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SAMELSON PRODUCTS IN p-REGULAR SO($2n$) AND ITS HOMOTOPY NORMALITY

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Abstract. A Lie group is called p-regular if it has the p-local homotopy type of a product of spheres. (Non)triviality of the Samelson products of the inclusions of the factor spheres into p-regular $\mathrm{SO}(2n)_{(p)}$ is determined, which completes the list of (non)triviality of such Samelson products in p-regular simple Lie groups. As an application, we determine the homotopy normality of the inclusion $\mathrm{SO}(2n-1) \to \mathrm{SO}(2n)$ in the sense of James at any prime p.

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1. Introduction and statement of the results. Let G be a compact connected Lie group. By the classical result of Hopf, it is well known that there is a rational homotopy equivalence

$$G \simeq_{(0)} S^{2n_1-1} \times \cdots \times S^{2n_\ell-1},$$

where $n_1 \leq \cdots \leq n_\ell$. The sequence $n_1 \leq \cdots \leq n_\ell$ is called the type of G. Here is the list of the types of simple Lie groups.

SU(n)	$2, 3, \ldots, n$	G_2	2, 6
SO($2n+1$)	$2, 4, \ldots, 2n$	F_4	2, 6, 8, 12
Sp(n)	$2, 4, \ldots, 2n$	E_6	2, 5, 6, 8, 9, 12
SO($2n$)	$2, 4, \ldots, 2n-2, n$	E_7	2, 6, 8, 10, 12, 14, 18
		E_8	2, 8, 12, 14, 18, 20, 24, 30

Serre generalizes the above rational homotopy equivalence to a p-local homotopy equivalence such that when G is semi-simple and $G_{(p)}$ is simply connected, there is a p-local homotopy equivalence

$$G \simeq_{(p)} S^{2n_1-1} \times \cdots \times S^{2n_\ell-1} \tag{1}$$

if and only if $p \geq n_\ell$, in which case G is called p-regular. In this paper, we are interested in the standard multiplicative structure of the p-localization $G_{(p)}$ when G is p-regular, and then we assume that G is a simple Lie group in the above table and is p-regular throughout this section. Recall that for a homotopy associative H-space X with inverse and maps $\alpha\colon A \to X, \beta\colon B \to X$, the correspondence

$$A \wedge B \to X, \quad (x, y) \mapsto \alpha(x)\beta(y)\alpha(x)^{-1}\beta(y)^{-1}$$

is called the Samelson product of α, β in X and is denoted by $\langle\alpha, \beta\rangle$. One easily sees that in investigating the multiplicative structure of $G_{(p)}$, the Samelson products $\langle\epsilon_i, \epsilon_j\rangle$ play the fundamental role as in [**9**], where ϵ_i is the inclusion $S^{2n_i-1} \to S^{2n_1-1}_{(p)} \times \cdots \times S^{2n_\ell-1}_{(p)} \simeq G_{(p)}$ into the ith factor. So, it is our task to determine (non)triviality of these Samelson products. In this direction, Bott [**2**] studied the order of a certain class of Samelson products in SU(n) and Sp(n), for example.

We here make a remark on the choice of ϵ_i which depends on the p-local homotopy equivalence (1). Recall from [**14**, Theorem 13.4] that

$$\pi_*(S^{2m-1}_{(p)}) = 0 \quad \text{for} \quad 2m-1 < * < 2m+2p-4. \tag{2}$$

Then, we see that $\pi_{2n_i-1}(G_{(p)})$ is a free $\mathbb{Z}_{(p)}$-module for all i, and so $\pi_{2n_i-1}(G_{(p)}) \cong \mathbb{Z}_{(p)}$ for all i and $G \neq \mathrm{SO}(2n)$ since the entries of the type are distinct for $G \neq \mathrm{SO}(2n)$ as in the above table. Hence, for $G \neq \mathrm{SO}(2n)$, we may choose any generator of $\pi_{2n_i-1}(G_{(p)}) \cong \mathbb{Z}_{(p)}$ as ϵ_i. For $G = \mathrm{SO}(2n)$, we will make an explicit choice of ϵ_i below.

We first consider the Samelson products $\langle\epsilon_i, \epsilon_j\rangle$ in $G_{(p)}$ when G is the classical group except for SO($2n$).

THEOREM 1.1. *Let G be the p-regular classical group except for* SO($2n$)*, and let ϵ_i be a generator of $\pi_{2n_i-1}(G_{(p)}) \cong \mathbb{Z}_{(p)}$ for the type $\{n_1, \ldots, n_\ell\}$ of G. Then,*

$$\langle\epsilon_i, \epsilon_j\rangle \neq 0 \quad \textit{if and only if} \quad n_i + n_j > p.$$

Proof. If $G = \mathrm{SU}(n), \mathrm{Sp}(n)$, non-triviality of the Samelson products follows from the result of Bott [**2**] and triviality follows from the fact that $\pi_{2*}(G_{(p)}) = 0$ for $* < p$ which is deduced from (2). Since there is a homotopy equivalence as loop spaces $\mathrm{Sp}(n)_{(p)} \simeq \mathrm{SO}(2n+1)_{(p)}$ due to Friedlander [**3**], the case of $\mathrm{SO}(2n+1)_{(p)}$ is the same as $\mathrm{Sp}(n)_{(p)}$. □

We next consider the Samelson products $\langle\epsilon_i, \epsilon_j\rangle$ in $G_{(p)}$ when G is the exceptional Lie group. Some of these Samelson products are calculated in [**5,9**], and (non)triviality of all these Samelson products is determined in [**6**] as follows.

THEOREM 1.2 ([**6**]). *Let G be a p-regular compact connected exceptional simple Lie group, and let ϵ_i be a generator of $\pi_{2n_i-1}(G_{(p)}) \cong \mathbb{Z}_{(p)}$ for the type $\{n_1, \ldots, n_\ell\}$ of G. Then,*

$$\langle\epsilon_i, \epsilon_j\rangle \neq 0 \quad \textit{if and only if} \quad n_i + n_j = n_k + p - 1 \textit{ for some } k.$$

Thus, the only remaining case is SO($2n$). The purpose of this paper is to show that a sufficient condition for non-triviality of the Samelson products $\langle\epsilon_i, \epsilon_j\rangle$ in $G_{(p)}$ (Lemma 2.1) used in [**4–6**, **10**] is actually a necessary and sufficient condition, and to apply it to determination of (non)triviality of all the Samelson products $\langle\epsilon_i, \epsilon_j\rangle$ in $\mathrm{SO}(2n)_{(p)}$. The difficulty of this case is caused by the middle dimensional sphere $S^{2n-1}_{(p)}$ in $\mathrm{SO}(2n)_{(p)}$ which vanishes by the inclusion $\mathrm{SO}(2n) \to \mathrm{SO}(2n+1)$. We choose the maps ϵ_i. Let ϵ_i

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be the composite

$$S^{4i-1} \to \mathrm{SO}(2n-1)_{(p)} \xrightarrow{\mathrm{incl}} \mathrm{SO}(2n)_{(p)},$$

for $i = 1, \dots, n-1$, where the first arrow is a generator of $\pi_{4i-1}(\mathrm{SO}(2n-1)_{(p)}) \cong \mathbb{Z}_{(p)}$. Let $\theta\colon S^{2n-1} \to \mathrm{SO}(2n)_{(p)}$ be the map corresponding to the adjoint of the fibre inclusion of the canonical homotopy fibre sequence

$$S^{2n} \to B\mathrm{SO}(2n) \to B\mathrm{SO}(2n+1).$$

There are only two results on Samelson products in SO(2n) involving θ: Mahowald [**12**] showed that the Samelson product $\langle\theta, \theta\rangle \in \pi_{4n-2}(\mathrm{SO}(2n))$ has order $(2n-1)!/8$ or $(2n-1)!/4$ according as n is even or odd. Hamanaka and Kono [**4**] showed that the Samelson product $\langle\epsilon_{\frac{p-1}{2}}, \theta\rangle \in \pi_{4n-2}(\mathrm{SO}(2n)_{(p)})$ is non-trivial when $p \le 2n-1$. Our main result determines (non)triviality of all Samelson products of ϵ_i and θ in p-regular SO(2n).

THEOREM 1.3. *Let ϵ_i, θ be the above maps into* $\mathrm{SO}(2n)_{(p)}$ *for p-regular* SO(2n)*. All non-trivial Samelson products of ϵ_i, θ in* $\mathrm{SO}(2n)_{(p)}$ *are*

$$\langle\epsilon_i, \epsilon_j\rangle \quad \textit{for} \quad 2i+2j > p \quad \textit{and} \quad \langle\epsilon_{n-1}, \theta\rangle = \langle\theta, \epsilon_{n-1}\rangle,\ \langle\theta, \theta\rangle \quad \textit{for} \quad p = 2n-1.$$

Recall that an H-map $f\colon X \to Y$ between homotopy associative H-spaces with inverse is homotopy normal in the sense of James [**7**] if the Samelson product $\langle f, 1_Y\rangle$ can be compressed to X through f up to homotopy. This is a generalization of the inclusion of a normal subgroup. James proved that O(n) is not homotopy normal in O($n+1$) when $n \ge 2$ using the mod 2 cohomology. His proof implies that the 2-localization $\mathrm{SO}(n)_{(2)}$ is not homotopy normal in $\mathrm{SO}(n+1)_{(2)}$ when $n \ge 2$. As an application of Theorem 1.3 we will prove:

THEOREM 1.4. *The inclusion $\iota_{(p)}$:* $\mathrm{SO}(2n-1)_{(p)} \to \mathrm{SO}(2n)_{(p)}$ *is homotopy normal if and only if $p > 2n-1$.*

For $p > 2n-1$, we can prove the following stronger result.

THEOREM 1.5. *For $p > 2n-1$, the map $\iota_{(p)} \cdot \theta$:* $\mathrm{SO}(2n-1)_{(p)} \times S^{2n-1}_{(p)} \to \mathrm{SO}(2n)_{(p)}$ *is an H-equivalence, where $S^{2n-1}_{(p)}$ is a homotopy associative and homotopy commutative H-space.*

Note that we do not need to assume that SO(2$n-1$) is p-regular in the last two theorems.

2. Detecting Samelson products by the Steenrod operations.

Let G be a p-torsion free connected finite loop space of type $n_1 \le \cdots \le n_\ell$ throughout this section where the type of a finite loop space is similarly defined. We set notation for G. Since G is p-torsion free, we have

$$H^*(BG_{(p)}; \mathbb{Z}/p) = \mathbb{Z}/p[x_1, \dots, x_\ell], \qquad |x_i| = 2n_i.$$

We fix this presentation of the mod p co-homology of $BG_{(p)}$. Note that

$$H^*(G_{(p)}; \mathbb{Z}/p) = \Lambda(e_1, \dots, e_\ell)$$

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for the suspension e_i of x_i. For each i, we take a non-trivial element $\epsilon_i \in \pi_{2n_i-1}(G_{(p)})$ which is not divisible by non-units in $\mathbb{Z}_{(p)}$ such that

$$(\Sigma\epsilon_i)^* \circ \iota_1^*(x_j) = \begin{cases} h_i \Sigma u_{2n_i-1} & i = j \\ 0 & i \neq j \end{cases}, \tag{3}$$

for some $h_i \in \mathbb{Z}_{(p)}$, where $\iota_1\colon \Sigma G_{(p)} \to BG_{(p)}$ is the canonical map and u_k is a generator of $H^k(S^k; \mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}$. We note that $G_{(p)}$ is a product of spheres if and only if $h_1, \ldots, h_\ell$ are units. The following lemma is first used in [**10**] and is the main tool in the proof of Theorem 1.2 given in [**6**]. Here, we reproduce the proof for completeness of the present paper.

LEMMA 2.1 ([**10**, Proof of Theorem 1.1]). *Suppose that h_i and h_j are units in $\mathbb{Z}_{(p)}$. If $\mathcal{P}^1 x_k$ is decomposable and includes the term cx_ix_j ($c \neq 0$), the Samelson product $\langle \epsilon_i, \epsilon_j \rangle$ is non-trivial.*

Proof. Suppose $\langle \epsilon_i, \epsilon_j \rangle = 0$ under the assumption that $\mathcal{P}^1 x_k$ includes the term cx_ix_j ($c \neq 0$). Let $\bar{\epsilon}_m\colon S^{2n_m} \to BG_{(p)}$ be the adjoint of ϵ_m. Then, by (3), we have $\bar{\epsilon}_m^*(x_m) = h_m u_{2m}$. By adjointness of Samelson products and Whitehead products, the Whitehead product $[\bar{\epsilon}_i, \bar{\epsilon}_j]$ in $BG_{(p)}$ is trivial, and then there is a map $\mu\colon S^{2n_i} \times S^{2n_j} \to BG_{(p)}$ satisfying $\mu|_{S^{2n_i} \vee S^{2n_j}} = \bar{\epsilon}_i \vee \bar{\epsilon}_j$. So we get $\mu^*(x_i) = h_i(u_{2n_i} \otimes 1)$ and $\mu^*(x_j) = h_j(1 \otimes u_{2n_i})$, and hence

$$ch_ih_ju_{2n_i} \otimes u_{2n_j} = \mu^*(cx_ix_j) = \mu^*(\mathcal{P}^1 x_k) = \mathcal{P}^1 \mu^*(x_k) = 0,$$

where the second and the last equality follows from the decomposability of $\mathcal{P}^1 x_k$ and triviality of $\mathcal{P}^1$ on $H^*(S^{2n_i} \times S^{2n_j}; \mathbb{Z}/p)$, respectively. This is a contradiction to $ch_ih_j \neq 0$. □

In this lemma, the assumption on the decomposability of $\mathcal{P}^1 x_k$ cannot be removed. Here is a counterexample.

EXAMPLE 2.2. We consider SU(4) at the prime 3. Recall that $H^*(B\mathrm{SU}(4); \mathbb{Z}/3) = \mathbb{Z}/3[c_2, c_3, c_4]$, where c_i denotes the ith universal Chern class. By inspection, we have

$$\mathcal{P}^1 c_2 = c_2^2 + c_4.$$

For a degree reason, the inclusion $\epsilon_1\colon S^3 = \mathrm{SU}(2) \to \mathrm{SU}(4)$ satisfies $(\Sigma\epsilon_1)^* \circ \iota_1^*(c_2) = \Sigma u_3$ as in (3), but the Samelson product $\langle \epsilon_1, \epsilon_1 \rangle$ is trivial since SU(2) commutes up to homotopy with itself in SU(4).

We elaborate Lemma 2.1 to prove that its converse is true when $G_{(p)}$ is a product of spheres. The following lemma is useful to detect the non-triviality of a Samelson product when $G_{(p)}$ (not necessarily p-regular) is decomposed into a product of a sphere and some space. The proof is independent of Lemma 2.1.

LEMMA 2.3. *For integers $1 \leq i, j, k \leq \ell$, suppose that there is a map $\pi_k\colon G_{(p)} \to S_{(p)}^{2n_k-1}$ such that $\pi_k^*(u_{2n_k-1}) = e_k$, h_i and h_j are units in $\mathbb{Z}_{(p)}$, and $n_i + n_j = n_k + p - 1$. Then, $\pi_k \circ \langle \epsilon_i, \epsilon_j \rangle \neq 0$ if and only if $\mathcal{P}^1 x_k$ includes the term cx_ix_j with $c \neq 0$.*

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Proof. We prove both implications simultaneously. We may suppose that $h_i = h_j = h_k = 1$. Let $P^2G_{(p)}$ be the projective plane of $G_{(p)}$, i.e. there is a cofibre sequence

$$\Sigma G_{(p)} \wedge G_{(p)} \xrightarrow{H} \Sigma G_{(p)} \xrightarrow{\rho_1} P^2G_{(p)}, \tag{4}$$

where H is the Hopf construction. By [**11**, Section 4], the canonical map $\iota_1 \colon \Sigma G_{(p)} \to BG$ extends to a map $\iota_2 \colon P^2G \to BG$, i.e. $\iota_2 \circ \rho_1 = \iota_1$. Put $\bar{x}_i = \iota_2^*(x_i)$. Then, we have $\rho_1^*(\bar{x}_i) = \Sigma e_i$. By [**11**, Section 3], we also have $\delta_1^*(\Sigma^2 e_i \otimes e_j) = \bar{x}_i\bar{x}_j$ for the connecting map $\delta_1 \colon P^2G_{(p)} \to \Sigma^2 G_{(p)} \wedge G_{(p)}$ of the cofibre sequence (4). Consider the map

$$\Phi = \Sigma\langle \epsilon_i, \epsilon_j\rangle - [\Sigma\epsilon_i, \Sigma\epsilon_j] \colon \Sigma S^{2n_i-1} \wedge S^{2n_j-1} \to \Sigma G_{(p)},$$

where $[-,-]$ denotes the Whitehead product. Note that Φ induces a trivial map on mod p cohomology since $H^*(G_{(p)};\mathbb{Z}/p)$ is primitively generated and the Whitehead product becomes trivial after suspending. The map Φ is connected with the Hopf construction H through the map constructed by Morisugi [**13**, Theorem 5.1] such that there is a map $\xi \colon S^{2n_i-1} \wedge S^{2n_j-1} \to G_{(p)} \wedge G_{(p)}$ satisfying

$$\Phi = H \circ \Sigma\xi \quad \text{and} \quad \xi^*(e_s \otimes e_t) = \begin{cases} u_{2n_i-1} \otimes u_{2n_j-1} & (s,t) = (i,j), (j,i) \\ 0 & \text{otherwise.} \end{cases}$$

Then, we get a homotopy commutative diagram

$$\begin{array}{ccccc}
\Sigma G_{(p)} & \xrightarrow{\rho_2} & C_\Phi & \xrightarrow{\delta_2} & \Sigma^2 S^{2n_i-1} \wedge S^{2n_j-1} \\
\| & & \downarrow{\scriptstyle \lambda_1} & & \downarrow{\scriptstyle \Sigma^2\xi} \\
\Sigma G_{(p)} & \xrightarrow{\rho_1} & P^2G_{(p)} & \xrightarrow{\delta_1} & \Sigma^2 G_{(p)} \wedge G_{(p)}
\end{array}$$

whose rows are homotopy cofibrations, implying that

$$\rho_2^* \circ \lambda_1^*(\bar{x}_k) = \Sigma e_k \quad \text{and} \quad \lambda_1^*(\bar{x}_s\bar{x}_t) = \begin{cases} \delta_2^*(\Sigma^2 u_{2n_i-1} \otimes u_{2n_j-1}) & (s,t) = (i,j), (j,i) \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

where $\delta_2^*(\Sigma^2 u_{2n_i-1} \otimes u_{2n_j-1})$ is non-trivial element since Φ is trivial on mod p cohomology. We have

$$\pi_k \circ \langle \epsilon_i, \epsilon_j\rangle = c\alpha_1 \quad (c \in \mathbb{Z}/p),$$

where α_1 is a generator of $\pi_{2n_k+2p-4}(S^{2n_k-1}) \cong \mathbb{Z}/p$ [**14**, Proposition 13.6]. Note that $\pi_k \circ \langle \epsilon_i, \epsilon_j\rangle$ is nontrivial if and only if $c \neq 0$. Then, for the map

$$\widehat{\Phi} = c\Sigma\alpha_1 - [\Sigma\pi_k \circ \epsilon_i, \Sigma\pi_k \circ \epsilon_j] \colon \Sigma S^{2n_i-1} \wedge S^{2n_j-1} \to \Sigma S^{2n_k-1}_{(p)},$$

there is a homotopy commutative diagram

$$\begin{array}{ccccc}
\Sigma G_{(p)} & \xrightarrow{\rho_2} & C_\Phi & \xrightarrow{\delta_2} & \Sigma^2 S^{2n_i-1} \wedge S^{2n_j-1} \\
\downarrow{\scriptstyle \Sigma\pi_k} & & \downarrow{\scriptstyle \lambda_2} & & \| \\
\Sigma S^{2n_k-1} & \xrightarrow{\rho_3} & C_{\widehat{\Phi}} & \xrightarrow{\delta_3} & \Sigma^2 S^{2n_i-1} \wedge S^{2n_j-1}
\end{array}$$

whose rows are homotopy cofibrations. Since α_1 is detected by the Steenrod operation $\mathcal{P}^1$ and $\Sigma\hat{\Phi} = c\Sigma^2\alpha_1$, the mod p cohomology of $C_{\hat{\Phi}}$ is given by

$$\widetilde{H}^*(C_{\hat{\Phi}}; \mathbb{Z}/p) = \langle a_{2n_k}, a_{2n_i+2n_j}\rangle, \quad \mathcal{P}^1 a_{2n_k} = c a_{2n_i+2n_j}$$

such that $\delta_3^*(\Sigma^2 u_{2n_i-1} \otimes u_{2n_j-1}) = a_{2n_i+2n_j}$ and $\rho_3^*(a_{2n_k}) = \Sigma u_{2n_k-1}$. Then, by (5), we get $\rho_2^* \circ \lambda_2^*(a_{2n_k}) = \Sigma e_k = \rho_2^* \circ \lambda_1^*(\bar{x}_k)$. By the homotopy cofibre sequence $\Sigma G_{(p)} \xrightarrow{\rho_2} C_\Phi \xrightarrow{\delta_2} \Sigma^2 S^{2n_i-1} \wedge S^{2n_j-1}$, one can see that the inclusion $\rho_2\colon \Sigma G_{(p)} \to C_\Phi$ is injective in the mod p cohomology of dimension $2n_k$, and then we obtain $\lambda_2^*(a_{2n_k}) = \lambda_1^*(\bar{x}_k)$. Now consider an element $\mathcal{P}^1 x_k$ in $H^*(BG_{(p)}; \mathbb{Z}/p)$, which is expressed as a polynomial of $x_1, \ldots, x_\ell$. Denote the coefficient of the term $x_i x_j$ in $\mathcal{P}^1 x_k$ by d. Then, we have

$$\lambda_1^*(\mathcal{P}^1 \bar{x}_k) = d\delta_2^*(\Sigma^2 u_{2n_i-1} \otimes u_{2n_j-1}) + \text{ a linear combination of } \lambda_1^*(\bar{x}_1), \ldots, \lambda_1^*(\bar{x}_\ell)$$

by (5). On the other hand, we also have

$$\lambda_1^*(\mathcal{P}^1 \bar{x}_k) = \mathcal{P}^1 \lambda_1^*(\bar{x}_k) = \mathcal{P}^1 \lambda_2^*(a_{2n_k}) = c\delta_2^*(\Sigma^2 u_{2n_i-1} \otimes u_{2n_j-1}).$$

Since $\delta_2^*(\Sigma^2 u_{2n_i-1} \otimes u_{2n_j-1})$ is non-trivial and is not contained in the span of $\lambda_1^*(\bar{x}_1), \ldots, \lambda_1^*(\bar{x}_\ell)$, we have $c = d$. Thus, $\mathcal{P}^1 x_k$ must include the term $c x_i x_j$. Therefore, we have established the lemma. □

THEOREM 2.4. *Suppose $p \geq n_\ell - n_1 + 2$. Then, the Samelson product $\langle \epsilon_i, \epsilon_j\rangle$ in $G_{(p)}$ is non-trivial if and only if for some k, $\mathcal{P}^1 x_k$ includes the term $c x_i x_j$ with $c \neq 0$.*

Proof. By the result of Kumpel [**8**], we can choose each ϵ_i such as $h_i = 1$. Then, the composite

$$S^{2n_1-1} \times \cdots \times S^{2n_\ell-1} \xrightarrow{\epsilon_1 \times \cdots \times \epsilon_\ell} G_{(p)} \times \cdots \times G_{(p)} \to G_{(p)}$$

induces a p-local homotopy equivalence where the second map is the multiplication, and we identify $G_{(p)}$ with $S^{2n_1-1}_{(p)} \times \cdots \times S^{2n_\ell-1}_{(p)}$ by this p-local homotopy equivalence. Under this assumption, h_i is a unit of $\mathbb{Z}_{(p)}$ for any i. By this decomposition, we can find a projection $\pi_k\colon G_{(p)} \to S^{2n_i-1}_{(p)}$ such that $\pi_k^* u_{2n_i-1} = e_i$ for each i. By Lemma 2.3, if $\mathcal{P}^1 x_k$ includes the term $c x_i x_j$ with $c \neq 0$, then the Samelson product $\langle \epsilon_i, \epsilon_j\rangle$ in $G_{(p)}$ is non-trivial. As in [**9**], if $\langle \epsilon_i, \epsilon_j\rangle$ is non-trivial, then for some $1 \leq k \leq \ell$ we have $n_k + p - 1 = n_i + n_j$ and $\pi_k \circ \langle \epsilon_i, \epsilon_j\rangle$ is non-trivial. Again by Lemma 2.3, this implies that $\mathcal{P}^1 x_k$ includes the term $c x_i x_j$ with $c \neq 0$. □

3. Proofs of the results. Let p be an odd prime and $p_i, e_n \in H^*(B\mathrm{SO}(2n)_{(p)}; \mathbb{Z}/p)$ be the mod p reduction of the ith universal Pontrjagin class for $i = 1, \ldots, n-1$ and the Euler class, respectively. Then,

$$H^*(B\mathrm{SO}(2n)_{(p)}; \mathbb{Z}/p) = \mathbb{Z}/p[p_1, \ldots, p_{n-1}, e_n]$$

and the maps ϵ_i and θ correspond to p_i and e_n, respectively, in the sense of (3). In particular, we take ϵ_i so that $h_i = 1$ for $i \leq \frac{p-1}{2}$ and θ so that $(\Sigma\theta)^* \circ \iota_1^*(e_n) = \Sigma u_{2n-1}$ and $(\Sigma\theta)^* \circ \iota_1^*(p_i) = 0$ for any i.

LEMMA 3.1. *The following statements hold.*

(1) *The element $\mathcal{P}^1 p_i$ does not include the quadratic term $ce_n p_j$ ($c \neq 0$) for any i and j.*
(2) *If $p = 2n-1$, the element $\mathcal{P}^1 p_1$ is decomposable and includes the term $(-1)^{\frac{p-1}{2}} e_n^2$.*

Proof. Since $p_i \in H^*(B\mathrm{SO}(2n)_{(p)}; \mathbb{Z}/p)$ is contained in the image from $H^*(B\mathrm{SO}(2n+1)_{(p)}; \mathbb{Z}/p)$, if a quadratic term of $\mathcal{P}^1 p_i$ includes e_n, it must be a multiple of e_n^2 and $i = n - \frac{p-1}{2} \geq 1$. Thus, the first statement holds. Recall that for a maximal torus T of SO$(2n)$ and the natural map $\iota\colon BT_{(p)} \to B\mathrm{SO}(2n)_{(p)}$, we have

$$H^*(BT_{(p)}; \mathbb{Z}/p) = \mathbb{Z}/p[t_1, \ldots, t_n], \quad |t_i| = 2$$

such that $\iota^*(p_i)$ is the ith elementary symmetric polynomial in $t_1^2, \ldots, t_n^2$ and $\iota^*(e_n) = t_1 \cdots t_n$. In particular, ι is injective in the mod p cohomology. Suppose $p = 2n-1$. We have

$$\iota^*(\mathcal{P}^1 p_1) = \mathcal{P}^1(t_1^2 + \cdots + t_n^2) = 2((t_1^2)^{\frac{p+1}{2}} + \cdots + (t_n^2)^{\frac{p+1}{2}}).$$

Then, we obtain

$$\mathcal{P}^1 p_1 \equiv (-1)^{\frac{p-1}{2}} e_n^2 \mod (p_1, \ldots, p_{n-1})^2$$

by the Newton formula. Therefore, the second statement holds. □

LEMMA 3.2. *The element $\mathcal{P}^1 e_n$ is decomposable and the following congruence hold:*

$$\mathcal{P}^1 e_n \equiv (-1)^{\frac{p-1}{2}} \frac{p-1}{2} e_n p_{\frac{p-1}{2}} \mod (p_1, \ldots, p_{n-1})^2.$$

Proof. We set $\iota\colon BT_{(p)} \to B\mathrm{SO}(2n)_{(p)}$ as in the proof of Lemma 3.1. We have

$$\iota^*(\mathcal{P}^1 e_n) = \mathcal{P}^1 \iota^*(e_n) = \mathcal{P}^1(t_1 \cdots t_n) = t_1 \cdots t_n((t_1^2)^{\frac{p-1}{2}} + \cdots + (t_n^2)^{\frac{p-1}{2}}).$$

Then, the proof is completed by the Newton formula. □

Proof of Theorem 1.3 Assume $p > 2n-2$. Since the inclusion $\mathrm{SO}(2n-1)_{(p)} \to \mathrm{SO}(2n)_{(p)}$ has a left homotopy inverse, it follows from Theorem 1.1 that the Samelson product $\langle \epsilon_i, \epsilon_j \rangle$ is non-trivial if and only if $2i + 2j > p$. To detect the Samelson products $\langle \epsilon_i, \theta \rangle = \langle \theta, \epsilon_i \rangle$ and $\langle \theta, \theta \rangle$ by Theorem 2.4, we need the information about the quadratic terms of $\mathcal{P}^1 p_i$ and $\mathcal{P}^1 e_n$ including e_n. Now these informations have already been obtained in Lemma 3.1 and 3.2. Therefore, the proof of Theorem 1.3 is completed. □

Lemma 3.2 implies non-triviality of the Samelson product $\langle \epsilon_{\frac{p-1}{2}}, \theta \rangle$ not only when SO$(2n)$ is p-regular but also when SO$(2n)$ is not p-regular as follows.

COROLLARY 3.3. *The Samelson product $\langle \epsilon_{\frac{p-1}{2}}, \theta \rangle = \langle \theta, \epsilon_{\frac{p-1}{2}} \rangle$ in $\pi_{2n+2p-4}(\mathrm{SO}(2n)_{(p)})$ is non-trivial for any odd prime p. More precisely, the image of $\langle \epsilon_{\frac{p-1}{2}}, \theta \rangle$ under the homomorphism induced by the projection $\mathrm{SO}(2n)_{(p)} \to S^{2n-1}_{(p)}$ generates $\pi_{2n+2p-4}(S^{2n-1}_{(p)}) \cong \mathbb{Z}/p$.*

Proof. Note that, for the projection $\pi : \mathrm{SO}(2n)_{(p)} \to S^{2n-1}_{(p)}$, we have $(\Sigma\pi)^* \Sigma u_{2n-1} = \iota_1^*(e_n)$. Then, the corollary follows from Lemma 2.3 and 3.2. □

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We next prove Theorem 1.4. Let X be a homotopy associative H-space with inverse. For maps $\alpha\colon A\to X$ and $\beta\colon B\to X$, let $\{\alpha,\beta\}$ denote the composite

$$A\times B\xrightarrow{\alpha\times\beta}X\times X\to X,$$

where the last arrow is the commutator map. Then, for the projection $q\colon A\times B\to A\wedge B$, we have $q^*(\langle\alpha,\beta\rangle)=\{\alpha,\beta\}$ and the induced map $q^*\colon[A\wedge B,X]\to[A\times B,X]$ is injective. In particular, $\langle\alpha,\beta\rangle$ is trivial if and only if so is $\{\alpha,\beta\}$.

LEMMA 3.4 (cf. [**9**, Proposition 1]). *For maps $\varphi_i\colon A_i\to X$ $(i=1,2)$ and $\beta\colon B\to X$, if $\{\varphi_2,\beta\}$ is trivial, then*

$$\{\varphi_1\cdot\varphi_2,\beta\}=\{\varphi_1,\beta\}\circ\rho_2,$$

where $\rho_2\colon A_1\times A_2\times B\to A_1\times B$ denotes the projection.

Proof. In the group of the homotopy set $[A_1\times A_2\times B,X]$, we have

$$\{\varphi_1\cdot\varphi_2,\beta\}=[(\varphi_1\circ\pi_1)\cdot(\varphi_2\circ\pi_2),\beta\circ\pi],$$

where $\pi_i\colon A_1\times A_2\times B\to A_i$ for $i=1,2$ and $\pi\colon A_1\times A_2\times B\to B$ denote the projections and $[-,-]$ means the commutator. In a group G, we have

$$[xy,z]=x[y,z]x^{-1}[x,z],$$

for $x,y,z\in G$. Then, the proof is completed by $[\varphi_1\circ\pi_1,\beta\circ\pi]=\{\varphi_1,\beta\}\circ\rho_2$. □

Proof of Theorem 1.4 Let $\iota\colon\mathrm{SO}(2n-1)\to\mathrm{SO}(2n)$ denote the inclusion and $\pi\colon\mathrm{SO}(2n)\to S^{2n-1}$ the projection. For $p=2$ and $n\geq 2$, as remarked in Section 1, the 2-localization $\iota_{(2)}\colon\mathrm{SO}(2n-1)_{(2)}\to\mathrm{SO}(2n)_{(2)}$ is not homotopy normal by the argument by James [**7**, Proof of Theorem (3.1)].

If $2<p\leq 2n-1$, then the Samelson product

$$\pi_{(p)}\circ\langle\iota_{(p)},1_{\mathrm{SO}(2n)_{(p)}}\rangle\circ(\epsilon_{\frac{p-1}{2}}\wedge\theta)=\pi_{(p)}\circ\langle\epsilon_{\frac{p-1}{2}},\theta\rangle$$

is non-trivial in $\pi_{2n+2p-4}(S^{2n-1}_{(p)})$ by Corollary 3.3. This implies that $\iota_{(p)}$ is not homotopy normal.

Suppose $p>2n-1$. Note that the identity map of $\mathrm{SO}(2n)_{(p)}$ is identified with the map $\iota_{(p)}\cdot\theta\colon\mathrm{SO}(2n-1)_{(p)}\times S^{2n-1}_{(p)}\to\mathrm{SO}(2n)_{(p)}$. Then, it follows from Lemma 3.4 that $\iota_{(p)}$ is homotopy normal if the Samelson product $\langle\iota_{(p)},\theta\rangle$ is trivial. Note also that $\iota_{(p)}$ is identified with the map $\epsilon_1\cdots\epsilon_{n-1}\colon S^3_{(p)}\times\cdots\times S^{4n-5}_{(p)}\to\mathrm{SO}(2n-1)_{(p)}$. Then, it is sufficient to show that $\{\epsilon_1\cdots\epsilon_{n-1},\theta\}$ is trivial. By Lemma 3.4, this is equivalent to that $\langle\epsilon_i,\theta\rangle$ are trivial for all i. Thus, $\iota_{(p)}$ is homotopy normal by Theorem 1.3. □

We finally prove Theorem 1.5. Let X, Y be homotopy associative H-spaces with inverse. Recall that the H-deviation $d(f)$ of a map $f\colon X\to Y$ is defined by

$$d(f)\colon X\wedge X\to Y,\quad (x_1,x_2)\mapsto f(x_1x_2)f(x_2)^{-1}f(x_1)^{-1}.$$

By definition, f is an H-map if and only if the H-deviation $d(f)$ is trivial.

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LEMMA 3.5. *Let X_1, X_2, Y be homotopy associative H-spaces with inverse, and $\lambda_i\colon X_i \to Y$ be H-maps for $i = 1, 2$. Then, the map $\lambda_1 \cdot \lambda_2\colon X_1 \times X_2 \to Y$ is an H-map if and only if the Samelson product $\langle \lambda_1, \lambda_2 \rangle$ is trivial.*

Proof. For $x_i, x_i' \in X_i$ ($i = 1, 2$), we have

$$\begin{aligned} d(\lambda_1 \cdot \lambda_2)(x_1, x_2, x_1', x_2') &\simeq \lambda_1(x_1x_1')\lambda_2(x_2x_2')\lambda_2(x_2')^{-1}\lambda_1(x_1')^{-1}\lambda_2(x_2)^{-1}\lambda_1(x_1)^{-1} \\ &\simeq \lambda_1(x_1)(\langle \lambda_1, \lambda_2 \rangle(x_1', x_2))\lambda_1(x_1)^{-1} \end{aligned}$$

since λ_1, λ_2 are H-maps. Then, since λ_1 is an H-map, $d(\lambda_1 \cdot \lambda_2)$ is trivial if and only if so is $\langle \lambda_1, \lambda_2 \rangle$, completing the proof. □

Proof of Theorem 1.5. Obviously, the map $\iota_{(p)} \cdot \theta$ is a homotopy equivalence, so it remains to show that it is an H-map. By definition, we have $d(\theta) \in \pi_{4n-2}(\mathrm{SO}(2n)_{(p)})$, and then by [**14**, Proposition 13.6] and $p > 2n - 1$, $d(\theta)$ is trivial, implying that θ is an H-map. The inclusion $\iota_{(p)}$ is clearly an H-map, and in the proof of Theorem 1.4 the Samelson product $\langle \iota_{(p)}, \theta \rangle$ is shown to be trivial for $p > 2n - 1$. Thus, by Lemma 3.5, $\iota_{(p)} \cdot \theta$ is an H-map. Note that we have not fixed an H-structure of $S^{2n-1}_{(p)}$. There is a one to one correspondence between H-structures on $S^{2n-1}_{(p)}$ and $\pi_{4n-2}(S^{2n-1}_{(p)})$. By [**14**, Proposition 13.6] and $p > 2n - 1$, $\pi_{4n-2}(S^{2n-1}_{(p)}) = 0$, so there is only one H-structure on $S^{2n-1}_{(p)}$. By [**1**], $S^{2n-1}_{(p)}$ has a homotopy associative and homotopy commutative H-structure. Then, $S^{2n-1}_{(p)}$ must be a homotopy associative and homotopy commutative H-space. □

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REFERENCES

1. J. F. Adams, The sphere, considered as an H-space mod p, *Quart. J. Math.* **12** (1961), 52–60.

2. R. Bott, A note on the Samelson products in the classical groups, *Comment. Math. Helv.* **34** (1960), 249–256.

3. E. M. Friedlander, Exceptional isogenies and the classifying spaces of simple Lie groups, *Ann. Math.* **101** (1975), 510–520.

4. H. Hamanaka and A. Kono, A note on the Samelson products in $\pi_*(\mathrm{SO}(2n))$ and the group $[\mathrm{SO}(2n), \mathrm{SO}(2n)]$, *Topology Appl.* **154**(3) (2007), 567–572.

5. H. Hamanaka and A. Kono, A note on Samelson products and mod p cohomology of classifying spaces of the exceptional Lie groups, *Topology Appl.* **157**(2) (2010), 393–400.

6. S. Hasui, D. Kishimoto and A. Ohsita, Samelson products in p-regular exceptional Lie groups, *Topology Appl.* **178**(1) (2014), 17–29.

7. I. M. James, On homotopy theory of classical groups, *Ann. Acad. Brasil. Cienc.* **39** (1967), 39–44.

8. P. G. Kumpel, Mod p-equivalences of mod p H-spaces, *Quart. J. Math.* **23** (1972), 173–178.

9. S. Kaji and D. Kishimoto, Homotopy nilpotency in p-regular loop spaces, *Math. Z.* **264**(1) (2010), 209–224.

10. A. Kono and H. Ōshima, Commutativity of the group of self homotopy classes of Lie groups, *Bull. London Math. Soc.* **36** (2004), 37–52.

11. J. Lin, H-spaces with finiteness conditions, in *Handbook of algebraic topology* (James I. M., Editors) (Elsevier, North-Holland, 1995), 1095–1141, Chapter 22.

12. M. Mahowald, A Samelson product in SO($2n$), *Bol. Soc. Math. Mexicana* **10** (1965), 80–83.

13. K. Morisugi, Hopf construction, Samelson products and suspension maps, *Contemporary Math.* **239** (1999), 225–238.

14. H. Toda, *Composition methods in homotopy groups of spheres*, Ann. of Math. Studies, vol. 49 (Princeton University Press, Princeton N.J., 1962).