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On the condition number of a Kreiss matrix

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Abstract. In 2005, N. Nikolski proved among other things that for any $r \in (0,1)$ and any $K \geq 1$, the condition number $CN(T) = \|T\| \cdot \|T^{-1}\|$ of any invertible n-dimensional complex Banach space operators T satisfying the Kreiss condition, with spectrum contained in $\{r \leq |z| < 1\}$, satisfies the inequality $CN(T) \leq CK(T)\|T\|n/r^n$ where $K(T)$ denotes the Kreiss constant of T and $C > 0$ is an absolute constant. He also proved that for $r \ll 1/n$, the latter bound is asymptotically sharp as $n \to \infty$. In this note, we prove that this bound is actually achieved by a family of explicit $n \times n$ Toeplitz matrices with arbitrary singleton spectrum $\{\lambda\} \subset \mathbb{D} \setminus \{0\}$ and uniformly bounded Kreiss constant. Independently, we exhibit a sequence of Jordan blocks with Kreiss constants tending to ∞ showing that Nikolski's inequality is still asymptotically sharp as K and n go to ∞.

1 Introduction

Let T be an invertible bounded operator on a complex Banach space X. In numerical analysis, it is a matter of interest to estimate the quantity $CN(T) := \|T\|\|T^{-1}\|$, called the condition number of T. Roughly speaking, the condition number of T measures the greatest loss of precision that the linear system $AT = x$ can exhibit over all inputs and their potential errors. The condition numbers of matrices are also intimately related to many problems from matrix analysis, such as the study of the distribution of the eigenvalues of classical random matrices appearing in mathematical statistics. We refer the reader to, e.g., [10, Chapter 3] or [3, 15], and the references therein.

Let us denote by $\mathcal{L}(X)$ the algebra of all bounded operators on X, and for $n \in \mathbb{N}$, let the notation $\mathcal{B}(n)$ (resp. $\mathcal{H}(n)$) stand for the set of all n-dimensional complex Banach spaces (resp. Hilbert spaces). When X belongs to $\mathcal{H}(n)$, one can easily deduce from the polar decomposition that

$$CN(T) \leq \prod_i |\lambda_i(T)|^{-1} \|T\|^n, \quad T \in \mathcal{L}(X), \tag{1.1}$$

where $\lambda_i(T)$, $i = 1, \ldots, n$, are the eigenvalues of T counting multiplicity. This inequality is clearly sharp since equality occurs if T is the $n \times n$ identity matrix. When turning to the Banach setting, the analogous problem becomes more involved and was first

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considered in the 70s by B. L. Van der Waerden, W. A. Coppel, and J. J. Schäffer. In view of (1.1), it is natural to seek for estimates of the quantity

(1.2)
$$C(n) := \inf\left\{C > 0 : CN(T) \le C\prod_i |\lambda_i(T)|^{-1}\|T\|^n,\ T \in \mathcal{L}(X) \text{ invertible},\ X \in \mathcal{B}(n)\right\}.$$

A normalization reduces this problem to that of estimating the norm of the inverse of *contractive* invertible operators (that is, those with norm less than or equal to 1). Namely, denoting by $\mathcal{C}(X) \subset \mathcal{L}(X)$ the class of contractive matrices on X, $C(n)$ coincides with the quantity

$$\inf\left\{C > 0 : \|T^{-1}\| \le C\prod_i |\lambda_i(T)|^{-1},\ T \in \mathcal{C}(X),\ T \text{ invertible},\ X \in \mathcal{B}(n)\right\}$$

$$= \sup\left\{\prod_i |\lambda_i(T)|\|T^{-1}\| : T \in \mathcal{C}(X),\ T \text{ invertible},\ X \in \mathcal{B}(n)\right\}, \tag{1.3}$$

which turns out to be easier to handle. In 1970, Schäffer [14] proved the general estimate $C(n) \le \sqrt{en}$, and showed that the inequality

$$\prod_i |\lambda_i(T)|\|T^{-1}\| \le 2 \tag{1.4}$$

holds for invertible contractions T acting on $\mathbb{C}^n$ endowed with the ℓ^1-norm or with the l^∞-norm, (and that (1.4) is sharp in both cases). This led him to make the conjecture—nowadays known as *Schäffer's conjecture*—that $C(n) = 2$, $n \in \mathbb{N}$. The latter was disproved first by Gluskin, Meyer, and Pajor [4], J. Bourgain [4],[1] and later by Queffélec [11], who obtained that $C(n) \ge c\sqrt{n}$ for some absolute constant $c > 0$, proving that the initial upper estimate given by Schäffer is optimal (see below for more precise statements with respect to our motivations).

In order to introduce our motivations and for the sake of general interest, we shall briefly survey recent extensions and refinements of the solutions to Schäffer's conjecture that have been developed at the light of new theoretical approaches, and that are natural to the study of the condition number of matrices. It will imply the introduction of a number of notations that, hopefully, will be still kept limited.

First of all, it is a natural problem to wonder how does the constant $C(n)$ behave, if in its equivalent definition (1.3) one replaces the class of all invertible contractions by some other classical sets of invertible operators. Generally speaking, let $\mathcal{P}$ be a property, and for $T \in \mathcal{L}(X)$, let the notation $T \in \mathcal{P}$ mean that the operator T satisfies the property $\mathcal{P}$. We set

$$C(n, \mathcal{P}) := \sup\left\{\prod_i |\lambda_i(T)|\|T^{-1}\| : T \in \mathcal{P},\ T \in \mathcal{L}(X) \text{ invertible},\ X \in \mathcal{B}(n)\right\}. \tag{1.5}$$

Thus, for example, $C(n) = C(n, \mathcal{C})$ if $T \in \mathcal{C}$ means "T is a contraction." For properties $\mathcal{P}$ that define too large classes of operators, it may happen that $C(n, \mathcal{P}) = \infty$. It is obviously the case if $\mathcal{P}$ is satisfied by any invertible operator acting on any Banach

[1] The same article [4] contains an appendix with a stronger estimate due to Bourgain.

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space (indeed, consider all the matrices λI_n, $|\lambda| > 1$, where I_n is the identity matrix of size n). In this situation, studying $C(n,\mathcal{P})$ is not relevant for estimating condition numbers. On the contrary, if $\mathcal{P}$ is a property for which $C(n,\mathcal{P}) < \infty$, then by definition it holds

$$CN(T) \le C(n,\mathcal{P}) \prod_i |\lambda_i(T)|^{-1} \|T\|$$

for any $T \in \mathcal{P}$. In this case, estimating $C(n,\mathcal{P})$ becomes especially relevant for properties $\mathcal{P}$ that are satisfied by operators that are not necessarily contractions since, for such operators, it may give upper estimates of condition numbers that are better than that given by Schäffer's inequality (the latter is valid for any invertible operator).

In 2005, Nikolski [8] developed a new approach to achieve estimates of $C(n,\mathcal{P})$ by using functional calculus on spaces of functions holomorphic on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. The idea is that for certain properties $\mathcal{P}$ and any Banach space X, an invertible operator $T \in \mathcal{L}(X)$ satisfies $\mathcal{P}$ if and only if $\|f(T)\| \le K\|f\|_{A_\mathcal{P}}$ for any f in some algebra $A_\mathcal{P}$ of holomorphic functions on $\mathbb{D}$ and for some absolute constant $K > 0$. To make it shorter, we will say in this case that property $\mathcal{P}$ obeys an $A_\mathcal{P}$-functional calculus. Then, for such properties $\mathcal{P}$, estimating $\|T^{-1}\|$ from above for $T \in \mathcal{P}$ reduces to estimating $\|f\|_{A_\mathcal{P}}$ or $\|f\|_{A_\mathcal{P}/m_T A_\mathcal{P}}$ from above for any $f \in A_\mathcal{P}$ satisfying the Bézout identity $zf + m_T h = 1$, where m_T is the minimal polynomial of T and h is any function in $A_\mathcal{P}$. For example, it is well known and easily seen that the property $\mathcal{C}$ of being a contraction obeys a W-functional calculus (with constant 1) where W refers to the Wiener space consisting of all functions f analytic in $\mathbb{D}$, such that $\|f\|_W := \sum_{k\ge 0} |\hat{f}(k)| < \infty$, and where $\hat{f}(k)$ denotes the kth Taylor coefficient of f. It is also readily checked that given $K \ge 1$, the property $\mathcal{PB}_K$ satisfied by all Banach space power bounded operators T such that $\sup_n \|T^n\| \le K$ obeys a W-functional calculus (with constant K), which leads to the same estimates for $C(n,\mathcal{PB}_K)$ and $C(n,\mathcal{C})$, up to an absolute constant. We also refer the reader to [18, Paragraph 2.3] for more explanations of the above Nikolski's strategy. Another standard class of operators obeying a functional calculus over a function algebra is that of Kreiss operators. It is defined, for $K \ge 1$, by the property $\mathcal{K}_K$ given for any Banach operator T by $T \in \mathcal{K}_K$ if and only if the following resolvent estimate holds:

$$\left\|(\zeta - T)^{-1}\right\| \le K(|\zeta| - 1)^{-1}, \quad |\zeta| > 1. \tag{1.6}$$

For a given operator T, the infimum of all constants K satisfying (1.6) is called the *Kreiss constant* of T. This class of operators satisfying $\mathcal{K}_K$ is an important one in numerical analysis [9, 17]. Note that every contraction is a Kreiss operator with Kreiss constant less than or equal to 1 and that there exist Kreiss operators that are not contractive. In [8], Nikolski made use of the fact that $\mathcal{K}_K$ obeys a Besov functional calculus, according to a result by Vitse [22], to obtain the following theorem.

Theorem 1.1

1. *[8, Theorem 3.26] Let X be a complex Banach space, and let $T \in \mathcal{L}(X)$ be a Kreiss operator with Kreiss constant $K \ge 1$. Then*

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(1.7) $$\|T^{-1}\| \leq CK \frac{n}{\prod_{i=1}^{n} |\lambda_i|},$$

where $(\lambda_1, \ldots, \lambda_n) \in \mathbb{D}^n$ *denotes the eigenvalues of* T *and* C *an absolute constant.*

2. *(Particular case of [8, Theorem 3.31]) For any* $(\lambda_1, \ldots, \lambda_n) \in \mathbb{D}^n$*, there exists an invertible Kreiss operator* T *with eigenvalues* $(\lambda_1, \ldots, \lambda_n)$ *such that*

(1.8) $$\left\|T^{-1}\right\| \geq c \frac{n}{\prod_{i=1}^{n} |\lambda_i|} \left(c' - \prod_{j=1}^{n} (1 + |\lambda_i|) \right),$$

where $c > 0$ *and* $c' > 1$ *are absolute constants.*

The first part of this theorem implies

(1.9) $$C(n, \mathcal{K}_K) \leq CKn \quad \text{and} \quad CN(T) \leq CKn \prod_i |\lambda_i(T)|^{-1} \|T\|$$

for any $T \in \mathcal{K}_K$. In passing, we point out that for any given $K \geq 1$, there exists an absolute constant C' such that $\|T\| \leq C'K$ for any $T \in \mathcal{K}_K$ (to see it, one can use the Riesz–Dunford functional calculus). Moreover, letting the λ_i's tend to 0 sufficiently fast in (1.8), one can deduce that for fixed K, the inequality $C(n, \mathcal{K}_K) \leq CKn$ is made sharp with respect to $n \to \infty$ for sequences of operators with spectrum shrinking to 0. It follows from the approach used by Nikolski that these extremal operators are model operators acting on model spaces (with adequate norms). Then three questions naturally arise.

Question 1 Is it possible to find operators with fixed degenerate spectrum that are extremal for $C(n, \mathcal{K}_K) \leq CKn$, $n \to \infty$?

Question 2 Can one choose these operators among classes of structured matrices (e.g., Toeplitz matrices)?

Question 3 Is the inequality $C(n, \mathcal{K}_K) \leq CKn$ sharp simultaneously when K and n go to ∞?

The aim of this note is to propose a solution to these three questions. For contractions, the same questions have already been investigated for a long time. A brief exposition of these investigations and a statement of our results requires the introduction of the following natural quantities:

- $C_n^{\mathcal{H}}(r, \mathcal{P}) := \sup\left\{\left\|T^{-1}\right\| : T \in \mathcal{P},\ T \in \mathcal{L}(X) \text{ invertible},\ X \in \mathcal{H}(n),\ r_{\min}(T) \geq r\right\}$;
- $C_n^{\mathcal{B}}(r, \mathcal{P}) := \sup\left\{\left\|T^{-1}\right\| : T \in \mathcal{P},\ T \in \mathcal{L}(X) \text{ invertible},\ X \in \mathcal{B}(n),\ r_{\min}(T) \geq r\right\}$;
- $\tau_n^{\mathcal{H}}(r, \mathcal{P}) := \sup\left\{\left\|T^{-1}\right\| : T \in \mathcal{P} \cap \mathcal{T},\ T \in \mathcal{L}(X) \text{ invertible},\ X \in \mathcal{H}(n),\ r_{\min}(T) \geq r\right\}$;
- $\tau_n^{\mathcal{B}}(r, \mathcal{P}) := \sup\left\{\left\|T^{-1}\right\| : T \in \mathcal{P} \cap \mathcal{T},\ T \in \mathcal{L}(X) \text{ invertible},\ X \in \mathcal{B}(n),\ r_{\min}(T) \geq r\right\}$,

where $r_{\min}(T)$ stands for the smallest eigenvalues of T, where $\mathcal{T}$ denotes the property of being a Toeplitz operator, and where the notation $\mathcal{P} \cap \mathcal{P}'$ means that $\mathcal{P}$ and $\mathcal{P}'$ are simultaneously satisfied.

If $\mathcal{P} = \mathcal{C}$, and in the Hilbert setting, it is known that for any $r \in (0,1)$, $C_n^{\mathcal{H}}(r, \mathcal{C}) = r^{-n}$. This result probably dates back to the 19th century and is sometimes attributed to L. Kronecker. A description of matrices achieving the supremum was obtained by Nikolski [8, Theorem 2.1], using the abovementioned functional calculus approach. In the Banach setting, none of the results given in [4, 12] lead to sharp estimates of $C_n^{\mathcal{B}}(r, \mathcal{C})$. Indeed, while Schäffer's inequality [14] gives the upper estimates

$$C_n^{\mathcal{B}}(r, \mathcal{C}) \leq \sqrt{en} r^{-n}, \tag{1.10}$$

the lower estimates obtained in [4, 11] are not enough to prove that (1.10) is sharp with respect to r, n: Gluskin–Meyer–Pajor [4], Bourgain (see [4]), and Queffélec [11], respectively, stated

$$C_n^{\mathcal{B}}\left(1-\frac{1}{n}, \mathcal{C}\right) \geq c\sqrt{\frac{n}{\log n}\frac{1}{\log\log n}}, \quad C_n^{\mathcal{B}}\left(1-\frac{1}{n}, \mathcal{C}\right) \geq c\sqrt{\frac{n}{\log n}}, \quad C_n^{\mathcal{B}}\left(1-\frac{1}{n}, \mathcal{C}\right) \geq c\sqrt{n},$$

where c is an absolute constant. In all these estimates, it appears that r and n are correlated (in particular, the spectrum of the extremal operators is not fixed).

When $\mathcal{P} = \mathcal{K}_K$, we recall that the first point of Theorem 1.1 gives that for any $r \in (0,1)$,

$$C_n^{\mathcal{B}}(r, \mathcal{K}_K) \leq CK\frac{n}{r^n}, \tag{1.11}$$

where C is an absolute constant, whereas the second one yields the asymptotic sharpness of (1.11) as $n \to \infty$, only for $r \ll 1/n$.

A somewhat natural approach to attack third the question of exhibiting operators with fixed spectrum that are asymptotically extremal for (1.10) and (1.11) may consist in looking for such operators in classes of structured matrices such as Jordan blocks, which are special cases of Toeplitz matrices. More generally, Toeplitz or Hankel matrices, which play a crucial role in matrix analysis and operator theory, are natural candidates. Yet, in the Banach setting, the proofs given in [4, 11] are far from providing with explicit examples achieving $C(n, \mathcal{C})$—and a fortiori with extremal Toeplitz matrices with degenerate spectrum. In the Hilbert setting, we recall that Nikolski characterized matrices that are extremal for $C_n^{\mathcal{H}}(r, \mathcal{C})$ for any r in [8] and asked for the existence of Toeplitz matrices satisfying this characterization. Szehr and the last author obtained in [19] the equalities

$$\tau_n^{\mathcal{H}}(r, \mathcal{C}) = c_n^{\mathcal{H}}(r, \mathcal{C}) = \frac{1}{r^n},$$

for any $r \in (0,1)$ (see also [23] where the weaker estimate $\tau_n^{\mathcal{H}}(r, \mathcal{C}) \geq 2^{-1}r^{-n}$ is obtained) and, in 2021, they proved that for some absolute constant $c > 0$,

$$C_n^{\mathcal{B}}(r, \mathcal{C}) \geq \tau_n^{\mathcal{B}}(r, \mathcal{C}) \geq c\frac{\sqrt{n}}{r^n}$$

for any $r \in (0,1)$ and any n large enough, so that (1.10) is sharp. Moreover, they provided with explicit examples of extremal Banach space Toeplitz operators with arbitrary degenerate spectrum [20]. Worth insisting, these extremal Toeplitz operators may be chosen with spectrum equal to $\{\lambda\}$, λ arbitrary in $\mathbb{D} \setminus \{0\}$.

Our main contribution in this note is to follow their strategy and to obtain an analogous result for Kreiss operators, giving a solution to Questions 1 and 2. More precisely, we will prove the following result.

Theorem 1.2 (For a more precise statement, see Theorem 2.1) *For a fixed $K \geq 1$, there exists an absolute constant $c > 0$ such that*

$$C_n^{\mathcal{B}}(r, \mathcal{K}_K) \geq \tau_n^{\mathcal{B}}(r, \mathcal{K}_K) \geq c\frac{n}{r^n}$$

for any $r \in (0,1)$ and n large enough.

Moreover, the extremal Toeplitz matrices in the second inequality can be chosen with a degenerate spectrum arbitrary in $\mathbb{D} \setminus \{0\}$.

This implies the sharpness of (1.11) as $n \to \infty$ for any $r \in (0,1)$. With respect to numerical analysis, this theorem says that degenerate Toeplitz matrices may be ill-conditioned in high dimensions. The outline of the proof will be similar to that proposed by Szehr and Zarouf in [20] for contractions, but the techniques will differ. We mention that Theorem 2.1 was announced in [2], without a proof.

Our second result concerns the asymptotic sharpness of Nikolski's upper bound (5.1) when K is permitted to grow unboundedly as $n \to \infty$. It is indeed a natural question to ask whether the dependency on K in (5.1) can be improved or not. We will show that (5.1) is also sharp in the following sense: there exists a sequence of Jordan blocks $T_n \in \mathcal{L}(X)$, where $X = (\mathbb{C}^n, \|\cdot\|_{l^1})$, such that $K(T_n) \to \infty$ as $n \to \infty$ and

$$\frac{|\det(T_n)| \, \|T_n^{-1}\|}{K(T_n)} \geq cn,$$

for some absolute constant $c > 0$ (see Proposition 4.1). We mention that the use of Jordan blocks as extremal matrices in the context of Kreiss matrices is rather classical (see, for example, [21]).

The organization of the paper is as follows. The next section contains the prerequisites and the statement of Theorem 2.1. The latter is proved in the third section. Section 4 contains the proof of the asymptotic sharpness of (5.1) with respect to K and n (Proposition 4.1). In the last section, we present—for the interested reader—a short and simple proof of the upper estimate (5.1) in Theorem 1.1.

Notation Throughout the paper, we will use the notation $f \lesssim g$ meaning that $f \leq cg$, where c is some absolute constant. The notation $f \simeq g$ will mean that $f \lesssim g$ and $g \lesssim f$.

2 Background and statement of the main result

Let us denote by $H(\mathbb{D})$ the space of analytic functions in $\mathbb{D}$ and by H^∞ the Banach algebra of bounded analytic functions in $\mathbb{D}$, endowed with the supremum norm on $\mathbb{D}$. The standard Hardy space H^2 is defined as the subspace of $H(\mathbb{D})$ consisting of those functions f for which

$$\|f\|_{H^2}^2 := \sup_{0 \leq r < 1} \int_{\mathbb{T}} |f(rz)|^2 \mathrm{d}m(z) < \infty,$$

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where m is the normalized Lebesgue measure on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Endowed with $\|\cdot\|_{H^2}$, H^2 is a Hilbert space with inner product given by

$$\langle f, g\rangle = \sum_{k\geq 0} \hat{f}(k)\overline{\hat{g}(k)}. \tag{2.1}$$

By Fatou's theorem, H^2 can be identified with the (Hilbert) subspace of $L^2(\partial\mathbb{D})$ so that $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\overline{g(e^{it})}dt$, where f and g in the right-hand side of the last equality denote, without possible confusion, the almost everywhere radial limits of f and g, respectively.

Given $\lambda \in \mathbb{D}$, we denote by b_λ the Blaschke factor associated with λ, namely

$$b_\lambda := \frac{z-\lambda}{1-\bar{\lambda}z}.$$

Let $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$ be a finite sequence. We shall consider the *model space* K_{B_σ} given by

$$K_{B_\sigma} := \left(B_\sigma H^2\right)^\perp = H^2 \ominus B_\sigma H^2,$$

where

$$B_\sigma := \prod_{k=1}^{n} b_{\lambda_k}$$

is the finite Blaschke product associated with $\sigma = \{\lambda_1, \dots, \lambda_n\}$. Any such space K_{B_σ} is an n-dimensional subspace of H^2. Now, for $1 \leq k \leq n$, let $f_k := \frac{1}{1-\bar{\lambda}_k z}$ and observe that $\|f_k\|_{H^2} = \left(1-|\lambda_k|^2\right)^{-1/2}$. Then set

$$e_1 = \frac{f_1}{\|f_1\|_{H^2}} \quad \text{and} \quad e_k = \prod_{j=1}^{k-1} b_{\lambda_j}\frac{f_k}{\|f_k\|_{H^2}}, \quad k = 2, \dots, n.$$

It is known that $(e_k)_{1\leq k\leq n}$ defines an orthonormal basis of K_{B_σ}, called the Malmquist–Walsh basis [6, p. 117].

The central object of Nikolski's approach in [8] is the *model operator* M_{B_σ} defined by

$$M_{B_\sigma} : \begin{cases} K_{B_\sigma} & \to & K_{B_\sigma}, \\ f & \mapsto & P_{B_\sigma}(zf), \end{cases}$$

where P_{B_σ} denotes the orthogonal projection on K_{B_σ}. The matrix representation $\widehat{M_{B_\sigma}}$ of M_{B_σ} with respect to the Malmquist–Walsh basis $(e_k)_{1\leq k\leq n}$ is as follows (see [18, Proposition III.4]):

$$\left(\widehat{M_{B_\sigma}}\right)_{ij} = \begin{cases} 0, & \text{if } i < j, \\ \lambda_i, & \text{if } i = j, \\ (1-|\lambda_i|^2)^{1/2}(1-|\lambda_j|^2)^{1/2}\prod_{\mu=j+1}^{i-1}\left(-\bar{\lambda}_\mu\right), & \text{if } i > j, \end{cases} \tag{2.2}$$

where $\left(\widehat{M_{B_\sigma}}\right)_{ij}$ stands for the i, j entry of $\widehat{M_{B_\sigma}}$. The reader may consult [6, 7] for a complete description of model spaces and model operators.

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For the proof of Theorem 2.1, we shall focus on the case where $\sigma = \{\lambda, \dots, \lambda\}$ (namely, $\lambda_1 = \cdots = \lambda_n = \lambda$ with the previous notations). In this case,

$$K_{B_\sigma} = \operatorname{span}\left\{ \frac{z^{k-1}}{(1-\bar{\lambda}z)^n} : k = 1, \dots, n \right\}$$

and the Malmquist–Walsh basis $\beta_\lambda := (e_k)_{1 \le k \le n}$ is given by

$$e_k(z) := \frac{(1-|\lambda|^2)^{1/2}}{1-\bar{\lambda}z} \left(\frac{z-\lambda}{1-\bar{\lambda}z} \right)^{k-1}, \quad k = 1, \dots, n.$$

Moreover, one can also show that K_{B_σ} coincides as a set with the n-dimensional Banach space consisting of all rational functions of degree at most n with poles located at $1/\bar{\lambda}$. We may and shall equip K_{B_σ} with the Banach norm

$$\|f\|_{\mathcal{B}_\infty} := \sup_{z \in \mathbb{D}} \left(1-|z|^2\right) |f(z)| < \infty, \quad f \in K_{B_\sigma}.$$

Then $(K_{B_\sigma}, \|\cdot\|_{\mathcal{B}_\infty})$ is a (Banach) subspace of the Besov space $\mathcal{B}_\infty$ defined as

$$\mathcal{B}_\infty = \left\{ f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}_\infty} = \sup_{z \in \mathbb{D}} \left(1-|z|^2\right) |f(z)| < \infty \right\}.$$

Our main theorem states as follows.

Theorem 2.1 *Let $\lambda \in \mathbb{D} \setminus \{0\}$ be fixed, and let T_λ denote the operator acting on $(K_{B_\sigma}, \|\cdot\|_{\mathcal{B}_\infty})$ whose matrix with respect to β_λ is given by*

$$M_\lambda := \begin{pmatrix} \lambda & 1-|\lambda|^2 & -\bar{\lambda}(1-|\lambda|^2) & \dots & (-\bar{\lambda})^{n-2}(1-|\lambda|^2) \\ 0 & \lambda & 1-|\lambda|^2 & \ddots & \vdots \\ 0 & \ddots & \lambda & \ddots & -\bar{\lambda}(1-|\lambda|^2) \\ \vdots & \ddots & \ddots & \ddots & 1-|\lambda|^2 \\ 0 & \dots & 0 & 0 & \lambda \end{pmatrix}.$$

Then T_λ satisfies the Kreiss condition and the inequality

$$\left\| T_\lambda^{-1} \right\|_* \ge c(\lambda) \frac{n}{|\lambda|^n}, \tag{2.3}$$

where $c(\lambda) > 0$ and where $\|\cdot\|_*$ is the operator norm induced by $\|\cdot\|_{\mathcal{B}_\infty}$. In particular, for any $r \in (0,1)$,

$$c_n^{\mathcal{B}}(r, \mathcal{K}_{\mathcal{K}}) \ge \tau_n^{\mathcal{B}}(r, \mathcal{K}_{\mathcal{K}}) \gtrsim \frac{n}{r^n},$$

as $n \to \infty$.

The proof of Theorem 2.1—displayed in the next section—is based on a duality argument already used in [20]. Considering Kreiss operators, we will here make use of the Besov functional calculus developed in [22] and the duality between the Besov spaces $\mathcal{B}_\infty$ and $\mathcal{B}_1$, where

$$\mathcal{B}_1 = \left\{ f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}_1} := \int_{\mathbb{D}} |(z^2 f(z))''| dA(z) < \infty \right\}, \tag{2.4}$$

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where A stands for the normalized Lebesgue measure on $\mathbb{D}$. This duality is given by the relation (with equivalence of the norms, see [22, p. 1815] for details)

$$\mathcal{B}_1^\star = \mathcal{B}_\infty \tag{2.5}$$

for the Cauchy duality (2.1).

3 Proof of Theorem 2.1

Proof of Theorem 2.1 In the whole section, $\sigma = \{\lambda, \dots, \lambda\} \subset \mathbb{D} \setminus \{0\}$ is fixed. First, using the orthonormality of the Malmquist–Walsh basis, we can see that the adjoint of M_{B_σ} acting on $(K_{B_\sigma}, \|\cdot\|_{\mathcal{B}_\infty})$ coincides with $T_{\bar{\lambda}}$. So, in order to prove that T_λ is a Kreiss operator, it is enough to show that $\widehat{M_{B_\sigma}}^*$—the adjoint of $\widehat{M_{B_\sigma}}$—satisfies the Kreiss condition, namely that there exists some constant $K > 0$ such that for any $|\zeta| > 1$,

$$\left\| \left(\zeta - \widehat{M_{B_\sigma}}^*\right)^{-1} \right\|_* \le K\,(|\zeta| - 1)^{-1}. \tag{3.1}$$

Let us fix ζ, $|\zeta| > 1$. By definition, the adjoint of M_{B_σ}coincides with the backward shift operator

$$S^* : f \mapsto \frac{f - f(0)}{z}$$

acting on K_{B_σ}.

Thus, using the Cauchy duality (2.5), we further have $\left\| \left(\zeta - \widehat{M_{B_\sigma}}^*\right)^{-1} \right\|_* := \left\| (\zeta - S^*)^{-1} \right\|_*$. Now, considering the shift operator $(Sf)(z) = zf(z)$ acting on the whole space $\mathcal{B}_1$ (and recalling that S is the adjoint of the operator S^* acting on $\mathcal{B}_\infty$), we have $\left\| (\zeta - S^*)^{-1} \right\|_{\mathcal{B}_\infty \to \mathcal{B}_\infty} = \left\| (\zeta - S)^{-1} \right\|_{\mathcal{B}_1 \to \mathcal{B}_1}$. The $\mathcal{B}_1$-functional calculus [22] then tells us that for any $f \in \mathcal{B}_1$,

$$\left\| (\zeta - S)^{-1} f \right\|_{\mathcal{B}_1} \le K_1 \|f\|_{\mathcal{B}_1} \left\| \frac{1}{\zeta - z} \right\|_{\mathcal{B}_1}$$

for some constant $K_1 > 0$ (see also [8, p. 143] and the references therein for more details on the last inequality). It remains to observe that

$$\left\| \frac{1}{\zeta - z} \right\|_{\mathcal{B}_1} \le K_2\,(|\zeta| - 1)^{-1},$$

whence there exists $K_3 > 0$ such that

$$\left\| (\zeta - S^*)^{-1} \right\|_{\mathcal{B}_\infty \to \mathcal{B}_\infty} = \left\| (\zeta - S)^{-1} \right\|_{\mathcal{B}_1 \to \mathcal{B}_1} \le K_3\,(|\zeta| - 1)^{-1}.$$

Since $\left\| (\zeta - S^*)^{-1} \right\|_* \le \left\| (\zeta - S^*)^{-1} \right\|_{\mathcal{B}_\infty \to \mathcal{B}_\infty}$, this yields (3.1) and the fact that T_λ is a Kreiss operator.

In order to derive (2.3), it is enough to show that $\left\| \left(\widehat{M_{B_\sigma}}^*\right)^{-1} \right\|_* \ge c(\lambda) \frac{n}{|\lambda|^n}$ for some constant $c(\lambda) > 0$. To do so, we apply $\left(\widehat{M_{B_\sigma}}^*\right)^{-1}$ to the test vector $X_0 = (0, \dots, 0, 1)$, i.e., to the rational function $R(z) = e_n(z)$. We set

$$g := \left(\widehat{M_{B_\sigma}}^*\right)^{-1} X_0^\top = (S^*)^{-1} e_n,$$

where $X_0^\top$ is the transpose of X_0. We have

$$S^* g = \frac{g - g(0)}{z} = e_n,$$

which means that

$$\begin{aligned} g &= z e_n + g(0) \\ &= \frac{(1-|\lambda|^2)^{1/2}}{1-\overline{\lambda} z} z b_\lambda^{n-1} + g(0) \\ &= (1-|\lambda|^2)^{1/2} \frac{z(z-\lambda)^{n-1}}{(1-\overline{\lambda} z)^n} + g(0). \end{aligned}$$

The condition $g \in K_{B_\sigma}$ imposes that g is a rational function with $\lim_{|z|\to\infty} g(z) = 0$, and hence

$$\begin{aligned} g(0) &= -(1-|\lambda|^2)^{1/2} \lim_{z\to+\infty} \frac{z(z-\lambda)^{n-1}}{(1-\overline{\lambda} z)^n} \\ &= (-1)^{n+1} \frac{(1-|\lambda|^2)^{1/2}}{\overline{\lambda}^n}. \end{aligned}$$

It follows that

$$\begin{aligned} \|g\|_{\mathcal{B}_\infty} &= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right) |g(z)| \\ &\geq |g(0)| \\ &= \frac{(1-|\lambda|^2)^{1/2}}{|\lambda|^n}. \end{aligned}$$

Let us now estimate $\|e_n\|_{\mathcal{B}_\infty}$ from the above. Defining $\widetilde{b_\lambda}(z) = -b_\lambda(z) = \frac{\lambda - z}{1-\overline{\lambda} z}$, we have $\widetilde{b_\lambda} \circ \widetilde{b_\lambda} = \mathrm{id}$ - where id denotes the identity function on $\mathbb{D}$—and the H^∞ norm is invariant under the composition by $\widetilde{b_\lambda}$. Therefore,

$$\begin{aligned} \|e_n\|_{\mathcal{B}_\infty} &= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right) \left| \frac{(1-|\lambda|^2)^{1/2}}{1-\overline{\lambda} z} (b_\lambda(z))^{n-1} \right| \\ &= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right) \left| \frac{(1-|\lambda|^2)^{1/2}}{1-\overline{\lambda} z} \widetilde{b_\lambda}(z)^{n-1} \right| \\ &= (1-|\lambda|^2)^{1/2} \sup_{z\in\mathbb{D}} \left(1-\left|\widetilde{b_\lambda}(z)\right|^2\right) \left| \frac{1}{1-\overline{\lambda}\widetilde{b_\lambda}(z)} z^{n-1} \right|. \end{aligned}$$

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Since

$$1-\left|\widetilde{b_\lambda}(z)\right|^2=\frac{(1-|\lambda|^2)(1-|z|^2)}{(1-\overline{\lambda}z)(1-\lambda\overline{z})}$$
$$\leq\frac{1+|\lambda|}{1-|\lambda|}(1-|z|^2)$$

and $|1-\overline{\lambda}\widetilde{b_\lambda}(z)|\geq(1-|\lambda|)$, we get

$$\|e_n\|_{\mathcal{B}_\infty}\leq\left(\frac{1+|\lambda|}{1-|\lambda|}\right)^{3/2}\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)|z|^{n-1}$$
$$\leq 2\left(\frac{1+|\lambda|}{1-|\lambda|}\right)^{3/2}\sup_{z\in\mathbb{D}}(1-|z|)\,|z|^{n-1}.$$

Studying the function $r\mapsto(1-r)r^{n-1}$ for $r\in(0,1)$, we can check that the supremum in the last inequality is attained at $|z|=1-1/n$. We conclude that

$$\|e_n\|_{\mathcal{B}_\infty}\leq\frac{2}{n-1}\left(\frac{1+|\lambda|}{1-|\lambda|}\right)^{3/2}\left(1-\frac{1}{n}\right)^n$$
$$\leq\frac{2}{e(n-1)}\left(\frac{1+|\lambda|}{1-|\lambda|}\right)^{3/2}.$$

In particular,

$$\frac{\|\left(\widehat{M_{B_\sigma}}^*\right)^{-1}X_0^\top\|_{\mathcal{B}_\infty}}{\|X_0^\top\|_{\mathcal{B}_\infty}}=\frac{\|g\|_{\mathcal{B}_\infty}}{\|e_n\|_{\mathcal{B}_\infty}}$$
$$\geq\frac{e(1-|\lambda|)^2}{2^{5/2}}\frac{(n-1)}{|\lambda|^n},$$

which completes the proof. ∎

4 On the sharpness of (5.1) with respect to *K* and *n*

In the previous sections, we were interested in the sharpness of the inequality $\det(T)\|T^{-1}\|\leq CK(T)n$ (see (5.1)) for Kreiss matrices T with Kreiss constant $K(T)$ less than some constant K. In this section, we will consider this inequality for *all* Kreiss matrices and prove that (5.1) is sharp as *n and K* tend to ∞. Relaxing the bound on the Kreiss constants will allow us to exhibit sequences of Jordan blocks that are extremal for (5.1).

For $a>1$ and $\lambda\in\mathbb{D}$ fixed, let J_λ stand for the Jordan block of size n

$$J_\lambda:=\begin{pmatrix}\lambda & a & 0 & \dots & 0\\ 0 & \lambda & a & \ddots & \vdots\\ 0 & \ddots & \lambda & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & a\\ 0 & \dots & 0 & 0 & \lambda\end{pmatrix}. \tag{4.1}$$

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From now on, we denote by $\|\cdot\|$ the operator norm induced by the ℓ^1- or the ℓ^∞-norm of $\mathbb{C}^n$. Observe that for any $z \in \mathbb{C}$, $z \neq \lambda$, the matrix $(zI_n - J_\lambda)^{-1}$ is well defined and is the Toeplitz matrix given by

$$(4.2)\qquad (zI_n - J_\lambda)^{-1} = \begin{pmatrix} \frac{1}{z-\lambda} & \frac{a}{(z-\lambda)^2} & \frac{a^2}{(z-\lambda)^3} & \cdots & \frac{a^{n-1}}{(z-\lambda)^n} \\ 0 & \frac{1}{z-\lambda} & \frac{a}{(z-\lambda)^2} & \ddots & \vdots \\ 0 & \ddots & \frac{1}{z-\lambda} & \ddots & \frac{a^2}{(z-\lambda)^3} \\ \vdots & \ddots & \ddots & \ddots & \frac{a}{(z-\lambda)^2} \\ 0 & \cdots & 0 & 0 & \frac{1}{z-\lambda} \end{pmatrix}.$$

Then

$$(4.3)\qquad \left\|(zI_n - J_\lambda)^{-1}\right\| = \frac{1}{|z-\lambda|^n}\,\frac{|z-\lambda|^n - |a|^n}{|z-\lambda| - |a|}.$$

In the following proposition, we exhibit sequences of Jordan blocks that are extremal for (5.1), with respect to the dimension n and the Kreiss constant K.

Proposition 4.1 *There exists* $(\lambda_n)_n$, $\lambda_n > 1$, *such that* $\lim_n K(J_{\lambda_n}) = \infty$ *and*

$$(4.4)\qquad \frac{|det(J_{\lambda_n})|\,\|J_{\lambda_n}^{-1}\|}{K(J_{\lambda_n})} \simeq n \quad as\ n \to \infty.$$

Proof By (4.3) with $z = 0$ and since $\det(J_\lambda) = \lambda^n$, we have $|\det(J_\lambda)|\,\|J_\lambda^{-1}\| = \frac{|\lambda|^n - |a|^n}{|\lambda| - |a|}$. We need to check that

$$K(J_{\lambda_n}) = \sup_{|z|>1} \frac{|z|-1}{|z-\lambda_n|^n}\,\frac{|z-\lambda_n|^n - |a|^n}{|z-\lambda_n| - |a|} \simeq \frac{1}{n}\,\frac{|\lambda_n|^n - |a|^n}{|\lambda_n| - |a|}$$

for some $(\lambda_n)_n$. From now on, we assume that $\lambda_n = 1/n$. Since the function $x \mapsto \frac{x^{-n}a^n - 1}{a-x}$ is decreasing on $[0, +\infty[$ and since $|z - \lambda_n| \geq |z| - \lambda_n$ for any $|z| > 1$, we have

$$K(J_{\lambda_n}) = \sup_{t>1} \frac{t-1}{(t-1/n)^n}\,\frac{(t-1/n)^n - |a|^n}{(t-1/n) - |a|}.$$

Setting $x := a/(t - 1/n)$ and

$$g(x) := \frac{(a + (1/n - 1)x)(x^n - 1)}{a(x-1)}$$

gives $K(J_{\lambda_n}) = \sup_{0<x<\frac{a}{1-1/n}} g(x)$. Studying the derivative and the second derivative of g easily leads to observe that g' vanish only once in the interval $[0, \frac{a}{1-1/n}]$. Moreover, a computation shows that $g'\left(a = \frac{a}{1+1/n-1/n}\right) > 0$, while $g'\left(\frac{na}{n+1} = \frac{a}{1+2/n-1/n}\right) < 0$ for n

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large enough. Thus, there exists $\delta_n \in (1/n, 2/n)$ such that g admits a maximum at $x_n := \frac{a}{1+\delta_n-1/n}$. Now,

$$\begin{aligned} K(J_{\lambda_n}) = g(x_n) &= \frac{\delta_n\left(\left(\frac{a}{1+\delta_n-1/n}\right)^n - 1\right)}{a-(1+\delta_n-1/n)} \\ &\simeq \frac{a^n}{n} \\ &\simeq \frac{1}{n}\frac{|\lambda_n|^n - |a|^n}{|\lambda_n|-|a|}, \end{aligned}$$

as $n \to \infty$, as desired. ∎

Remark 4.2 In contrast with the estimate $C(n, \mathcal{K}_K) \simeq Kn$ (as $n \to \infty$) considered in the previous sections, it follows from the previous proposition that

$$C(n, \mathcal{K}) := \sup\left\{\prod_i |\lambda_i(T)|\|T^{-1}\| : T \in \mathcal{K},\ T \in \mathcal{L}(X) \text{ invertible},\ X \in \mathcal{B}(n)\right\} = \infty.$$

Now, Proposition 4 shows that some sequence of Jordan blocks asymptotically achieves—up to numerical factor independent of K and n—the supremum

$$M(n, \mathcal{K}) := \sup\left\{\frac{\prod_i |\lambda_i(T)|\|T^{-1}\|}{K(T)} : T \in \mathcal{K},\ T \in \mathcal{L}(X) \text{ invertible},\ X \in \mathcal{B}(n)\right\},$$

and provides with an elementary proof of the sharpness of the estimate $M(n, \mathcal{K}) \lesssim n$ as $n \to \infty$ (which is a consequence of the first part of Theorem 1.1). Note that Theorem 2.1 obviously also leads to the latter assertion, but not for Jordan blocks and with more sophisticated arguments.

We shall also notice that, by the Kreiss Matrix Theorem [5, 13, 16], properties $\mathcal{PB}$ and $\mathcal{K}$ are equivalent, so $C(n, \mathcal{PB}) = C(n, \mathcal{K}) = \infty$. One can wonder whether an estimate similar to (4.4) can be obtained with $\sup_k \|T_n^k\|$ instead of $K(T_n)$, for some (simple) sequence $(T_n)_n$ instead of $(J_{\lambda_n})_n$.

5 A short and simple proof of Inequality (5.1)

In fact, the statement of Theorem 3.26 in [8] is slightly stronger than the first part of Theorem 1.1. More precisely, Nikolski proves the following.

Theorem 5.1 *Let X be a complex Banach space, and let $T \in \mathcal{L}(X)$ be a Kreiss operator with Kreiss constant $K \geq 1$. Let us denote by $m_T = \prod_{i=1}^d (z - \lambda_i)$ its minimal polynomial, and assume that $(\lambda_1, \ldots, \lambda_d) \in \mathbb{D}^d$. Then*

$$\|T^{-1}\| \leq CK \frac{d}{\prod_{i=1}^d |\lambda_i|}, \tag{5.1}$$

where C is an absolute constant.

We will display below a short proof of this more precise result that follows Nikolski's approach and is obtained as a combination of Vitse's [22] functional calculus for Kreiss operators and the Bonsall–Walsh inequality [1] for rational functions. In comparison with the proof of [8, Theorem 3.26], its simplicity lies in the choice of a very simple test function.

A short proof of Inequality (5.1) Let T, $(\lambda_1, \dots, \lambda_d)$, m_T, and K be as in the statement of Theorem 1.1. We denote by B the Blaschke product associated with the sequence $(\lambda_1, \dots, \lambda_d)$ and introduce the *test function* f given by

$$f(z) = \frac{B(0) - B(z)}{zB(0)}.$$

Observe that f is a rational function (analytic in $\overline{\mathbb{D}}$) that interpolates the function $1/z$ on the set $(\lambda_1, \dots, \lambda_d)$. More precisely,

$$zf(z) - 1 = h(z)m_T(z), \quad z \in \mathbb{D},$$

where $h(z) = \frac{1}{B(0)} \prod_{i=1}^{d} (\overline{\lambda_i} z - 1)^{-1}$, and therefore $Tf(T)$ is the identity matrix. Now, by [22, Theorem 2.4(3)],

$$\begin{aligned} \|T^{-1}\| = \|f(T)\| &\leq \frac{16Kd}{\pi} \left\| \frac{B(0) - B(z)}{zB(0)} \right\|_{H^\infty} \\ &\leq \frac{16Kd}{\pi \prod_{i=1}^{d} |\lambda_i|} \max_{z \in \mathbb{T}} |B(0) - B(z)| \\ &\leq \frac{32K}{\pi} \frac{d}{\prod_{i=1}^{d} |\lambda_i|}. \end{aligned} \tag{5.2}$$

■

Note that the above proof gives the explicit constant $\frac{32}{\pi}$ in (5.1). Yet we expect that it is not optimal.

Remark 5.1 For completeness, let us give an insight into the first inequality of (5.2). It is obtained as a combination of Vitse's functional calculus and the Bonsall–Walsh inequality: applying [22, Theorem 2.4(1)] to the function f, we get

$$\|T^{-1}\| = \|f(T)\| \leq 2K \|f\|_{\mathcal{B}_1},$$

where $\mathcal{B}_1$ is the analytic Besov algebra defined in Section 2, and it remains to apply the Bonsall–Walsh inequality [1] to the rational function f:

$$\|f\|_{\mathcal{B}_1} \leq \frac{8}{\pi} \deg f \, \|f\|_{H^\infty},$$

where $\deg f$ stands for the degree of f.

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