Second Meeting, December 9th, 1892.

JOHN ALISON, Esq., M.A., F.R.S.E., President, in the Chair.

On certain results involving Areal and Trilinear co-ordinates.

By Professor A. H. ANGLIN.

We propose to obtain certain results involving areal and trilinear co-ordinates, by a uniform method of changing to Cartesian co-ordinates with two sides of the triangle of reference as axes.

Taking ABC as the triangle of reference, change to Cartesian co-ordinates with CA and CB as axes. Then, if $\bar{x}$, $\bar{y}$ denote the Cartesian, x, y, z the areal, and α, β, γ the trilinear co-ordinates of any point, we have at once

$$\bar{x} = bx, \qquad \bar{y} = ay;$$

and

$$\bar{x} = \alpha/\sin C, \quad \bar{y} = \beta/\sin C.$$

1. *To find the distance between two points.*

If r be the distance between two points whose areal co-ordinates are given, substituting in the usual expression in oblique Cartesians for the square of the distance, we get

$$r^2 = b^2(x_1 - x_2)^2 + a^2(y_1 - y_2)^2 + 2ab\,(x_1 - x_2)(y_1 - y_2)\cos C.$$

$$\therefore \quad -r^2 = c^2(x_1 - x_2)(y_1 - y_2) - a^2(y_1 - y_2)(x_1 - x_2 + y_1 - y_2)$$
$$- b^2(x_1 - x_2)(x_1 - x_2 + y_1 - y_2).$$

Thus, by the invariable relation $x + y + z = 1$ in areals, we have

$$-r^2 = a^2(y_1 - y_2)(z_1 - z_2) + b^2(z_1 - z_2)(x_1 - x_2) + c^2(x_1 - x_2)(y_1 - y_2) \ldots \quad (1).$$

The corresponding in trilinears may be obtained independently in like manner, or deduced from the foregoing, the result being

$$-r^2 = \{(\beta_1 - \beta_2)(\gamma_1 - \gamma_2)\sin A + \ldots\}/\sin A \sin B \sin C \quad \ldots \quad (1)'$$

The distance between two points may also be expressed in other interesting forms.

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Since $$(x_1-x_2)+(y_1-y_2)=-(z_1-z_2),$$
we have $$2(x_1-x_2)(y_1-y_2)=(z_1-z_2)^2-(x_1-x_2)^2-(y_1-y_2)^2.$$

Substituting in

$$r^2=b^2(x_1-x_2)^2+a^2(y_1-y_2)^2+(a^2+b^2-c^2)(x_1-x_2)(y_1-y_2),$$

we shall get

$$2r^2=(b^2+c^2-a^2)(x_1-x_2)^2+\text{two similar expressions},$$

or

$$r^2=bc\cos A(x_1-x_2)^2+ca\cos B(y_1-y_2)^2+ab\cos C(z_1-z_2)^2 \quad \ldots \quad (2).$$

The corresponding in trilinears will be found to be

$$r^2=\frac{\sin 2A(\alpha_1-\alpha_2)^2+\sin 2B(\beta_1-\beta_2)^2+\sin 2C(\gamma_1-\gamma_2)^2}{2\sin A\sin B\sin C} \quad \ldots \quad (2)'.$$

Further, since

$$x_1-x_2=(x_1y_2-x_2y_1)-(z_1x_2-z_2x_1)=Z-Y \text{ suppose},$$

with like equivalents for y_1-y_2 and z_1-z_2, substituting in (1) and (2) we shall get the additional forms

$$\begin{aligned} r^2 &= a^2(X-Y)(X-Z)+b^2(Y-Z)(Y-X)+c^2(Z-X)(Z-Y) \quad \ldots \quad (3)\\ &= bc.\cos A(Y-Z)^2+ca\cos B(Z-X)^2+ab\cos C(X-Y)^2\\ &= a^2X^2+b^2Y^2+c^2Z^2-2bcYZ\cos A-2caZX\cos B-2abXY\cos C, \end{aligned}$$

which are sometimes useful.

2. *To find the perpendicular distance of a point from a straight line.*

Let the equation to the line in areal co-ordinates be

$$lx+my+nz=0,$$

and x', y', z' the co-ordinates of the point.

Reducing the problem to the oblique Cartesian system, we have to find the perpendicular from the point (bx', ay') on the line

$$a(l-n)\bar{x}+b(m-n)\bar{y}+nab=0.$$

Now the perpendicular from (x', y') on the line $Ax+By+C=0$ in oblique Cartesians being

$$(Ax'+By'+C)\sin\omega/\sqrt{A^2+B^2-2AB\cos\omega},$$

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the required perpendicular

$$=\frac{ab\sin C\{(l-n)x'+(m-n)y'+n\}}{\sqrt{\{a^2(l-n)^2+b^2(m-n)^2-2ab(l-n)(m-n)\cos C\}}}$$

$$=\qquad 2(lx'+my'+nz')/d,$$

where

$$d^2=a^2l^2+b^2m^2+c^2n^2-2bcmn\cos A-\ \ldots$$
$$=(l-m)(l-n)a^2+(m-n)(m-l)b^2+(n-l)(n-m)c^2.$$

[The corresponding expression in Trilinears may be deduced from this, or obtained independently, as follows :—

If the point be (α', β', γ'). and the line $l\alpha+m\beta+n\gamma=0$, changing to Cartesians, we seek the perpendicular from the point ($\alpha'\operatorname{cosec}C$, $\beta'\operatorname{cosec}C$) on the line

$$(cl-an)\bar{x}+(cm-bn)\bar{y}+2n\Delta\operatorname{cosec}C=0,$$

which

$$=\frac{(cl-an)\alpha'+(cm-bn)\beta'+2n\Delta}{\sqrt{\{(cl-an)^2+(cm-bn)^2-2(cl-an)(cm-bn)\cos C\}}}$$

$$=\qquad (l\alpha'+m\beta'+n\gamma')/d,$$

where

$$d^2=l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C.]$$

3. *The perpendicular from a given point on the line joining two other given points may be noticed.*

The equation to the line joining the points (x_1, y_1, z_1), (x_2, y_2, z_2) being

$$\Sigma(y_1z_2-y_2z_1)x+(z_1x_2-z_2x_1)y+(x_1y_2-x_2y_1)z=0,$$

the perpendicular on it from the point (x_3, y_3, z_3) becomes

$$2\Delta\{(y_1z_2-y_2z_1)x_3+\ \ldots\ \}/d,$$

where

$$d^2=(X-Y)(X-Z)a^2+(Y-Z)(Y-X)b^2+(Z-X)(Z-Y)c^2.$$

Thus, by reference to the third expression for the distance between two points, we see that the perpendicular is

$$2\Delta(x_1y_2z_3)/d,$$

where d is the distance between the points (x_1, y_1, z_1), (x_2, y_2, z_2).

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In Trilinears, the corresponding expression will be found to be

$$abc(\alpha_1\beta_2\gamma_3)/4\Delta^2 d.$$

These results also follow directly from the consideration that, in Cartesians, the perpendicular is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y & 1 \end{vmatrix} \frac{\sin\omega}{d};$$

and on transformation to the other systems we readily obtain the above expressions.

4. *To find the area of a triangle in terms of the co-ordinates of its angular points.*

(1) Independently of the corresponding expression in Cartesians.

If Δ' denote the area, since twice area = side × perpendicular, we have by the foregoing

$$2\Delta' = 2\Delta(x_1y_2z_3) \cdot \frac{d}{d}$$

$$\therefore \qquad \Delta' = \Delta(x_1y_2z_3),$$

involving the areal co-ordinates; and

$$\Delta' = abc(\alpha_1\beta_2\gamma_3)/8\Delta^2,$$

involving the trilinear co-ordinates of the points.

(2) Directly from the expression in Cartesians.

We have

$$2\Delta' = \begin{vmatrix} \bar{x}_1 & \bar{y}_1 & 1 \\ \bar{x}_2 & \bar{y}_3 & 1 \\ \bar{x}_3 & \bar{y}_3 & 1 \end{vmatrix} \sin C = ab\sin C \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= 2\Delta \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}, \text{ since } x + y + z = 1.$$

Thus $\qquad \Delta' = \Delta(x_1y_2z_3).$

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[The corresponding in Trilinears may be deduced from this, or obtained independently thus :—

$$2\Delta' = \begin{vmatrix} \alpha_1 & \beta_1 & 1 \\ \alpha_2 & \beta_2 & 1 \\ \alpha_3 & \beta_3 & 1 \end{vmatrix} \frac{\sin C}{\sin^2 C}, \text{ by direct substitution ;}$$

$$\therefore \quad 2\Delta' \sin C \cdot 2\Delta = \begin{vmatrix} \alpha_1, & \beta_1 & a\alpha_1 + b\beta_1 + c\gamma_1 \\ \alpha_2, & \beta_2, & a\alpha_2 + b\beta_2 + c\gamma_2 \\ \alpha_3, & \beta_3, & a\alpha_3 + b\beta_3 + c\gamma_3 \end{vmatrix}, \text{ since } a\alpha + b\beta + c\gamma = 2\Delta$$

$$= c \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

Thus $\Delta' = abc(\alpha_1\beta_2\gamma_3)/8\Delta^2$.]

5. *To find the condition that two lines may be at right angles.*

In Trilinears, if the equations to the lines be of the form $l\alpha + m\beta + n\gamma = 0$, on changing to Cartesians they will be of the form

$$(cl - an)\bar{x} + (cm - bn)\bar{y} + 2n\Delta \operatorname{cosec} C = 0.$$

Hence, the condition that two lines in oblique Cartesians may be at right angles, becomes

$$(cl - an)(cl' - an') + (cm - bn)(cm' - bn')$$
$$- \{(cl - an)(cm' - bn') + (cl' - an')(cm - bn)\}\cos C = 0.$$

Now the co-efficient of $ll' + mm' + nn'$ is c^2, while that of $mn' + m'n$ is $-(bc - ac\cos C)$, that is $-c^2\cos A$; and those of $nl' + n'l$ and $lm' + l'm$ are $-c^2\cos B$ and $-c^2\cos C$ respectively.

Hence the condition becomes

$$ll' + mm' + nn' - (mn' + m'n)\cos A - (nl' + n'l)\cos B$$
$$- (lm' + l'm)\cos C = 0.$$

[The corresponding for areals may be obtained in like manner, or deduced from the foregoing by writing la, mb, nc for l, m, n respectively ; and is

$$a^2ll' + \ldots - (mn' + m'n)bc\cos A - \ldots = 0.]$$

6. *To find the condition that two lines may be parallel.*

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The equation $lx+my+nz=0$ in areals becomes

$$a(l-n)\bar{x}+b(m-n)\bar{y}+nab=0$$

in Cartesians; and the condition for parallelism of two lines is therefore $$(l-n)(m'-n')-(l'-n')(m-n)=0,$$

that is, $$mn'-m'n+nl'-n'l+lm'-l'm=0,$$

or $$\begin{vmatrix} l, & m & n \\ l' & m' & n' \\ 1 & 1 & 1 \end{vmatrix}=0;$$

the corresponding in trilinears being

$$\begin{vmatrix} l, & m & n \\ l' & m' & n' \\ a & b & c \end{vmatrix}=0.$$

7. *To find the angle between two lines.*

If ϕ be the angle between two lines in oblique Cartesians whose equations are of the form $Ax+By+C=0$,

$$\tan\phi=\frac{(AB'-A'B)\sin\omega}{AA'+BB'-(AB'+A'B)\cos\omega}$$

Expressing the trilinear equations in Cartesians and substituting, we get

$$\tan\phi=\frac{c\{a(mn'-m'n)+\ \dots\}\sin C}{c^2\{ll'+\ \dots\ -(mn'+m'n)\cos A-\ \dots\}}$$

$$=\frac{(mn'-m'n)\sin A+\ \dots}{ll'+\ \dots\ -(mn'+m'n)\cos A\ -\dots}$$

In areals this becomes

$$\tan\phi=\frac{2\Delta(mn'-m'n+\ \dots)}{a^2ll'+\ \dots\ -(mn'+m'n)bc\cos A\ -\dots}$$

[The expressions for $\sin\phi$ may be worth noticing; and may be deduced from those for $\tan\phi$, or obtained independently, thus:—

In oblique Cartesians

$$\sin\phi=\frac{(AB'-A'B)\sin\omega}{\sqrt{A^2+B^2-2AB\cos\omega}.\ \sqrt{A'^2+B'^2-2A'B'\cos\omega}}$$

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which for areals becomes

$$\frac{2\Delta(mn' - m'n + \dots)}{\sqrt{(l-m)(l-n)a^2 + \dots} \cdot \sqrt{(l'-m')(l'-n')a^2 + \dots}}$$

and for trilinears

$$\frac{(mn' - m'n)\sin A + \dots}{\sqrt{l^2 + \dots - 2mn\cos A - \dots}\ \sqrt{l'^2 + \dots - 2m'n'\cos A - \dots}}\ ]$$

8. We will now consider the general equation of the second degree, and obtain certain results by the same method.

To find the conditions that the equation may represent a circle.

Let the equation in areals be

$$ux^2 + vy^2 + wz^2 + 2u'yz + 2v'zx + 2w'xy = 0.$$

Changing to Cartesians, and removing z by the relation $x + y + z = 1$, this becomes

$$(2v' - w - u)\frac{\bar{x}^2}{b^2} + (2u' - v - w)\frac{\bar{y}^2}{a^2} + 2(u' + v' - w - w')\frac{\bar{x}\bar{y}}{ab}$$
$$+ \dots = 0.$$

Now the general equation

$$Ax^2 + 2Bxy + Cy^2 + \dots = 0$$

in oblique Cartesians will represent a circle if $A = C = B\sec\omega$; hence the required conditions are

$$\frac{2u' - v - w}{a^2} = \frac{2v' - w - u}{b^2} = \frac{u' + v' - w - w'}{ab\cos C},$$

each of which ratios

$$= \frac{2w' - u - v}{a^2 + b^2 - 2ab\cos C} = \frac{2w' - u - v}{c^2}.$$

[The corresponding for trilinears may be likewise obtained, or at once deduced, when we get

$$2bcu' - c^2v - b^2w = 2cav' - a^2w - c^2u = 2abw' - b^2u - a^2v.]$$

9. *To find the condition that the equation may represent an ellipse, parabola, or hyperbola.*

The above equation in Cartesians will represent these curves

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respectively according as $B^2 - AC$ is negative, zero, or positive. If we take the general equation in trilinears

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0,$$

and change to Cartesians, removing γ by the relation $a\alpha + b\beta + c\gamma = 2\Delta$, it becomes

$$(a^2w + c^2u - 2cav')x^2 + (c^2v + b^2w - 2bcu')y^2 \\ + 2(abw + c^2w' - cau' - bcv')xy + \ldots = 0.$$

Thus the required condition is that

$$-(abw + c^2w' - cau' - bcv')^2 + (a^2w + c^2u - 2cav')(c^2v + b^2w - 2bcu'),$$

or, with a known notation,

$$Ua^2 + Vb^2 + Wc^2 + 2U'bc + 2V'ca + 2W'ab$$

is positive, zero, or negative; or that

$$\begin{vmatrix} u, & w', & v', & a \\ w', & v, & u', & b \\ v', & u', & w, & c \\ a, & b, & c & 0 \end{vmatrix} \text{ is negative, zero, or positive.}$$

The corresponding in areals may be obtained in a similar way, or deduced from the preceding by putting $a = b = c = 1$, when the condition is that

$$(2u' - v - w)(2v' - w - u) - (u' + v' - w - w')^2,$$

or,
$$U + V + W + 2U' + 2V' + 2W'$$

is positive, zero, or negative; or that

$$\begin{vmatrix} u, & w', & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} \text{ is negative, zero, or positive.}$$

The condition that the equation in areals may represent a parabola can also be expressed under another interesting form.

Since

$$2(u' + v' - w - w') = (2u' - v - w) + (2v' - w - u) - (2w' - u - v),$$

the expression

$$4(u' + v' - w - w')^2 - 4(2u' - v - w)(2v' - w - u)$$

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is equal to the product of the four expressions

$$\sqrt{2u' - v - w} \pm \sqrt{2v' - w - u} \pm \sqrt{2w' - u - v}.$$

Hence the equation represents a parabola if any one of these expressions is zero.

10. *To find the condition that the equation may represent a rectangular hyperbola.*

The equation $Ax^2 + 2Bxy + Cy^2 + \dots = 0$ in oblique Cartesians will represent a rectangular hyperbola if the lines $Ax^2 + 2Bxy + Cy^2 = 0$ are at right angles, the condition for which is that

$$A + C - 2B\cos\omega = 0.$$

Hence, for trilinears, the required condition is that

$$a^2w + c^2u - 2cav' + c^2v + b^2w - 2bcu'$$
$$- 2(abw + c^2w' - cau' - bcv')\cos C = 0,$$

$$\therefore \quad c^2(u + v + w) - 2c(b - a\cos C)u' - \dots \quad = 0,$$

that is,

$$u + v + w - 2u'\cos A - 2v'\cos B - 2w'\cos C = 0.$$

[For areals, the condition is that

$$a^2u + b^2v + c^2w - 2bc\cos A . u' - \dots = 0,$$

or

$$a^2(u + u' - v' - w') + b^2(v + v' - w' - u') + c^2(w + w' - u' - v') = 0.]$$

11. *To find expressions for the product, and sum of squares of the semi-axes, when the equation represents a central conic.*

If the general Cartesian equation $a'x^2 + 2hxy + b'y^2 + 2gx + 2fy + c = 0$ become $Ax^2 + By^2 + C = 0$ when the conic is referred to its principal axes, the product of the semi-axes is $C/\sqrt{AB}$, and the sum of their squares is $-C(A+B)/AB$; and if the original axes be oblique these are respectively equal to

$$D\sin\omega/(a'b' - h^2)^{\frac{3}{2}} \text{ and } -D(a' + b' - 2h\cos\omega)/(a'b' - h^2)^2,$$

$$\text{where } D = \text{the discriminant } \begin{vmatrix} a' & h & g \\ h & b' & f \\ g & f & c \end{vmatrix}.$$

Transforming the general equation

$$ux^2 + vy^2 + wz^2 + 2u'yz + 2v'zx + 2w'xy = 0$$

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in areals to the Cartesian system, it becomes

$$(w+u-2v')\frac{\bar{x}^2}{b^2}+(v+w-2u')\frac{\bar{y}^2}{a^2}+2(w+w'-u-v')\frac{\bar{x}\bar{y}}{ab}$$
$$+2(v'-w)\frac{\bar{x}}{b}+2(u'-w)\frac{\bar{y}}{a}+w=0\,;$$

whence, substituting, we shall get

$$a^2b^2D=\begin{vmatrix} u & w' & v' \\ w' & v' & u' \\ v' & u' & w \end{vmatrix}\equiv H,$$

$$a^2b^2(a'b'-h^2)=-\begin{vmatrix} u & w' & v' & 1 \\ w' & v & u' & 1 \\ v' & u' & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}\equiv -K$$

and $$a^2b^2(a'+b'-2h\cos\omega)$$
$$=(u+u'-v'-w')a^2+(v+v'-w'-u')b^2+(w+w'-u'-v')c^2\equiv I.$$

Thus, the product of the semi-axes $= \quad 2\Delta H/(-K)^{\frac{3}{2}}$,
and the sum of their squares $\quad = -HI/K^2$.

[Proceeding in like manner with the general equation

$$u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta=0$$

in trilinears, we shall get

$$D=a^2b^2c^4.H,$$

$$a'b'-h^2=-c^2\begin{vmatrix} u, & w', & v', & a \\ w', & v, & u', & b \\ v', & u', & w, & c \\ a, & b, & c, & 0 \end{vmatrix}=-c^2K' \text{ suppose},$$

$$a'+b'-2h\cos\omega=c^2(u+v+w-2u'\cos A-2v'\cos B-2w'\cos C)$$
$$=c^2.I' \text{ suppose}.$$

Thus, the product of the semi-axes $= \quad 2abc\Delta H/(-K')^{\frac{3}{2}}$,
and the sum of their squares $\quad = -a^2b^2c^2HI'/K'^2$.]

12. *Particular forms of the general equation.*

The general equation in areals represents :—

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Two straight lines if $H=0$;
an ellipse, parabola, or hyperbola according as $K<=>0$;
a rectangular hyperbola if $I=0$;
while in trilinears the corresponding conditions are

$$H=0,\ K<=>0,\ I'=0 \text{ respectively.}$$

We append the values of these functions for three particular forms of the general equation.

(1) For the circumscribed conic in areals, $lyz+mzx+nxy=0$

$$4H=lmn;\ 4K=l^2+m^2+n^2-2mn-2nl-2lm;$$
$$2I=a^2(l-m-n)+b^2(m-n-l)+c^2(n-l-m),$$

or
$$-I=lbc\cos A+mca\cos B+nab\cos C;$$

and the condition for a parabola is equivalent to

$$\sqrt{l}\pm\sqrt{m}\pm\sqrt{n}=0.$$

For the same conic in trilinears, $l\beta\gamma+m\gamma a+na\beta=0$,

$$4H=lmn;\ 4K'=al^2+b^2m^2+c^2n^2-2bcmn-2canl-2ablm;$$
$$-I'=l\cos A+m\cos B+n\cos C;$$

and condition for a parabola becomes

$$\sqrt{al}\pm\sqrt{bm}\pm\sqrt{cn}=0.$$

(2) For the inscribed conic, or conics touching sides of triangle of reference in areals, $\sqrt{lx}\pm\sqrt{my}\pm\sqrt{nz}=0$.

$$H=-4l^2m^2n^2;\ K=-4lmn(l+m+n);$$
$$I=(l+m+n)(a^2l+b^2m+c^2n)-a^2mn-b^2nl-c^2lm.$$

For the same conics in trilinears, $\sqrt{la}\pm\sqrt{m\beta}\pm\sqrt{n\gamma}=0$

$$H=-4l^2m^2n^2;\ K=-4lmn(bcl+cam+abn);$$
$$I=l^2+m^2+n^2+mn+nl+lm.$$

(3) For the conic with respect to which the triangle of reference is self-conjugate, the equation to which in areals is $lx^2+my^2+nz^2=0$,

$$H=lmn;-K=mn+nl+lm,$$
$$I=la^2+mb^2+nc^2;$$

and for the same conic in trilinears $la^2+m\beta^2+n\gamma^2=0$

$$H=lmn;\ K'=mna^2+nlb^2+lmc^2;$$
$$I'=l+m+n.$$

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