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A Classification of Three-dimensional Real Hypersurfaces in Non-flat Complex Space Forms in Terms of their Generalized Tanaka–Webster Lie Derivative

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Abstract. On a real hypersurface M in a non-flat complex space form there exist the Levi–Civita and the k-th generalized Tanaka–Webster connections. The aim of this paper is to study three dimensional real hypersurfaces in non-flat complex space forms, whose Lie derivative of the structure Jacobi operator with respect to the Levi–Civita connection coincides with the Lie derivative of it with respect to the k-th generalized Tanaka-Webster connection. The Lie derivatives are considered in direction of the structure vector field and in direction of any vector field orthogonal to the structure vector field.

1 Introduction

A *complex space form* is an n-dimensional Kähler manifold of constant holomorphic sectional curvature c. A complete and simply connected complex space form is analytically isometric to a complex projective space $\mathbb{C}P^n$ if $c > 0$, a complex Euclidean space $\mathbb{C}^n$ if $c = 0$, or a complex hyperbolic space $\mathbb{C}H^n$ if $c < 0$. Furthermore, the complex projective and complex hyperbolic spaces are called *non-flat complex space forms*, and the symbol $M_n(c)$, $n \geq 2$, is used to denote them when it is not necessary to distinguish them.

Let M be a *connected real hypersurface* of $M_n(c)$ without boundary. Let ∇ be the Levi–Civita connection on M and J the complex structure of $M_n(c)$. Take a locally defined unit normal vector field N on M and denote it by $\xi = -JN$. This is a tangent vector field to M called the *structure vector field* on M. If it is an eigenvector of the shape operator A of M, the real hypersurface is called a *Hopf hypersurface* and the corresponding eigenvalue is $\alpha = g(A\xi, \xi)$. Moreover, the complex structure J induces on M an almost contact metric structure (ϕ, ξ, η, g), where ϕ is the tangential component of J and η is an one-form given by $\eta(X) = g(X, \xi)$ for any X tangent to M.

The classification of homogeneous real hypersurfaces in $\mathbb{C}P^n$, $n \geq 2$, was obtained by Takagi, and they were divided into six type of real hypersurfaces (see [13–15]).

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Among them the three dimensional real hypersurfaces in $\mathbb{C}P^2$ are geodesic hyperspheres of radius r, $0 < r < \frac{\pi}{2}$, which are called *real hypersurfaces of type* (A) and tubes of radius r, $0 < r < \frac{\pi}{4}$, over the complex quadric, which are called *real hypersurfaces of type* (B). All of them are Hopf hypersurfaces with constant principal curvatures (see [6]). In case of $\mathbb{C}H^n$, the study of Hopf hypersurfaces with constant principal curvatures, was initiated by Montiel in [8] and completed by Berndt in [1]. Such hypersurfaces in $\mathbb{C}H^2$ are open subsets of horospheres, geodesic hyperspheres, or tubes over totally geodesic complex hyperbolic hyperplane $\mathbb{C}H^1$ (type (A)), or tubes over totally geodesic real hyperbolic space $\mathbb{R}H^2$ (type (B)).

The *Jacobi operator* R_X of a Riemannian manifold $\widetilde{M}$ with respect to a unit vector field X is given by $R_X = R(\cdot, X)X$, where R is the curvature tensor field on $\widetilde{M}$. It is a self-adjoint endomorphism of the tangent space $T\widetilde{M}$ and is related to Jacobi vector fields, which are solutions of the second-order differential equation $\nabla_{\dot{\gamma}}(\nabla_{\dot{\gamma}} Y) + R(Y, \dot{\gamma})\dot{\gamma} = 0$ along a geodesic γ in $\widetilde{M}$ (known as the Jacobi equation). In the case of real hypersurfaces in $M_n(c)$ the Jacobi operator with respect to the structure vector field ξ, R_ξ, is called the *structure Jacobi operator* on M and it plays an important role their study.

Apart from the Levi–Civita connection on a non-degenerate, pseudo-Hermitian CR-manifold, a canonical affine connection is defined, called the *Tanaka–Webster connection* (see [16, 18]). As a generalization of this connection, Tanno [17] defined the *generalized Tanaka-Webster connection* for contact metric manifolds by

$$\widehat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y.$$

Using the naturally extended affine connection of Tanno's generalized Tanaka–Webster connection, Cho defined the *k-th generalized Tanaka–Webster connection* $\widehat{\nabla}^{(k)}$ on a real hypersurface M in $M_n(c)$ given by

$$\widehat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{1.1}$$

for any vector fields X, Y tangent to M where k is a nonnull real number (see [2, 3]). Then the following relations hold:

$$\widehat{\nabla}^{(k)}\eta = 0, \quad \widehat{\nabla}^{(k)}\xi = 0, \quad \widehat{\nabla}^{(k)} g = 0, \quad \widehat{\nabla}^{(k)}\phi = 0.$$

In particular, if the shape operator of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, the k-th generalized Tanaka-Webster connection coincides with the Tanaka-Webster connection.

The Lie derivative of a tensor field T of type (1,1) with respect to the generalized Tanaka–Webster connection is denoted by $\widehat{\mathcal{L}}^{(k)}_X T$, called *k-th generalized Tanaka–Webster Lie derivative with respect to X* and is given by

$$(\widehat{\mathcal{L}}^{(k)}_X T)Y = \widehat{\nabla}^{(k)}_X TY - \widehat{\nabla}^{(k)}_{TY} X - T\widehat{\nabla}^{(k)}_X Y + T\widehat{\nabla}^{(k)}_Y X,$$

where X, Y are tangent to M .

Many geometric conditions with respect to the k-th generalized Tanaka-Webster connection on real hypersurfaces have been studied. One of them is the classification of real hypersurfaces in $M_n(c)$, $n \geq 2$, whose k-th generalized Tanaka-Webster Lie derivative agrees with the ordinary Lie derivative when applied to the tensor field T

of type (1,1), *i.e.*, $(\widehat{\mathcal{L}}_X^{(k)}T)Y = (\mathcal{L}_X T)Y$, for all X, Y tangent to M. Because of (1.1), the last relation implies

$$g\big((\phi A + A\phi)X, TY\big)\xi - (\phi A - k\phi)(X \wedge TY)\xi = g\big((\phi A + A\phi)X, Y\big)T\xi - T(\phi A - k\phi)(X \wedge Y)\xi, \tag{1.2}$$

and the wedge product is given by

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y,$$

for all X, Y Z tangent to M.

Real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, whose structure Jacobi operator satisfies relation $\widehat{\mathcal{L}}_\xi^{(k)}R_\xi = \mathcal{L}_\xi R_\xi$ are classified. Furthermore, the non-existence of real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, whose structure Jacobi operator satisfies relation $\widehat{\mathcal{L}}_X^{(k)}R_\xi = \mathcal{L}_X R_\xi$, for any X orthogonal to ξ is proved.

The purpose of this paper is to extend the previous results to the case of three dimensional real hypersurfaces in $M_2(c)$. First, we study real hypersurfaces in $M_2(c)$ satisfying relation

$$\widehat{\mathcal{L}}_\xi^{(k)}R_\xi = \mathcal{L}_\xi R_\xi \tag{1.3}$$

and obtain the following theorem.

Theorem 1.1 *Every real hypersurface in $M_2(c)$, whose structure Jacobi operator satisfies relation (1.3) is a Hopf hypersurface. Moreover, M is locally congruent either to a real hypersurface of type* (A)*, or to a Hopf hypersurface with $A\xi = 0$.*

Next we study three dimensional real hypersurfaces in $M_2(c)$, whose structure Jacobi operator satisfies relation

$$\widehat{\mathcal{L}}_X^{(k)}R_\xi = \mathcal{L}_X R_\xi \tag{1.4}$$

for all X orthogonal to ξ, and the following theorem is proved.

Theorem 1.2 *There do not exist real hypersurfaces in $M_2(c)$ whose structure Jacobi operator satisfies relation* (1.4).

The following corollary is an immediate consequence of the above theorems.

Corollary 1.3 *There do not exist real hypersurfaces in $M_2(c)$ such that $\widehat{\mathcal{L}}_X^{(k)}R_\xi = \mathcal{L}_X R_\xi$, for all $X \in TM$.*

This paper is organized as follows. Section 2 includes basic results about real hypersurfaces in non-flat complex space forms. Section 3 provides the proof of Theorem 1.1. Finally, in Section 4, the proof of Theorem 1.2 is given.

2 Preliminaries

Throughout this paper, all manifolds, vector fields etc. are assumed to be of class C^∞, all manifolds are assumed to be connected, and the real hypersurfaces M are supposed to be without boundary. Furthermore, all the material mentioned in this section is valid for all real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ without regard to the Lie derivative conditions.

Thus, let M be a real hypersurface immersed in a non-flat complex space form $(M_n(c), G)$ with complex structure J of constant holomorphic sectional curvature c, let N be a locally defined unit normal vector field on M, and let $\xi = -JN$ be the structure vector field of M. For a vector field X tangent to M, relation

$$JX = \phi X + \eta(X)N$$

holds, where ϕX and $\eta(X)N$ are respectively the tangential and the normal component of JX. The Riemannian connections $\overline{\nabla}$ in $M_n(c)$ and ∇ in M are related for any vector fields X, Y on M by

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N,$$

where g is the Riemannian metric induced from the metric G.
The *shape operator* A of the real hypersurface M in $M_n(c)$ with respect to N is given by

$$\overline{\nabla}_X N = -AX.$$

The real hypersurface M has an almost contact metric structure (ϕ, ξ, η, g) induced from J of $M_n(c)$, where ϕ is the *structure tensor*, which is a tensor field of type (1,1) and η is an 1-form such that

$$g(\phi X, Y) = G(JX, Y), \qquad \eta(X) = g(X, \xi) = G(JX, N).$$

Moreover, the following relations hold:

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \quad \eta(\xi) = 1,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \phi Y) = -g(\phi X, Y).$$

The fact that J is parallel implies $\overline{\nabla} J = 0$, and this leads to

$$\nabla_X \xi = \phi AX, \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi.$$

The ambient space $M_n(c)$ is of constant holomorphic sectional curvature c, and this results in the Gauss and Codazzi equations being respectively given by

(2.1)
$$\begin{aligned} R(X,Y)Z &= \frac{c}{4}\big[g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\big] \\ &\quad + g(AY, Z)AX - g(AX, Z)AY, \\ (\nabla_X A)Y - (\nabla_Y A)X &= \frac{c}{4}[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi], \end{aligned}$$

where R denotes the Riemannian curvature tensor on M, and X, Y, Z are any vector fields on M.

The tangent space $T_P M$ at every point $P \in M$ can be decomposed as

$$T_P M = \operatorname{span}\{\xi\} \oplus \mathbb{D},$$

where $\mathbb{D} = \ker\eta = \{X \in T_PM : \eta(X) = 0\}$ and is called (*maximal*) *holomorphic distribution* (*if* $n \geq 3$). Due to the above decomposition, the vector field $A\xi$ can be written as $A\xi = \alpha\xi + \beta U$, where $\beta = |\phi\nabla_\xi\xi|$ and $U = -\frac{1}{\beta}\phi\nabla_\xi\xi \in \ker(\eta)$ is a unit vector field, provided that $\beta \neq 0$.

Next, the following results concern any non-Hopf real hypersurface M in $M_2(c)$ with local orthonormal basis $\{U, \phi U, \xi\}$ at a point P of M.

Lemma 2.1 *Let M be a non-Hopf real hypersurface in $M_2(c)$. The following relations hold on M:*

$$(2.2)\quad \begin{aligned} &AU = \gamma U + \delta\phi U + \beta\xi, && A\phi U = \delta U + \mu\phi U, && A\xi = \alpha\xi + \beta U\\ &\nabla_U\xi = -\delta U + \gamma\phi U, && \nabla_{\phi U}\xi = -\mu U + \delta\phi U, && \nabla_\xi\xi = \beta\phi U,\\ &\nabla_U U = \kappa_1\phi U + \delta\xi, && \nabla_{\phi U}U = \kappa_2\phi U + \mu\xi, && \nabla_\xi U = \kappa_3\phi U,\\ &\nabla_U\phi U = -\kappa_1 U - \gamma\xi, && \nabla_{\phi U}\phi U = -\kappa_2 U - \delta\xi, && \nabla_\xi\phi U = -\kappa_3 U - \beta\xi, \end{aligned}$$

where $\alpha, \beta, \gamma, \delta, \mu, \kappa_1, \kappa_2, \kappa_3$ are smooth functions on M and $\beta \neq 0$.

Remark 2.2 The proof of Lemma 2.1 is included in [12].

Because of Lemma 2.1, the Codazzi equation for $X \in \{U, \phi U\}$ and $Y = \xi$ implies the following relations:

$$(2.3)\qquad \xi\delta = \alpha\gamma + \beta\kappa_1 + \delta^2 + \mu\kappa_3 + \frac{c}{4} - \gamma\mu - \gamma\kappa_3 - \beta^2,$$

$$(2.4)\qquad \xi\mu = \alpha\delta + \beta\kappa_2 - 2\delta\kappa_3,$$

$$(2.5)\qquad (\phi U)\alpha = \alpha\beta + \beta\kappa_3 - 3\beta\mu,$$

$$(2.6)\qquad (\phi U)\beta = \alpha\gamma + \beta\kappa_1 + 2\delta^2 + \frac{c}{2} - 2\gamma\mu + \alpha\mu,$$

and for $X = U$ and $Y = \phi U$,

$$(2.7)\qquad U\delta - (\phi U)\gamma = \mu\kappa_1 - \kappa_1\gamma - \beta\gamma - 2\delta\kappa_2 - 2\beta\mu.$$

Furthermore, the combination of the Gauss equation (2.1) with the formula for Riemannian curvature $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, taking into account relations of Lemma 2.1, implies

$$(2.8)\qquad U\kappa_2 - (\phi U)\kappa_1 = 2\delta^2 - 2\gamma\mu - \kappa_1^2 - \gamma\kappa_3 - \kappa_2^2 - \mu\kappa_3 - c.$$

Relation (2.1) implies that the structure Jacobi operator R_ξ is given by

$$R_\xi(X) = \frac{c}{4}[X - \eta(X)\xi] + \alpha AX - \eta(AX)A\xi$$

for any vector field X tangent to M, where $\alpha = \eta(A\xi) = g(A\xi, \xi)$.

Moreover, the structure Jacobi operator for $X = U$, $X = \phi U$ and $X = \xi$ due to (2.2) is given by

$$(2.9)\qquad \begin{aligned} R_\xi(U) &= \Big(\frac{c}{4} + \alpha\gamma - \beta^2\Big)U + \alpha\delta\phi U,\\ R_\xi(\phi U) &= \alpha\delta U + \Big(\frac{c}{4} + \alpha\mu\Big)\phi U \quad\text{and}\quad R_\xi(\xi) = 0. \end{aligned}$$

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The following theorem in the case of $\mathbb{C}P^n$ is due to Maeda [7] and in the case of $\mathbb{C}H^n$ is due to Ki and Suh [5] (see also [10, Corollary 2.3]).

Theorem 2.3 *Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 2$, with $A\xi = \alpha\xi$.*

(i) *α is constant.*

(ii) *If W is a vector field that belongs to $\mathbb{D}$ such that $AW = \lambda W$, then*

$$\Big(\lambda - \frac{\alpha}{2}\Big) A\phi W = \Big(\frac{\lambda\alpha}{2} + \frac{c}{4}\Big)\phi W.$$

(iii) *If the vector field W satisfies $AW = \lambda W$ and $A\phi W = \nu\phi W$, then*

$$\lambda\nu = \frac{\alpha}{2}(\lambda + \nu) + \frac{c}{4}. \tag{2.10}$$

Remark 2.4 In case of three dimensional Hopf hypersurfaces we can always consider a local orthonormal basis $\{W, \phi W, \xi\}$ at some point $P \in M$ such that $AW = \lambda W$ and $A\phi W = \nu\phi W$. Thus, relation (2.10) is satisfied. Furthermore, the structure Jacobi operator of Hopf hypersurfaces, whose shape operator is given by the previous relations for $X = W$ and $X = \phi W$ is given by

$$R_\xi(W) = \Big(\frac{c}{4} + \alpha\lambda\Big) W \quad \text{and} \quad R_\xi(\phi W) = \Big(\frac{c}{4} + \alpha\nu\Big)\phi W. \tag{2.11}$$

We also mention the following theorem, which plays an important role in the study of real hypersurfaces in $M_n(c)$. This is due to Okumura [11] in the case of $\mathbb{C}P^n$ and to Montiel and Romero [9] in the case of $\mathbb{C}H^n$. It provides the classification of real hypersurfaces in $M_n(c)$, $n \geq 2$, whose shape operator A commutes with the structure tensor field ϕ.

Theorem 2.5 *Let M be a real hypersurface of $M_n(c)$, $n \geq 2$. Then $A\phi = \phi A$, if and only if M is locally congruent to a homogeneous real hypersurface of type (A). More precisely:*

In case of $\mathbb{C}P^n$:

(A_1) *a geodesic hypersphere of radius r, where $0 < r < \frac{\pi}{2}$,*

(A_2) *a tube of radius r over a totally geodesic $\mathbb{C}P^k$,$(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$.*

In case of $\mathbb{C}H^n$:

(A_0) *a horosphere in $\mathbb{C}H^n$,* i.e., *a Montiel tube,*

(A_1) *a geodesic hypersphere or a tube over a totally geodesic complex hyperbolic hyperplane $\mathbb{C}H^{n-1}$,*

(A_2) *a tube over a totally geodesic $\mathbb{C}H^k$ $(1 \leq k \leq n-2)$.*

Remark 2.6 In the case of three-dimensional real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$, type (A_2) hypersurfaces do not occur.

Finally, we mention the following proposition (see [4]), which is used in the proof of Theorems 1.1 and 1.2.

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Proposition 2.7 *There do not exist real hypersurfaces in $M_2(c)$, whose structure Jacobi operator vanishes.*

3 Proof of Theorem 1.1

Let M be a non-Hopf real hypersurface in $M_2(c)$ whose structure Jacobi operator satisfies relation (1.3). More analytically, the previous relation, due to (1.2) for $T = R_\xi$ and $X = \xi$, and since $R_\xi(\xi) = 0$ implies

$$g\big(\phi A\xi, R_\xi(Y)\big)\xi - (\phi A - k\phi)\big(\xi \wedge R_\xi(Y)\big)\xi = -R_\xi(\phi A - k\phi)(\xi \wedge Y)\xi, \tag{3.1}$$

for all Y tangent to M.

We consider $\mathcal{N}$ the open subset of M such that

$$\mathcal{N} = \{P \in M : \beta \neq 0, \text{ in a neighborhood of } P.\}$$

Lemma 2.1 holds on $\mathcal{N}$, and the inner product of relation (3.1) for $Y = U$ with ξ due to the first of (2.9) yields $\alpha\delta = 0$.

Suppose that $\alpha \neq 0$. Then the above relation implies $\delta = 0$ and relations (2.2) and (2.9) become, respectively,

$$AU = \gamma U + \beta\xi, \qquad A\phi U = \mu\phi U, \qquad A\xi = \alpha\xi + \beta U, \tag{3.2}$$

$$R_\xi(U) = \Big(\frac{c}{4} + \alpha\gamma - \beta^2\Big)U, \qquad R_\xi(\phi U) = \Big(\frac{c}{4} + \alpha\mu\Big)\phi U, \qquad R_\xi(\xi) = 0. \tag{3.3}$$

Because of (3.2) and the second relation of (3.3), the inner product of (3.1) for $Y = \phi U$ with ξ implies

$$\mu = -\frac{c}{4\alpha} \Longrightarrow R_\xi(\phi U) = 0.$$

Moreover, relation (3.1) for $Y = \phi U$, taking into account that $R_\xi(\phi U) = 0$ and the first of (3.3) results in $(\mu - k)R_\xi(U) = 0$. If $\mu \neq k$ then $R_\xi(U) = 0$. So the structure Jacobi operator vanishes identically, which is impossible because of Proposition 2.7.

Thus, $\mu = k$. Furthermore, the inner product of (3.1) for $Y = U$ with ϕU, due to the first relation of (3.3) and $R_\xi(\phi U) = 0$, implies

$$(\gamma - k)g(R_\xi(U), U) = 0.$$

If $\gamma \neq k$, then $g(R_\xi(U), U) = 0$, and this results in $R_\xi(U) = 0$, which implies that the structure Jacobi operator vanishes identically, which is impossible because of Proposition 2.7.

So $\gamma = k$. Differentiation of the last relation with respect to ϕU yields $(\phi U)\gamma = 0$. Thus, since $\delta = 0$ and $\mu = \gamma = k$ relation (2.7) implies $k = 0$, which is a contradiction.

Therefore, we have $\alpha = 0$ on M, and relation (2.9) becomes

$$R_\xi(U) = \Big(\frac{c}{4} - \beta^2\Big)U, \quad R_\xi(\phi U) = \frac{c}{4}\phi U, \quad \text{and} \quad R_\xi(\xi) = 0. \tag{3.4}$$

Because of the second relation of (3.4), the inner product of relation (3.1) for $Y = \phi U$ with ξ gives $c = 0$, which is a contradiction.

Thus, $\mathcal{N}$ is empty and the following proposition is proved.

Proposition 3.1 *Every real hypersurface in $M_2(c)$ whose structure Jacobi operator satisfies relation (1.3) is a Hopf hypersurface.*

Due to the above proposition, relations in Theorem 2.3 and remark 2.4 hold. Taking into account (2.11), relation (3.1) for $Y = W$ and $Y = \phi W$ implies, respectively,

$$k\alpha(\lambda - \nu) = \lambda\alpha(\lambda - \nu) \quad \text{and} \quad k\alpha(\lambda - \nu) = \nu\alpha(\lambda - \nu). \tag{3.5}$$

If there is a point where $\lambda \neq \nu$, relation (3.5) yields $k\alpha = \alpha\lambda$ and $k\alpha = \nu\alpha$, which implies $\alpha(\lambda - \nu) = 0$. So, $\alpha = 0$.

If $\lambda = \nu$ at all points, then this implies $(A\phi - \phi A)X = 0$ for any X tangent to M. So due to Theorem 2.5, M is locally congruent to a real hypersurface of type (A), and this completes the proof of Theorem 1.1. ∎

4 Proof of Theorem 1.2

Because of (1.2), since $T = R_\xi$ and $X \in \mathbb{D}$ because of $R_\xi(\xi) = 0$, relation (1.4) implies

$$g\big((\phi A + A\phi)X, R_\xi(Y)\big)\,\xi = -R_\xi(\phi A - k\phi)(X \wedge Y)\xi \tag{4.1}$$

for all X orthogonal to ξ and for all vectors Y tangent to M.

First we prove the following proposition.

Proposition 4.1 *There do not exist Hopf hypersurfaces in $M_2(c)$ whose structure Jacobi operator satisfies relation* (1.4).

Proof Let M be a Hopf hypersurface. Then we have $A\xi = \alpha\xi$, where α is constant, and remark 2.4 holds. Relation (4.1) for (X, Y) being (W, ξ), $(\phi W, \xi)$, $(W, \phi W)$ and $(\phi W, W)$ taking into account relation (2.11) implies respectively

$$(\lambda - k)\Big(\alpha\nu + \frac{c}{4}\Big) = 0, \tag{4.2}$$

$$(\nu - k)\Big(\alpha\lambda + \frac{c}{4}\Big) = 0, \tag{4.3}$$

$$(\lambda + \nu)\Big(\alpha\nu + \frac{c}{4}\Big) = 0, \tag{4.4}$$

$$(\lambda + \nu)\Big(\alpha\lambda + \frac{c}{4}\Big) = 0. \tag{4.5}$$

There are three possibilities to consider:

(a) Suppose $\alpha = 0$. Then relations (4.2) and (4.3) give $\lambda = \nu = k$. So, relation (4.4) implies $k = 0$, which is a contradiction.

(b) Suppose $\alpha \neq 0$ and there is a point where $\lambda \neq \nu$. If $\lambda \neq k$, then relation (4.2) implies $\alpha\nu + \frac{c}{4} = 0$. So $\alpha\lambda + \frac{c}{4} \neq 0$ and relation (4.3) yields $\nu = k$. Furthermore, relation (4.5) gives $\lambda + \nu = 0$. So, $\lambda = -k$, and the Hopf hypersurface has three constant principal curvatures and must be an open subset of a type (B) hypersurface. But type (B) hypersurfaces satisfy $\lambda\nu + \frac{c}{4} = 0$ and substitution of the last relation in (2.10) leads to a contradiction.

(c) $\alpha \neq 0$ and $\lambda = \nu$. Because of (4.4) this implies that either $\lambda = 0$ or $\alpha\lambda = -\frac{c}{4}$. Substitution of the previous in (2.10) leads to a contradiction, and this completes the proof of the proposition. ∎

Next we examine non-Hopf hypersurfaces in $M_2(c)$ whose structure Jacobi operator satisfies relation (4.1). Since M is a non-Hopf hypersurface, we have that $\beta \neq 0$

and relation (2.2) holds. Relation (4.1) for $X = Y = U$, $X = U$ and $Y = \phi U$ and for $X = \phi U$ and $Y = U$ implies, respectively,

$$(\gamma + \mu) g\big(R_\xi(U), \phi U\big) = 0, \tag{4.6}$$

$$(\gamma + \mu) g\big(R_\xi(\phi U), \phi U\big) = 0, \tag{4.7}$$

$$(\gamma + \mu) g\big(R_\xi(U), U\big) = 0. \tag{4.8}$$

If $\gamma + \mu \neq 0$, then relations (4.6), (4.7), and (4.8) result in

$$g\big(R_\xi(U), \phi U\big) = g\big(R_\xi(\phi U), \phi U\big) = g\big(R_\xi(U), U\big).$$

The above relation leads to the conclusion that the structure Jacobi operator R_ξ vanishes identically and because of Proposition 2.7 this is impossible.

Thus on M, relation $\gamma + \mu = 0$ holds. Moreover, for $X = U$ and $Y = \xi$ and for $X = \phi U$ and $Y = \xi$ due to (2.9) and $\gamma + \mu = 0$ relation (1.4) implies

$$\delta\Big(\frac{c}{4} - \beta^2 + \alpha k\Big) = 0, \tag{4.9}$$

$$(\mu + k)\Big(\frac{c}{4} + \alpha\mu\Big) = -\alpha\delta^2, \tag{4.10}$$

$$(\mu - k)\Big(\frac{c}{4} - \alpha\mu - \beta^2\Big) = \alpha\delta^2, \tag{4.11}$$

$$\delta\Big(\frac{c}{4} + \alpha k\Big) = 0. \tag{4.12}$$

Suppose that $\delta \neq 0$. Then combination of relations (4.9) and (4.12) yields $\beta = 0$, which is a contradiction.

So, on M we have $\delta = 0$ and $\gamma = -\mu$, and relations (4.10) and (4.11) become

$$(\mu + k)\Big(\frac{c}{4} + \alpha\mu\Big) = 0 \quad \text{and} \quad (\mu - k)\Big(\frac{c}{4} - \alpha\mu - \beta^2\Big) = 0. \tag{4.13}$$

If $k + \mu \neq 0$, then $\frac{c}{4} + \alpha\mu = 0$ and the second of the above relation gives $\mu = k$, because if $\frac{c}{4} - \alpha\mu - \beta^2 = 0$, then relation (2.9) implies that the structure Jacobi operator R_ξ vanishes identically, which is impossible. Since $k = \mu$, we obtain $\xi\mu = 0$ and relation (2.4) implies $\kappa_2 = 0$. Furthermore, differentiation of $\gamma = -\mu$ with respect to ϕU gives

$$(\phi U)\mu = (\phi U)\gamma = 0.$$

Furthermore, differentiation of $\frac{c}{4} + \alpha\mu = 0$ with respect to ϕU because of the above relation and relation (2.5) gives $\kappa_3 = 3\mu - \alpha$. Since $(\phi U)\gamma = 0$, relation (2.7) implies $\kappa_1 = \beta/2$. So bearing in mind all the previous relations relation (2.3) gives $\beta^2/2 = c + 7\mu^2$. Differentiating the last relation with respect to ϕU yields $(\phi U)\beta = 0$ and relation (2.6) implies $\beta^2/2 + c/2 + 2\mu^2 = 0$. Moreover, since $\kappa_1 = \beta/2$ and $(\phi U)\beta = 0$, we conclude that $(\phi U)\kappa_1 = 0$ and due to $\gamma = -\mu$, $\kappa_1 = \beta/2$, $\kappa_3 = 3\mu - \alpha$ and $\kappa_2 = 0$ relation (2.8) results in $\beta^2/2 = 4\mu^2 - 2c$. Combination of the last one with $\beta^2/2 = c + 7\mu^2$ implies $c = -\mu^2$. Substitution of the latter in $\beta^2/2 + 2\mu^2 + c/2 = 0$ due to $\beta^2/2 = 4\mu^2 - 2c$ leads to $c = 0$, which is a contradiction.

Thus, on M we have $\mu + k = 0$. Summarizing on M the following relations hold:

$$\delta = 0 \quad \text{and} \quad \gamma = -\mu = k.$$

The second relation of (4.13) implies that $k\alpha = \beta^2 - c/4$.

Moreover, due to $\mu = -k$, relation (2.4) implies $\kappa_2 = 0$, and bearing in mind all the previous relations, relation (2.7) results in $\beta = 2\kappa_1$. Furthermore, because of $\gamma = -\mu$, $\beta = 2\kappa_1$, and $\mu = -k$, relation (2.6) implies $(\phi U)\beta = \beta^2/2 + c/2 + 2k^2$ and relation (2.8) taking into account $\gamma + \mu = 0$, $\kappa_2 = 0$ and $\beta = 2\kappa_1$ yields $(\phi U)\beta = -4k^2 + \beta^2/2 + 2c$. Combination of the last two relations of $(\phi U)\beta$ results in $c = 4k^2$. The last relation leads to a contradiction when the ambient space is $\mathbb{C}H^2$. So it remains to examine the case when the ambient space is $\mathbb{C}P^2$.

Since $c = 4k^2$ and $k \neq 0$, relation $k\alpha = \beta^2 - c/4$ implies $\alpha = \beta^2/k - k$. Differentiation of the latter with respect to ϕU taking into account relations (2.5) and (2.6) yields $\kappa_3 = 6k$. Furthermore, because of the last one and $\beta = 2\kappa_1$ relation (2.3) results in $\beta^2 = 22k^2$. So because of the above relations, relation (2.6) implies $\beta^2 + 2c = 0$. The last relation due to $c = 4k^2$ and $\beta^2 = 22k^2$ results in $k = 0$, which is impossible, and this completes the proof of Theorem 1.2. ∎

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