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# The Iitaka conjecture $C_{n,m}$ in dimension six

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Abstract

We prove that the Iitaka conjecture $C_{n,m}$ for algebraic fibre spaces holds up to dimension six, that is, when $n \leqslant 6$.

## 1. Introduction

We work over an algebraically closed field $k$ of characteristic zero. Let $X$ be a normal variety of dimension $n$ The canonical divisor $K_X$ is one of the most important objects associated with $X$ especially in birational geometry. If $Y \subseteq X$ is the smooth locus of $X$, the canonical sheaf $\omega_Y$ of $Y$ is defined as the exterior power $\bigwedge^n \Omega_Y$, where $\Omega_Y$ is the sheaf of regular differential 1-forms on $Y$. Since $Y$ is smooth, $\omega_Y \simeq \mathcal{O}_Y(K_Y)$ for some divisor $K_Y$, which is called the canonical divisor of $Y$. Obviously, $K_Y$ is not unique as a divisor but it is unique up to linear equivalence. By taking the closure of the irreducible components of $K_Y$, we obtain a divisor $K_X$ on $X$ that we call the canonical divisor of $X$. Since $X$ is normal, $X \setminus Y$ is a closed subset of codimension at least two. So, $K_X$ is uniquely determined up to linear equivalence.

If another normal variety $Z$ is in some way related to $X$, it is often crucial to find a relation between $K_X$ and $K_Z$. A classical example is when $Z$ is a smooth prime divisor on a smooth $X$, in which case we have $(K_X + Z)|_Z = K_Z$.

An algebraic fibre space is a surjective morphism $f\colon X \to Z$ of normal projective varieties, with connected fibres. A central problem in birational geometry is the Iitaka conjecture, which attempts to relate $K_X$ and $K_Z$ through invariants associated with them, namely the Kodaira dimension (see § 2 for the definition of Kodaira dimension).

Conjecture 1.1 (Iitaka). Let $f\colon X \to Z$ be an algebraic fibre space, where $X$ and $Z$ are smooth projective varieties of dimensions $n$ and $m$, respectively, and let $F$ be a general fibre of $f$. Then,

$$\kappa(X) \geqslant \kappa(F) + \kappa(Z).$$

This conjecture is usually denoted by $C_{n,m}$. A strengthened version was proposed by Viehweg (cf. [Vie83]) as follows, which is denoted by $C^+_{n,m}$.

Conjecture 1.2 (Iitaka–Viehweg). Under the assumptions of Conjecture 1.1,

$$\kappa(X) \geqslant \kappa(F) + \max\{\kappa(Z), \mathrm{var}(f)\}$$

when $\kappa(Z) \geqslant 0$.

Kawamata [Kaw85a] showed that these conjectures hold if the general fibre $F$ has a good minimal model, in particular, if the minimal model and the abundance conjectures hold in dimension $n - m$ for varieties of nonnegative Kodaira dimension. However, at the moment the

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minimal model conjecture for such varieties is known only up to dimension five [Bir08] and the abundance conjecture up to dimension three [Kaw92, Miy88] and some cases in higher dimensions, which will be discussed below. Viehweg [Vie83] proved $C^+_{n,m}$ when $Z$ is of general type. When $Z$ is a curve, $C_{n,m}$ was settled by Kawamata [Kaw82]. Kollár [Kol87] proved $C^+_{n,m}$ when $F$ is of general type. The latter also follows from Kawamata [Kaw85a] and the existence of good minimal models for varieties of general type by Birkar *et al.* [BCHM08]. We refer the reader to Mori [Mor87] for a detailed survey of the above conjectures and related problems. In this paper, we prove the following.

THEOREM 1.3. *The Iitaka conjecture $C_{n,m}$ holds when $n \leqslant 6$.*

THEOREM 1.4. *The Iitaka conjecture $C_{n,m}$ holds when $m = 2$ and $\kappa(F) = 0$.*

When $n \leqslant 5$ or when $n = 6$ and $m \neq 2$, $C_{n,m}$ follows immediately from theorems of Kawamata and deep results of the minimal model program.

The Iitaka conjecture is closely related to the following.

CONJECTURE 1.5 (Ueno). Let $X$ be a smooth projective variety with $\kappa(X) = 0$. Then, the Albanese map $\alpha\colon X \to A$ satisfies the following:

(1) $\kappa(F) = 0$ for the general fibre $F$;
(2) there is an etale cover $A' \to A$ such that $X \times_A A'$ is birational to $F \times A'$ over $A$;
(3) $\alpha$ is an algebraic fibre space.

The Ueno conjecture is often referred to as conjecture K, and Kawamata [Kaw81] showed that part (3) holds. See Mori [Mor87, § 10] for a discussion of this conjecture.

COROLLARY 1.6. *Part* (1) *of the Ueno conjecture holds when* $\dim X \leqslant 6$.

*Proof.* Immediate by Theorem 1.3. □

Concerning part (1) of the Ueno conjecture, recently Chen and Hacon [CH] showed that $\kappa(F) \leqslant \dim A$.

## 2. Preliminaries

*Nef divisors.* A Cartier divisor $L$ on a projective variety $X$ is called nef if $L \cdot C \geqslant 0$ for any curve $C \subseteq X$. If $L$ is a $\mathbb{Q}$-divisor, we say that it is nef if $lL$ is Cartier and nef for some $l \in \mathbb{N}$. We need a theorem about nef $\mathbb{Q}$-divisors due to Tsuji [Tsu00] and Bauer *et al.* [BCEKPRSW02].

THEOREM 2.1. *Let $L$ be a nef $\mathbb{Q}$-divisor on a normal projective variety $X$. Then, there is a dominant almost regular rational map $\pi\colon X \dashrightarrow Z$ with connected fibres to a normal projective variety, called the reduction map of $L$, such that*

(1) *if a fibre $F$ of $\pi$ is projective and $\dim F = \dim X - \dim Z$, then $L|_F \equiv 0$;*
(2) *if $C$ is a curve on $X$ passing through a very general point $x \in X$ with $\dim \pi(C) > 0$, then $L \cdot C > 0$.*

Here, by almost regular we mean that some of the fibres of $\pi$ are projective and away from the indeterminacy locus of $\pi$. Using the previous theorem, one can define the nef dimension $n(L)$ of the nef $\mathbb{Q}$-divisor $L$ to be $n(L) := \dim Z$. In particular, if $n(L) = 0$, the theorem says that $L \equiv 0$.

*Kodaira dimension.* For a divisor $D$ on a normal projective variety $X$ and $l \in \mathbb{N}$, we let

$$H^0(X, lD) = \{f \in k(X) \mid \operatorname{div}(f) + lD \geqslant 0\} \cup \{0\},$$

where $k(X)$ is the function field of $X$. It is well known that $H^0(X, lD)$ is a finite-dimensional vector space over the ground field $k$. If $r := H^0(X, lD) \neq 0$ and if $f_1, \ldots, f_r$ form a basis over $k$, we can define a rational map

$$\phi_{lD} \colon X \dashrightarrow \mathbb{P}^{\dim_k H^0(X,lD)-1}$$

by taking $x$ to $(f_1(x) : \cdots : f_r(x))$. Now, let

$$N(D) = \{l \in \mathbb{N} \mid H^0(X, lD) \neq 0\}.$$

If $N(D) \neq \emptyset$, we define the Kodaira dimension of $D$ to be

$$\kappa(D) = \max\{\dim \phi_{lD}(X) \mid l \in N(D)\}$$

but if $N(D) = \emptyset$ we let $\kappa(D) = -\infty$.

If $D$ is a $\mathbb{Q}$-divisor, one can define the Kodaira dimension $\kappa(D) := \kappa(lD)$ for any $l \in \mathbb{N}$ such that $lD$ is an integral divisor. This does not depend on $l$.

Campana and Peternell [CP07] made the following interesting conjecture.

Conjecture 2.2. Let $X$ be a smooth projective variety and suppose that $K_X \equiv A + M$, where $A$ and $M$ are effective and pseudo-effective $\mathbb{Q}$-divisors, respectively. Then, $\kappa(X) \geqslant \kappa(A)$.

They proved the conjecture in the case $M \equiv 0$ [CP07, Theorem 3.1]. This result is an important ingredient of the proofs below.

*Minimal models.* Let $X$ be a smooth projective variety. A projective variety $Y$ with terminal singularities is called a minimal model of $X$ if there is a birational map $\phi \colon X \dashrightarrow Y$, such that $\phi^{-1}$ does not contract divisors, $K_Y$ is nef, and finally there is a common resolution of singularities $f \colon W \to X$ and $g \colon W \to Y$ such that $f^*K_X - g^*K_Y$ is effective and its support contains the birational transform of any prime divisor on $X$ which is exceptional over $Y$. If in addition $lK_Y$ is base-point free for some $l \in \mathbb{N}$, we call $Y$ a good minimal model.

The minimal model conjecture asserts that every smooth projective variety $X$ has a minimal model or a Mori fibre space $Y$, in particular, if $X$ has nonnegative Kodaira dimension then it should have a minimal model $Y$. The abundance conjecture states that every minimal model is a good one.

Note that if the dimension of $X$ is at least three, a minimal model $Y$ could have singular points even though we assume $X$ to be smooth.

## 3. Proofs

*Proof of Theorem 1.4.* We are given that the base variety $Z$ has dimension two and that $\kappa(F) = 0$. We may assume that $\kappa(Z) \geqslant 0$, otherwise the theorem is trivial. Let $p \in \mathbb{N}$ be the smallest number such that $f_*\mathcal{O}_X(pK_X) \neq 0$. By Fujino and Mori [FM00, Theorem 4.5], there is a diagram

$$\begin{tikzcd} X' \arrow[r, "\sigma"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Z' \arrow[r, "\tau"] & Z \end{tikzcd}$$



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in which $g$ is an algebraic fibre space of smooth projective varieties, $\sigma$ and $\tau$ are birational, and there are $\mathbb{Q}$-divisors $B$ and $L$ on $Z'$ and a $\mathbb{Q}$-divisor $R = R^+ - R^-$ on $X'$ decomposed into its positive and negative parts satisfying the following:

(1) $B \geqslant 0$;

(2) $L$ is nef;

(3) $pK_{X'} = pg^*(K_{Z'} + B + L) + R$;

(4) $g_*\mathcal{O}_{X'}(iR^+) = \mathcal{O}_{Z'}$ for any $i \in \mathbb{N}$;

(5) $R^-$ is exceptional$/X$ and the codimension of $g(\operatorname{Supp} R^-)$ in $Z'$ is $\geqslant 2$.

Thus, for any sufficiently divisible $i \in \mathbb{N}$, we have

(6) $g_*\mathcal{O}_{X'}(ipK_{X'} + iR^-) = \mathcal{O}_{Z'}(ip(K_{Z'} + B + L))$.

If the nef dimension $n(L) = 2$ or if $\kappa(Z) = \kappa(Z') = 2$, then $ip(K_{Z'} + L)$ is big for some $i$ by Ambro [Amb04, Theorem 0.3]. So, $ip(K_{Z'} + B + L)$ is also big and by (6) and by the facts that $\sigma$ is birational and $R^- \geqslant 0$ is exceptional$/X$ we have

$$H^0(ipK_X) = H^0(ipK_{X'} + iR^-) = H^0(ip(K_{Z'} + B + L))$$

for sufficiently divisible $i \in \mathbb{N}$. Therefore, in this case $\kappa(X) = 2 \geqslant \kappa(Z)$.

If $n(L) = 1$, then the nef reduction map $\pi \colon Z' \to C$ is regular, where $C$ is a smooth projective curve, and there is a $\mathbb{Q}$-divisor $D'$ on $C$ such that $L \equiv \pi^* D'$ and $\deg D' > 0$ by [BCEKPRSW02, Proposition 2.11]. On the other hand, if $n(L) = 0$, then $L \equiv 0$. So, when $n(L) = 1$ or $n(L) = 0$, there is a $\mathbb{Q}$-divisor $D \geqslant 0$ such that $L \equiv D$. Now, letting $M := \sigma_* g^*(D - L)$, for sufficiently divisible $i \in \mathbb{N}$, we have

$$\begin{aligned} H^0(ip(K_X + M)) &= H^0(ip(K_{X'} + g^*D - g^*L) + iR^-) \\ &= H^0(ip(K_{Z'} + B + D)) \end{aligned}$$

and by Campana and Peternell [CP07, Theorem 3.1]

$$\kappa(X) \geqslant \kappa(K_X + M) = \kappa(K_{Z'} + B + D) \geqslant \kappa(Z). \qquad \square$$

*Proof of Theorem 1.3.* We assume that $\kappa(Z) \geqslant 0$ and $\kappa(F) \geqslant 0$, otherwise the theorem is trivial.

If $m = 1$, then the theorem follows from Kawamata [Kaw82]. On the other hand, if $n - m \leqslant 3$, then the theorem follows from Kawamata [Kaw85a] and the existence of good minimal models in dimension $\leqslant 3$. So, from now on we assume that $n = 6$ and $m = 2$; hence, $\dim F = 4$. By the flip theorem of Shokurov [Sho03] and the termination theorem of Kawamata *et al.* [KMM87, 5-1-15], $F$ has a minimal model (see also [Bir08]). If $\kappa(F) > 0$, by Kawamata [Kaw85b, Theorem 7.3] such a minimal model is good, so we can apply [Kaw85a] again. Another possible argument would be to apply Kollár [Kol87] when $F$ is of general type and to use the relative Iitaka fibration otherwise.

Now, assume that $\kappa(F) = 0$. In this case, though we know that $F$ has a minimal model, abundance is not yet known. Instead, we use Theorem 1.4. $\square$

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