$ is as usual understood to denote the positive value of the square root of x, so that the equation is manifestly self-contradictory).

The usual process of "solving" an equation begins with the *assumption* that there exists a solution, but this assumption is sometimes groundless.

The logical nature of the process of solution may be thus explained. We start with an equation, which we will call (1), involving a symbol x, and we deduce by a series of steps the conclusion $x=a$ or b or etc. This conclusion we may call (2). What we have *proved* is this—"Assuming that (1) has a 'root,' that root must be either a, b, or etc." We have proved in fact that no other value than these can be a root of the equation, so that when we have tested by substitution of these values for x in (1), whether any or all of them are roots, the investigation is complete. But we cannot be sure that a single root of (1) exists, unless we *either* verify by substitution *or* scrutinize the steps by which we deduced the conclusion (2) from the previous (1), to see whether the converse can also be asserted, *i.e.*, whether (1) is deducible from (2).

When one of the values a, b, c... in (2) does not satisfy (1), it is often called an *extraneous* solution. Such extraneous solutions can arise by multiplication of both sides of the equation by a factor involving x (which happens commonly in the process of "clearing of fractions" if inadvertently we multiply by something higher than the lowest common denominator of the terms). An extraneous solution may also arise in the process of squaring both sides of an equation. This is equivalent to multiplication by a factor. Thus, if $x=a$ be an equation and we deduce $x^2=a^2$, we have really deduced $x^2-a^2=0$ or $(x+a)(x-a)=0$ from the equation $x-a=0$, and this step is clearly *not* reversible, whichever way we look at it.

Take this example:

$$\sqrt{2x+1}+\sqrt{x}+5=0 \qquad - \quad - \quad - \quad - \qquad (1)$$

In this case we can see at once that there is no solution, since the sum of 3 positive quantities cannot $=0$. But proceed:

$$\sqrt{2x+1}=-5-\sqrt{x} \qquad - \quad - \quad - \quad - \qquad (2)$$

Squaring both sides

$$2x+1=25+10\sqrt{x}+x$$

$$\therefore \quad x-24=10\sqrt{x} \qquad - \quad - \quad - \quad - \quad - \qquad (3)$$

Squaring both sides,

$$x^2-48x+576=100x$$

$$\therefore \quad (x-144)(x-4)=0 \qquad - \quad - \quad - \quad - \qquad (4)$$

$$\therefore \quad x=144 \text{ or } 4 \qquad - \quad - \quad - \quad - \qquad (5)$$

Neither 4 nor 144 satisfies (1).

What we have proved is that if any value of x satisfies (1) it must either be 4 or 144. As neither of these does satisfy (1), the equation has no solution.

By retracing our steps we see that (5) involves (4), but (4) does not involve (3), though (3) is satisfied by 144. Also (3) does not involve (2), which is satisfied by neither 4 nor 144.

We may vary the method of solution thus:

$$(3) \text{ may be written } x-10\sqrt{x}-24=0 \qquad - \quad - \qquad (3')$$

$$\therefore \quad (\sqrt{x}-12)(\sqrt{x}+2) \qquad - \quad - \quad - \qquad (4')$$

$$\therefore \quad \sqrt{x}=12 \text{ or } -2 \qquad - \quad - \quad - \quad - \qquad (5')$$

$$\therefore \quad x=144 \text{ or } 4 \qquad - \quad - \quad - \quad - \qquad (6')$$

Here it is clear at stage (5′) that $\sqrt{x}=-2$ is no solution, so that we might reject it at once before proceeding to (6′). But rejection of a solution at an intermediate stage must be done warily, as the following example will show:

$$x+4+\sqrt{\frac{x+4}{x-4}}=\frac{12}{x-4} \tag{1}$$

Multiply both sides by $x-4$.

$$x^2-16+\sqrt{x^2-16}=12 \tag{2}$$

$$\text{or,}\quad y^2+y-12=0 \text{ where } y\equiv\sqrt{x^2-16}$$

$$\therefore\quad (y+4)(y-3)=0 \tag{3}$$

$$\therefore\quad \sqrt{x^2-16}=3 \text{ or } -4, \tag{4}$$

Reject $\sqrt{x^2-16}=-4$, which is a self contradictory equation.

$$\therefore\quad \sqrt{x^2-16}=3 \tag{5}$$

$$\therefore\quad x^2-16=9$$

$$x^2=25 \tag{6}$$

$$x=\pm 5 \tag{7}$$

Otherwise, from (2) deduce $\sqrt{x^2-16}=-x^2+28$.

Squaring both sides,

$$x^2-16=x^4-56x^2+784$$

$$\text{or,}\quad x^4-57x^2+800=0$$

$$\therefore\quad (x^2-25)(x^2-32)=0 \tag{3$'$}$$

$$\therefore\quad x^2=25 \text{ or } 32 \tag{4$'$}$$

$$\therefore\quad x=\pm 5 \text{ or } \pm 4\sqrt{2}.$$

By substitution, we find that the values 5 and $-4\sqrt{2}$ do, but the values -5 and $+4\sqrt{2}$ do not satisfy (1).

Thus (7) gives one "extraneous" root, and (what is more serious) omits a true root of (1). This root is one of the two values got by retaining the rejected equation $\sqrt{x^2-16}=-4$. The explanation is that the step from (1) to (2) depends on the assumption $(x-4)\sqrt{\frac{x+4}{x-4}}=\sqrt{x^2-16}$, which is only true if $x-4$ be positive. Now the value $-4\sqrt{2}$ for x makes $x-4$ negative, and therefore $(x-4)\sqrt{\frac{x+4}{x-4}}=-\sqrt{x^2-16}$.

Thus the true deduction from (1) is this: $x^2-16\pm\sqrt{x^2-16}=12$ according as $x-4\gtreqless 0$.

The former supposition gives $\sqrt{x^2-16}=-4$ or 3, or rejecting -4, $x=\pm 5$; and here -5 must be rejected since it makes $x-4<0$. Again, the supposition $x-4<0$ leads to $\sqrt{x^2-16}=4$ or -3. Rejecting -3, we get $x=\pm 4\sqrt{2}$, and here $+4\sqrt{2}$ must be rejected since it contradicts the supposition $x-4<0$.

R. F. MUIRHEAD