Canad. Math. Bull. Vol. **14** (1), 1971

AN EMBEDDING THEOREM FOR SEPARABLE LOCALLY CONVEX SPACES

BY

ROBERT H. LOHMAN

A well-known embedding theorem of Banach and Mazur [1, p. 185] states that every separable Banach space is isometrically isomorphic to a subspace of $C[0, 1]$, establishing $C[0, 1]$ as a universal separable Banach space. The embedding theorem one encounters in a course in topological vector spaces states that every Hausdorff locally convex space (l.c.s.) is topologically isomorphic to a subspace of a product of Banach spaces. The purpose of this note is to show that, with a modification of the usual embedding techniques, one can obtain a universal separable Hausdorff l.c.s. and hence prove an analogue of the Banach–Mazur theorem in a more general setting. While our result is probably known to some, apparently no mention of it has been made in the literature. We feel that the result is of sufficient interest to warrant its mention.

Let C denote a product of c copies of $C[0, 1]$ with the product topology, where c denotes the cardinal number of the continuum. Then C is a separable [3] l.c.s.

LEMMA. *Let E be a separable l.c.s. and $\mathfrak{u}$ a local base for E consisting of closed, convex neighborhoods of* 0. *Then* card $\mathfrak{u} \leq c$.

Proof. Let D be a countable dense set in E. If $U \in \mathfrak{u}$, $D \cap (\text{int } U)$ is dense in $\text{int } U$. U is convex implies $\text{int } U$ is dense in $\overline{U}$ [2, p. 110]. Since U is closed, $D \cap (\text{int } U)$ is dense in U. Letting $\mathfrak{M} = \{D \cap (\text{int } U) : U \in \mathfrak{u}\}$, we have $\mathfrak{u} = \{\overline{M} : M \in \mathfrak{M}\}$. Since D is countable, card $\mathfrak{u} \leq$ card $\mathfrak{M} \leq c$.

THEOREM. *Every separable Hausdorff l.c.s. E is topologically isomorphic to a subspace of C.*

Proof. Let $\mathfrak{u}$ be a local base for E consisting of closed, convex and circled neighborhoods of 0. By the lemma, card $\mathfrak{u} \leq c$. For each $U \in \mathfrak{u}$ let p_U denote the gauge functional of U. We let E_U denote E with the p_U-topology, $F_U = p_U^{-1}(0)$, $G_U = E_U/F_U$ with the quotient topology and let $\hat{G}_U$ denote the completion of G_U. Since E is separable, $\hat{G}_U$ is a separable Banach space. By the Banach–Mazur theorem, there is an into isometric isomorphism $R_U : \hat{G}_U \to C[0, 1]$. Let $I_U : E \to E_U$, $\pi_U : E_U \to G_U$ and $J_U : G_U \to \hat{G}_U$ denote the identity, quotient, and inclusion maps, respectively.

We denote by $C_{\mathfrak{u}}$ a product of card $\mathfrak{u}$ copies of $C[0, 1]$ with the product topology. Define $R : E \to C_{\mathfrak{u}}$ by

$$R(x)_U = R_U \circ J_U \circ \pi_U \circ I_U(x)$$

https://doi.org/10.4153/CMB-1971-023-1 Published online by Cambridge University Press

for all $x \in E$ and $U \in \mathfrak{u}$. Then R is a continuous algebraic isomorphism. To show R is relatively open it suffices [2, p. 46] to show that for each neighborhood V of 0 in E there exist a $U \in \mathfrak{u}$ and a neighborhood W of 0 in $C[0, 1]$ such that

$$(R_U \circ J_U \circ \pi_U \circ I_U)^{-1}(W) \subset V.$$

We may assume $V \in \mathfrak{u}$. Since R_V is relatively open, there is a neighborhood W of 0 in $C[0, 1]$ such that

$$W \cap R_V(\hat{G}_V) \subset R_V(\overline{J_V \circ \pi_V(4^{-1}V)}).$$

It is then easily checked that $(R_V \circ J_V \circ \pi_V \circ I_V)^{-1}(W) \subset V$. Regarding $C_{\mathfrak{u}}$ as a subspace of C, the proof is complete.

REFERENCES

1. S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932.

2. J. L. Kelley and I. Namioka, *Linear topological spaces*, Princeton Univ. Press, Princeton, N.J., 1963.

3. E. S. Pondiczery, *Power problems in abstract spaces*, Duke Math. J. **11** (1944), 835–837.

KENT STATE UNIVERSITY,
KENT, OHIO

https://doi.org/10.4153/CMB-1971-023-1 Published online by Cambridge University Press