Canad. Math. Bull. Vol. **50** (2), 2007 pp. 284–290

Second Order Mock Theta Functions

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Abstract. In his last letter to Hardy, Ramanujan defined 17 functions $F(q)$, where $|q| < 1$. He called them mock theta functions, because as q radially approaches any point $e^{2\pi ir}$ (r rational), there is a theta function $F_r(q)$ with $F(q) - F_r(q) = O(1)$. In this paper we establish the relationship between two families of mock theta functions.

1 Introduction

Mock theta functions were first studied by Ramanujan. G. H. Hardy [5, p. 534] defines such functions as follows

> a "mock θ-function" is a function defined by a q-series convergent for $|q| < 1$, for which we can calculate asymptotic formulae, when q tends to a "rational point" $e^{2r\pi i/s}$, of the same degree of precision as those furnished, for the ordinary θ-functions, by the theory of linear [fractional] transformation.

Ramanujan divided his list of mock theta functions into "third order," "fifth order," and "seventh order" functions, but did not say what he meant. Known identities for them make it clear that they are related to the numbers 3, 5 and 7, but as yet no generally accepted definition of order has been given. The designation "second order" in the title of this paper should therefore be regarded as tentative.

The purpose of this paper is to establish the relationship between two families of mock theta functions.

We use the standard notation for q-shifted factorials:

$$(a;q^k)_0 = 1,$$
$$(a;q^k)_n = (1-a)(1-aq^k)(1-aq^{2k})\cdots(1-aq^{(n-1)k}),$$
$$(a;q^k)_\infty = \prod_{m=0}^{\infty}(1-aq^{mk}).$$

When $k = 1$ we usually write $(a)_n$ and $(a)_\infty$ instead of $(a;q)_n$ and $(a;q)_\infty$, respectively. For nonnegative integers n we have

$$(a;q^k)_n = \frac{(a;q^k)_\infty}{(aq^{nk};q^k)_\infty},$$

and for other real n we take this as the definition of $(a;q^k)_n$.

Received by the editors January 29, 2005.
AMS subject classification: 11B65, 33D15.
Keywords: q-series, mock theta function, Mordell integral.

2 The Two Families

The mock theta functions in our first family are:

$$A(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}(-q\,;q^2)_n}{(q\,;q^2)_{n+1}^2}=\sum_{n=0}^{\infty}\frac{q^{n+1}(-q^2\,;q^2)_n}{(q\,;q^2)_{n+1}},$$

$$B(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}(-q^2\,;q^2)_n}{(q\,;q^2)_{n+1}^2}=\sum_{n=0}^{\infty}\frac{q^{n}(-q\,;q^2)_n}{(q\,;q^2)_{n+1}},$$

$$\mu(q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}(q\,;q^2)_n}{(-q^2\,;q^2)_n^2}.$$

The function μ appears in Ramanujan's "Lost" Notebook [6] in a number of identities (see also [1], equations $(3.1)_{\mathrm{R}}$–$(3.13)_{\mathrm{R}}$ with $a=1$). The other two functions are related to μ by the modular transformation formulas [1],

$$q^{-\frac{1}{8}}\mu(q)=\sqrt{\frac{8\pi}{\alpha}}\,q_1^{\frac{1}{2}}B(q_1)+\sqrt{\frac{2\alpha}{\pi}}J\Big(\frac{\alpha}{2}\Big),$$

$$q^{-\frac{1}{8}}A(q)=\sqrt{\frac{\pi}{16\alpha}}\,q_1^{-\frac{1}{8}}\mu(-q_1)-\sqrt{\frac{\alpha}{2\pi}}K(\alpha),$$

$$q^{-\frac{1}{8}}A(-q)=\sqrt{\frac{\pi}{2\alpha}}\,q_1^{\frac{1}{2}}B(-q_1)-\sqrt{\frac{\alpha}{8\pi}}J\Big(\frac{\alpha}{2}\Big),$$

where $q=e^{-\alpha}$ and $q_1=e^{-\beta}$ with $\alpha\beta=\pi^2$. The Mordell integrals J and K are defined by

$$J(\alpha)=\int_0^{\infty}\frac{e^{-\alpha x^2}}{\cosh\alpha x}\,dx,\qquad J(\alpha)=\sqrt{\frac{\pi^3}{\alpha^3}}J(\beta),$$

$$K(\alpha)=\int_0^{\infty}e^{-\frac{1}{2}\alpha x^2}\frac{\cosh\frac{1}{2}\alpha x}{\cosh\alpha x}\,dx,\qquad K(\alpha)=\sqrt{\frac{\pi^3}{\alpha^3}}K(\beta).$$

Note that some of the transformation formulas in [1] are stated incorrectly. The equality of the two series in the definitions of $A(q)$ and $B(q)$ follows from the identity

$$\sum_{n=0}^{\infty}\frac{(-q^2/z\,;q^2)_n q^{n(n+1)}z^n}{(q\,;q^2)_{n+1}^2}=\sum_{n=0}^{\infty}\frac{(-zq\,;q^2)_n q^n}{(q\,;q^2)_{n+1}}\tag{1}$$

with $z=q$ and $z=1$, respectively. This identity can be obtained from [3, p. 241, (III.9)], which states that

$${}_3\phi_2\left[\begin{matrix}a,b,c\\ d,e\end{matrix};q;\frac{de}{abc}\right]=\frac{(e/a)_\infty(de/bc)_\infty}{(e)_\infty(de/abc)_\infty}\,{}_3\phi_2\left[\begin{matrix}a,d/b,d/c\\ d,de/bc\end{matrix};q;\frac{e}{a}\right].$$

Replacing q by q^2 and putting $a = q^2$, $b = -q^2/\tau$, $c = -q^2/z$ and $d = e = q^3$ gives

$$\sum_{n=0}^{\infty} \frac{(-q^2/z\,;q^2)_n q^{n(n+1)} z^n}{(q\,;q^2)_{n+1}^2} = \lim_{\tau\to 0} \frac{1}{(1-q)^2} \sum_{n=0}^{\infty} \frac{(-q^2/\tau\,;q^2)_n(-q^2/z\,;q^2)_n \tau^n z^n}{(q^3\,;q^2)_n (q^3\,;q^2)_n}$$

$$= \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{(-zq\,;q^2)_n q^n}{(q^3\,;q^2)_n}$$

$$= \sum_{n=0}^{\infty} \frac{(-zq\,;q^2)_n q^n}{(q\,;q^2)_{n+1}}.$$

Observe that as $z \to 0$ in (1) we get a new q-series representing the third order mock theta function [8, p. 62]

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q\,;q^2)_{n+1}^2} = \sum_{n=0}^{\infty} \frac{q^n}{(q\,;q^2)_{n+1}}.$$

In our discovery [4] of the eighth order mock theta functions S_0, S_1, T_0 and T_1, Basil Gordon and I encountered the lower order mock theta functions U_0, U_1, V_0 and V_1. These functions are the members of our second family of mock theta functions. They are defined by [4, pp. 322–323]:

$$U_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q\,;q^2)_n}{(-q^4\,;q^4)_n} = 2\frac{(-q\,;q^2)_\infty}{(q^2\,;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(2n+1)}}{1+q^{4n}},$$

$$U_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q\,;q^2)_n}{(-q^2\,;q^4)_{n+1}} = \frac{(-q\,;q^2)_\infty}{(q^2\,;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(n+1)(2n+1)}}{1+q^{4n+2}},$$

$$V_0(q) = -1 + 2\sum_{n=0}^{\infty} \frac{q^{n^2}(-q\,;q^2)_n}{(q\,;q^2)_n} = -1 + 2\frac{(-q^2\,;q^4)_\infty}{(q^4\,;q^4)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n(2n+1)}}{1-q^{4n+1}},$$

$$V_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q\,;q^2)_n}{(q\,;q^2)_{n+1}} = (-q^4\,;q^4)_\infty (q^4\,;q^4)_\infty \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(2n+1)^2}}{1-q^{4n+1}}.$$

I will show that our two families of mock theta functions are related by the following identities:

$$U_0(q) - 2U_1(q) = \mu(q), \tag{2}$$

$$V_0(q) - V_0(-q) = 4qB(q^2), \tag{3}$$

$$V_1(q) + V_1(-q) = 2A(q^2). \tag{4}$$

It is interesting to note that if we change the signs on the left sides of these identities we obtain [4, p. 323]:

$$U_0(q) + 2U_1(q) = (q)_\infty(-q\,;q^2)_\infty^4, \tag{5}$$

$$V_0(q) + V_0(-q) = 2(-q^2\,;q^4)_\infty^4 (q^8\,;q^8)_\infty, \tag{6}$$

$$V_1(q) - V_1(-q) = 2q(-q^2\,;q^2)_\infty(-q^4\,;q^4)_\infty^2 (q^8\,;q^8)_\infty. \tag{7}$$

Note that [4, (1.11)] is stated incorrectly.

To prove (2) we begin with the generalized Lambert series

$$\begin{aligned} U_0(-q) - 2U_1(-q) &= 2\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\left\{ \sum_{n=-\infty}^{\infty} \frac{q^{n(2n+1)}}{1+q^{4n}} + \sum_{n=-\infty}^{\infty} \frac{q^{(n+1)(2n+1)}}{1+q^{4n+2}} \right\} \\ &= 2\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\left\{ \sum_{n=-\infty}^{\infty} \frac{q^{2n(2n+1)/2}}{1+q^{2(2n)}} + \sum_{n=-\infty}^{\infty} \frac{q^{(2n+1)(2n+2)/2}}{1+q^{2(2n+1)}} \right\} \\ &= 2\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)/2}}{1+q^{2n}}. \end{aligned}$$

By the Watson–Whipple transformation [3, p. 242, (III.17)],

$$\begin{aligned} &{}_8\phi_7\left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, b, c, d, e, f, \\ a^{\frac12}, -a^{\frac12}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{aq}{f} \end{matrix}; q; \frac{a^2q^2}{bcdef}\right] \\ &\qquad = \frac{\left(aq, \frac{aq}{de}, \frac{aq}{df}, \frac{aq}{ef}\right)_\infty}{\left(\frac{aq}{d}, \frac{aq}{e}, \frac{aq}{f}, \frac{aq}{def}\right)_\infty} {}_4\phi_3\left[\begin{matrix} \frac{aq}{bc}, d, e, f \\ \frac{aq}{b}, \frac{aq}{c}, \frac{def}{a} \end{matrix}; q; q\right] \end{aligned}$$

with $a \to 1^-$, $b = -c = i$, $d = -e = q^{1/2}$ and $f \to \infty$, we get

$$2\sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)/2}}{1+q^{2n}} = 1 + 2\sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}(1+q^n)}{1+q^{2n}} = 2\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n(q;q^2)_n}{(-q^2;q^2)_n},$$

where the convergence of the last sum is defined in the Cesàro sense (see [2, p. 205], for example), which in this case is equal to the limit of the average of consecutive partial sums. Hence

$$U_0(-q) - 2U_1(-q) = 2\sum_{n=0}^{\infty} \frac{(-1)^n(q;q^2)_n}{(-q^2;q^2)_n},$$

and therefore

$$\begin{aligned} (8)\quad U_0(-q) - 2U_1(-q) + 4\alpha(q) &= 2\sum_{n=0}^{\infty} \frac{(-1)^n(q;q^2)_n}{(-q^2;q^2)_n} + 4\sum_{n=0}^{\infty} \frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}} \\ &= 2\sum_{n=0}^{\infty} \frac{(-1)^n(q;q^2)_n}{(-q^2;q^2)_n} + 2\sum_{n=1}^{\infty} \frac{(-1)^{-n}(q;q^2)_{-n}}{(-q^2;q^2)_{-n}} \\ &= 2\sum_{n=-\infty}^{\infty} \frac{(-1)^n(q;q^2)_n}{(-q^2;q^2)_n} \\ &= (-q;q^2)_\infty^5(q^2;q^2)_\infty, \end{aligned}$$

https://doi.org/10.4153/CMB-2007-028-9 Published online by Cambridge University Press

where the last equation was obtained from Ramanujan's sum [3, p. 239, (II.29)]

$$ {}_1\psi_1(a\,;b\,;q,z) = \frac{(q, b/a, az, q/az\,;q)_\infty}{(b, q/a, z, b/az\,;q)_\infty} $$

with q replaced by q^2 and then $a = q$, $b = -q^2$ and $z \to -1^+$. Equation (3.28) of [1] is equivalent to

$$ 4\alpha(q) = (-q\,;q^2)_\infty^5 (q^2\,;q^2)_\infty - \mu(-q). \tag{9} $$

Substituting (9) into (8) and replacing q by $-q$ completes the proof of (2).

Identities (3) and (4) can now be established by modular transformations. The first set of modular transformations [4, p. 333] and [1, p. 12] is:

$$ q^{-\frac{1}{8}} U_0(q) = \sqrt{\frac{\pi}{2\alpha}} V_0(q_1^{\frac{1}{2}}) + \sqrt{\frac{\alpha}{2\pi}} J\Big(\frac{\alpha}{2}\Big), $$

$$ q^{-\frac{1}{8}} U_1(q) = \sqrt{\frac{\pi}{8\alpha}} V_0(-q_1^{\frac{1}{2}}) - \sqrt{\frac{\alpha}{8\pi}} J\Big(\frac{\alpha}{2}\Big), $$

$$ q^{\frac{-18}{\mu}}(q) = \sqrt{\frac{8\pi}{\alpha}} q_1^{\frac{1}{2}} B(q_1) + \sqrt{\frac{2\alpha}{\pi}} J\Big(\frac{\alpha}{2}\Big). $$

Substituting these transformations into (2) and replacing q_1 by q^2 yields (3). Next we substitute the transformations

$$ q^{-\frac{1}{8}} U_0(-q) = \sqrt{\frac{4\pi}{\alpha}} q_1^{-\frac{1}{8}} V_1(q_1^{\frac{1}{2}}) + \sqrt{\frac{2\alpha}{\pi}} K(\alpha), $$

$$ q^{-\frac{1}{8}} U_1(-q) = -\sqrt{\frac{\pi}{\alpha}} q_1^{-\frac{1}{8}} V_1(-q_1^{\frac{1}{2}}) - \sqrt{\frac{\alpha}{2\pi}} K(\alpha), $$

$$ q^{-\frac{1}{8}} \mu(-q) = \sqrt{\frac{16\pi}{\alpha}} q_1^{-\frac{1}{8}} A(q_1) + \sqrt{\frac{8\alpha}{\pi}} K(\alpha) $$

into (2) (with q replaced by $-q$). Replacing q_1 by q^2 establishes (4).

The function

$$ \lambda(q) = \sum_{n=0}^{\infty} \frac{q^{n+1}(-q)_{2n}}{(-q^2\,;q^4)_{n+1}} $$

appears in the "Lost" Notebook [6, p. 8]. It is connected to $A(q)$ by [1, (3.26)]

$$ \lambda(q) = A(q^2) + q(-q^2\,;q^2)_\infty (-q^4\,;q^4)_\infty^2 (q^8\,;q^8)_\infty. $$

Adding equations (4) and (7) gives

$$ V_1(q) = A(q^2) + q(-q^2\,;q^2)_\infty (-q^4\,;q^4)_\infty^2 (q^8\,;q^8)_\infty, $$

which implies that $\lambda(q) = V_1(q)$.

https://doi.org/10.4153/CMB-2007-028-9 Published online by Cambridge University Press

Analogous to [1, (3.26)] is the equation

$$\kappa(q) = 2q\beta(q^2) + (-q^2;q^4)^4_\infty(q^8;q^8)_\infty,$$

where

$$\kappa(q) = \sum_{n=0}^{\infty} \frac{q^n(-1)_{2n}}{(-q^4;q^4)_n}.$$

Adding equations (3) and (6) gives

$$V_0(q) = 2qB(q^2) + (-q^2;q^4)^4_\infty(q^8;q^8)_\infty,$$

which implies that $\kappa(q) = V_0(q)$.

3 Concluding Remarks

In my many unsuccessful attempts to establish the relation between these two families of mock theta functions I derived the following q-series for $A(q)$, $B(q)$, $V_0(q)$ and $V_1(q)$:

$$A(q) = \frac{(-q^2;q^2)^2_\infty}{(q;q^2)^2_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+1}(-q^2;q^2)^2_n}{(q^2;q^2)_n(-q;q^2)_{n+1}},$$

$$B(q) = \frac{(-q^2;q^2)^2_\infty}{(q;q^2)^2_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n}(-q;q^2)^2_n}{(q^4;q^4)_n},$$

$$V_0(q) = -1 + 2\sum_{n=0}^{\infty} \frac{q^n(-q;q^4)_n}{(q^3;q^4)_n},$$

$$V_1(q) = \sum_{n=0}^{\infty} \frac{q^{n+1}(-q^3;q^4)_n}{(q^3;q^4)_{n+1}}.$$

I also encountered the false theta function[1]

$$\sum_{n=0}^{\infty} \frac{q^n(q)_{2n}}{(q^4;q^4)_n} = \sum_{n=0}^{\infty} \frac{q^n(q;q^2)_n}{(-q^2;q^2)_n} = \sum_{n=0}^{\infty}(-1)^{n(n-1)/2}q^{n(n+1)/2}.$$

Acknowledgements The author is grateful to Bruce Berndt for pointing out equation [3, (III.9)]. Support by the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged.

[1] A false theta function differs from an ordinary theta function in the signs of some of its q-series terms. Such functions were studied by L. J. Rogers [7, pp. 332–334]. (See also [8, p. 56].)

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https://doi.org/10.4153/CMB-2007-028-9 Published online by Cambridge University Press