Introduction

Rectifiable sets, measures, currents and varifolds are basic objects of geometric measure theory. In particular, during the last four decades, they have spread out to many areas of analysis and geometry. One of the goals of this survey is to show how rectifiability unifies surprisingly many different topics. Starting from the beginning and basic theories, I shall briefly describe many of these appearances. The table of contents should give a pretty good idea of what will follow. Here I just mention some of the milestones.

The pre-beginning is the right generalization of length, area, and so on. This was provided by Carathéodory in 1914. Lebesgue with his measure had given a generalization of volume in the Euclidean n-space and Carathéodory continued from this to define for integers $0 < m < n$ an outer measure generalizing the m-dimensional area. This measure $\mathcal{H}^m$ is now called m-dimensional Hausdorff measure, since Hausdorff generalized it further in 1919 to non-integral values of m. Once equipped with this tool, Besicovitch began in the 1920s to study the properties of $\mathcal{H}^1$ measurable planar sets E with $\mathcal{H}^1(E) < \infty$. In three papers he was able to reveal an amazing amount of structure. Such a set splits into a rectifiable and purely unrectifiable part. The former has properties similar to those of smooth surfaces and the latter completely opposite properties. Federer generalized most of Besicovitch's theory in 1947 to m-dimensional sets in $\mathbb{R}^n$. After the founding work of Besicovitch and Federer, the ingenious ideas of Marstrand and Preiss have been the most influential for the basic theory of rectifiability.

In the 1950s, De Giorgi described the structure of sets of finite perimeter in terms of rectifiable sets. This gave rectifiability a permanent place in the calculus of variations. First for sets of codimension one and then via Federer and Fleming's theory of normal and integral currents in 1960 for all dimensions. These are generalized surfaces and another class of them of fundamental importance over the years, rectifiable varifolds, was introduced by Almgren in

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the 1960s and developed by Allard in the 1970s. In the 1990s, Simon proved some fundamental results on the rectifiability of the singularities of minimal currents and harmonic maps.

Rectifiability has played a big role in complex and harmonic analysis. In the 1950s, Vitushkin anticipated this for the geometric description of removable sets of bounded complex analytic functions, which was fully confirmed much later in 1998 by David. In the 1990s, continuing from Jones's analyst's travelling salesman theorem, David and Semmes established the theory of uniform rectifiability and its connections to singular integrals and other topics of harmonic analysis.

Rectifiability has found a prominent place also outside Euclidean spaces. After some important work in the 1990s by Ambrosio, by Preiss and Tiser and by Kirchheim, Ambrosio and Kirchheim developed the theory of rectifiable sets and currents in metric spaces in 2000. In 2001 Franchi, Serapioni and Serra Cassano introduced the right notions of rectifiability in Heisenberg groups, which has led to an extensive theory in general Carnot groups.

This survey covers many topics, but all of them briefly and only scratching the surface. I am trying to give the reader a flavour of each of them without detailed proofs, often with ideas of the proofs, and often just presenting the results. There probably are many topics and articles that I have not, but should have, mentioned. And certainly this text is biased – I have concentrated more on topics I know best, including my own work.

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