# ON THE DISTRIBUTION OF THE TIME TO FIRST EMPTINESS OF A STORE WITH STOCHASTIC INPUT

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## 1. Introduction

Kendall [4] has given for the distribution of the time to first emptiness in a store with an input process which is homogeneous and has non-negative independent increments and an output of one unit per unit time the formula

$$g(t, z) = \frac{z}{t} k(t, t-z). \tag{1}$$

In this formula, $z$ is the initial content of the store, $g(t, z)$ is the density function of the time to first emptiness $\tau(z)$, defined by

$$P\{\tau(z) \leqq t\} = G(t, z) = \int_0^t g(u, z)du,$$

and $k(t, x)$ is the density function of the input process $\xi(t)$, defined by

$$P\{\xi(t) \leqq x\} = K(t, x) = \int_0^x k(t, y)dy.$$

Lloyd [5] has given the corresponding formula for the case of a discrete input in the form

$$q_n(z) = \frac{z}{z+n} p_n(z+n), \quad n = 0, 1, 2, \cdots$$

where

$$q_n(z) = P\{\tau(z) = z+n\},$$
$$p_n(z) = P\{\xi(t) = n\}.$$

However, as pointed out by Lloyd, Kendall did not establish formula (1) as giving the density function of $\tau(z)$. Kendall only showed that (1) satisfied the integral equation

$$g(t, z) = \int_0^{t-z} g(t-z, y)k(z, y)dy. \tag{2}$$

In fact, it can be easily shown that the integral equation (2) has the general solution

(3) $$g(t, z) = \int_{0-}^{t-z} k(t, t-z-x)dP(x)$$

where $P(x)$ is an arbitrary function of bounded variation. The particular solution (1) is obtained by taking $P(x) = U(x) - K_{10}(0, 2)$, where $K(t, x) = P\{\xi(t) \leqq x\}$, $K_{10}(0, x) = \partial K(t, x)/\partial t|_{t=0}$, and

$$U(x) = \begin{cases} 1 & \text{for } x \geqq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The last result can be obtained as follows: we must solve the equations

$$\int_{0-}^{t-z} k(t, t-z-x)dP(x) = \frac{z}{t} k(t, t-z).$$

We know that the Laplace transform of $k(t, x)$ can be written in the form

$$\int_0^\infty e^{-sx} k(t, x)dx = e^{-\alpha(s)t}$$

because of the additivity of the process $\xi(t)$.

Writing $t-z = u$, $P(x) = U(x) - Q(x)$, we find that

$$\int_{0-}^{u} k(t, u-x)dQ(x) = \frac{u}{t} k(t, u).$$

Take Laplace transforms. This yields

$$e^{-\alpha(s)t} \int_0^\infty e^{-sx} dQ(x) = \alpha'(s) e^{-\alpha(s)t},$$

i.e. the Laplace-Stieltjes transform of $Q(x)$ is $\alpha'(s)$.

But

$$\alpha'(s) = \int_0^\infty e^{-sx} k_{10}(t, x)dx\,|_{t=0} = \int_0^\infty e^{-sx} dK_{10}(0, x),$$

so that we can take $Q(x)$ to be $K_{10}(0, x)$. Our final answer is therefore

$$P(x) = U(x) - K_{10}(0, x).$$

In this paper we shall prove the following under very mild general restrictions:

(a) If the input process has a density function $k(t, x)$, then $\tau(z)$ has a density function $g(t, z)$, and

$$g(t, z) = \begin{cases} \dfrac{z}{t} k(t, t-z) & \text{if } \quad t \geqq z, \\ 0 & \qquad \text{otherwise.} \end{cases}$$

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(b) If $\xi(t)$ is a Compound Poisson process, then the distribution function, $G(t, z)$, of $\tau(z)$, is given by

$$(4)\quad G(t,z) = \begin{cases} \frac{z}{t} K(t, t-z) - \int_z^t \frac{z}{u^2} [u K_{10}(u, u-z) - K(u, u-z)] du & \text{if } t \geqq z \\ 0 & \text{otherwise.} \end{cases}$$

In particular, formula (4) will hold for discrete inputs, and we shall show that it reduces to Lloyd's formula in that case.

## 2. Definition and measurability of the time of first emptiness

We shall consider a store with an infinite capacity, having an input $\xi(t)$ in the time interval $(0, t]$, and a "planned output function" $\eta(t)$ during the same period. By this we mean that if the store did not become empty at any time during the period $(0, t]$, the realised output would be $\eta(t)$. We shall assume that $\xi(t)$ and $\eta(t)$ are arbitrary non-decreasing functions which are continuous to the right and bounded in any finite interval, and such that $\xi(0) = \eta(0) = 0$.

We shall set $\nu(t) = \xi(t) - \eta(t)$. Then $\nu(t)$ will be the "net planned input" to the store. We shall now further assume that $\nu(t)$ has no downwards discontinuities, i.e. that $\nu(t) - \nu(t-) \geqq 0$. Let $\nu^*(t) = -\inf_{0 \leqq u \leqq t} \nu(u)$. Then $\nu^*(t)$ is a non-decreasing function of $t$ with no discontinuities. Let $z$ be the initial content of the store. Kingman [3] has shown that the content, $\zeta(t)$, of the store at time $t$ can be defined by the following formula

$$\zeta(t) = \nu(t) + \max [z; \nu^*(t)].$$

Let now $\tau(z)$ be the time elapsing until the store becomes empty for the first time, i.e. the smallest value of $t$ for which $\zeta(t) = 0$. We shall first show that $\tau(z)$ is the smallest value of $t$ for which $\nu^*(t) = z$. In fact if $t_0$ is this smallest value, we have $\nu^*(t_0) = -\nu(t_0)$ and therefore $\zeta(t) = 0$. Moreover, as $\nu^*(t)$ is non-decreasing, we have, for all $t < t_0$, $\nu^*(t) < z$. This implies $-\nu(t) \leqq \nu^*(t) < z$, and consequently $\zeta(t) = \nu(t) + z > 0$.

Suppose now that the net planned input is a stochastic process $\nu(t, \omega)$. As $\nu$ is a function of bounded variation in $t$, $\nu(t, \omega)$ is separable. We now show that $\tau(z)$ is a random variable, i.e. a measurable function of $\omega$.

The event $\{\omega; \tau(z) \leqq t\}$ is given by

$$\begin{aligned} \{\omega; \tau(z) \leqq t\} &= \{\omega; \nu^*(t) \geqq z\} \\ &= \{\omega; \inf_{0 \leqq u \leqq t} \nu(u) \leqq -z\} \\ &= \bigcup_{0 \leqq u \leqq t} \{\omega; \nu(u) \leqq -z\}. \end{aligned}$$

It follows that the event $\{\omega;\ \tau(z) \leqq t\}$ is measurable, and therefore $\tau(z)$ is a (possibly defective) random variable. We now make the further assumption that $\nu(t)$ is a homogeneous stochastic process with independent increments, i.e. we shall assume that $\delta\nu = \nu(t+\delta t)-\nu(t)$ is a random variable which is independent of $\nu(t)$ and whose distribution depends on $\delta t$ only. In this case it is well known that we can write the Laplace-Stieltjes transform of the distribution function of $\nu(t)$ as

$$E[e^{-s\nu(t)}] = e^{-\phi(s)t}.$$

THEOREM 1. *Under the above assumptions, the Laplace-Stieltjes transform* $\Gamma(p, z) = E[\exp\{-p\tau(z)\}]$ is given by $\Gamma(p, z) = \exp\{-\theta z\}$, where $\theta$ satisfies the equation

$$p = -\phi(\theta).$$

PROOF. Because of the assumptions on the nature of the process $\nu(t)$, we obviously have

$$\tau(y+z) = \tau(y)+\tau(z) \tag{5}$$

where $\tau(y)$ and $\tau(z)$ are independent. It follows that the Laplace-Stieltjes transform of the distribution of $\tau(z)$, $\Gamma(p, z)$, where

$$\Gamma(p, z) = E[e^{-p\tau(z)}],$$

is of the form $\exp\{-\theta z\}$, where $\theta$ is some function of $p$. We also have the relation

$$\tau[\eta(z)] = z+\tau[\xi(z)] \tag{6}$$

for if the initial content of the store is $\eta(z)$, after a period of time of length $z$ the initial content has been exhausted, and the new content is the input in the period $[0, z]$, namely $\xi(z)$. We now extend the definition of $\tau(z)$ to negative values of the argument by setting $\tau(-z) = -\tau(z)$. Equation (5) then formally generalises to negative values of $y$ and $z$. Thus we can rewrite (6) as $\tau[\xi(z)-\eta(z)] = -z$ or $\tau[\nu(z)] = -z$.

From this we deduce

$$\begin{aligned} e^{pz} = E[\exp\{-p\tau[\nu(z)]\}] &= E\{E[\exp\{-p\tau[\nu(z)]\}|\nu(z)]\} \\ &= E[\exp\{-\theta\nu(z)\}] \\ &= \exp\{-\phi(\theta)z\}, \end{aligned}$$

so that finally $p = -\phi(\theta)$.

## 3. The Lagrange expansion of $\Gamma(p, z)$ in the case of an output of one unit per unit time

Let us now consider the special case where the input $\xi(t)$ is a homoge-

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neous process with independent increments and the planned output is given by $\eta(t) = t$.

We shall write

$$E[e^{-s\xi(t)}] = e^{-\alpha(s)t}.$$

We shall assume that $\alpha(s)$ can be expressed in the form

$$\alpha(s) = \int_{0-}^{\infty} (e^{-sx}-1)dM(x),$$

where $M(x)$ is a non-decreasing function such that $M(\infty) = 0$. We shall further assume that $\lim_{x\to 0} xM(x) = 0$. That $\exp\{-\alpha(s)t\}$ then corresponds to some process $\xi(t)$ with independent increments follows from the general theory of infinitely divisible distributions. See, for instance, Gnedenko and Kolmogorov [2].

Integrating by parts, we find

$$\alpha(s) = s\int_0^{\infty} e^{-sx}M(x)dx = s\beta(s), \text{ say.}$$

We shall also assume that $\alpha'(0)$ is finite, and consequently, as

$$\alpha'(0) = \lim_{s\to 0} \alpha(s)|s = \lim_{s\to 0}\int_0^{\infty} e^{-sx}M(x)dx,$$

the last limit will exist.

Finally, we note that in this case, the function which we had previously denoted by $\phi(s)$ is now equal to $\alpha(s)-s$, so that equation (7) becomes

$$p = \theta-\alpha(\theta). \tag{8}$$

We now introduce the following

THEOREM 2. There exist two real positive numbers $p_0$, $\sigma_0$ such that equation (8) has exactly one root $\theta$ satisfying $\text{Re}(\theta) > \sigma_0$, for all real values of $p$ satisfying $p > p_0$. Moreover, if $f(z)$ is a function analytic in $\text{Re}(z) > \sigma_0$, $f(\theta)$ is given by

$$f(\theta) = f(p)+\sum_{n=1}^{\infty}\frac{1}{n!}\frac{d^{n-1}}{dp^{n-1}}[f'(p)\{\alpha(p)\}^n].$$

PROOF. Let $s = \sigma+i\omega$. Then $\lim_{\sigma\to\infty}\beta(\sigma+i\omega) = 0$. Moreover, $|\beta(\sigma+i\omega)| \leqq \beta(\sigma_0)$ for all $\sigma \geqq \sigma_0$ and all $\omega$. It follows that we can choose $\sigma_0$ such that $|\beta(s)| \leqq \mu < \frac{1}{2}$ for all $s$ such that $\text{Re}(s) > \sigma_0$. We then have, in the same region, $|\alpha(s)| \leqq \mu|s|$. We now show that, if $p$ is real and $|s-p| > \mu p/(1-\mu)$, we have $|s-p| > |\alpha(s)|$ for all $s$ such that $\text{Re}\ (s) > \sigma_0$. In fact, we then have

$$|\alpha(s)| \leqq \mu|s| = \mu|s-p+p| \leqq \mu|s-p|+\mu p < |s-p|.$$

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Finally we note that if $p$ satisfies the inequality $p > (1-\mu)\sigma_0/(1-2\mu)$, all points such that $|s-p| \leqq \mu p/(1-\mu)$ will have an abscissa larger than $\sigma_0$, so that every point in $\text{Re}(s) > \sigma_0$ can be surrounded by a contour $C$ in the same region containing the circle $|s-p| = \mu p/(1-\mu)$. On this contour, we shall have $|s-p| > |\alpha(s)|$, and by applying Rouché's theorem, we conclude that the equation $s-p = \alpha(s)$ has only one root in $\text{Re}(s) > \sigma_0$. Moreover, any function $f(z)$ which is analytic in a region containing the contour $C$ can be expanded by using Lagrange's theorem, yielding the expansion given in the theorem.

COROLLARY.

$$\Gamma(p, z) = e^{-\theta z} = e^{-pz} - z\sum_{n=1}^{\infty} \frac{1}{n!}\frac{d^{n-1}}{dp^{n-1}}[e^{-pz}\{\alpha(p)\}^n]. \tag{9}$$

PROOF. The only point requiring checking is whether the root $\theta$ in the expression for $\Gamma(p, z)$ is the same as the one discussed in the theorem. This, however follows from the formula $\lim_{p\to\infty}\Gamma(p, z) = P\{\tau(z) = 0\} = 0$ for $z > 0$, which implies $\lim_{p\to\infty} \text{Re}(\theta) = +\infty$.

## 4. The inversion of $\Gamma(p, z)$ when the input has a density function

THEOREM 3. *Let the distribution of $\xi(t)$ have a density function $k(t, x)$. Moreover, let the Laplace-Stieltjes transform of $\xi(t)$ be of the form* $\exp[-\alpha(s)t]$ *where $\alpha(s) = s\int_0^\infty e^{-sx}M(x)dx$ and $\alpha'(0)$ is finite. Then $\tau(z)$ has a density function $g(t, x)$, which is given by*

$$g(t, x) = \frac{z}{t}k(t, t-z) \text{ for almost all } t,$$

*provided that*

$$\int_y^\infty e^{-pt}\frac{y}{t}k(t, t-y)dt$$

*is of bounded variation in $y$ in some neighbourhood of $y = z$.*

PROOF. We have

$$\int_0^\infty e^{-sx}k(t, x)dx = e^{-\alpha(s)t}, \quad \text{Re}(s) \geqq 0,$$

where $\text{Re}[\alpha(s)] \geqq 0$. We deduce that

$$\int_0^\infty\int_0^\infty e^{-pt-sx}\, k(t, x)dx = \frac{1}{p+\alpha(s)}, \quad \text{Re}(p) > 0. \tag{10}$$

Let us for the moment restrict $s$ and $p$ to real positive values, and change variables in (10) by replacing $x$ by $t-z$. We find that

$$\frac{1}{p+\alpha(s)} = \int_0^\infty \int_{-\infty}^t e^{-(p+s)t+sz} k(t, t-z) dz\, dt.$$

Write now $p$ for $p+s$. We obtain

$$\frac{1}{p-s+\alpha(s)} = \int_0^\infty e^{-pt} \left\{ \int_{-\infty}^t e^{sz} k(t, t-z) dz \right\} dt.$$

Differentiate both sides with respect to $s$. We have

$$\text{(11)} \qquad \frac{1-\alpha'(s)}{[p-s+\alpha(s)]^2} = \int_0^\infty e^{-pt} \left\{ \int_{-\infty}^t e^{sz} z k(t, t-z) dz \right\} dt,$$

where the double integral still converges absolutely.

Integrate both sides of (11) with respect to $p$ from $p$ to infinity. We obtain

$$\text{(12)} \qquad -\frac{1-\alpha'(s)}{p-s+\alpha(s)} = \int_0^\infty e^{-pt} \left\{ \int_{-\infty}^t e^{sz} \frac{z}{t} k(t, t-z) dz \right\} dt.$$

Let us put

$$g^*(t, z) = \begin{cases} \dfrac{z}{t} k(t, t-z) & \text{for } z \leqq t, \\ 0 & \text{otherwise.} \end{cases}$$

Then we can write (12) as

$$\text{(13)} \qquad -\frac{1-\alpha'(s)}{p-s+\alpha(s)} = \int_0^\infty e^{-pt} \int_{-\infty}^{+\infty} e^{sz} g^*(t, z) dz\, dt.$$

As the double integral converges absolutely, we can use Fubini's theorem to interchange the integrals, thus obtaining

$$-\frac{1-\alpha'(s)}{p-s+\alpha(s)} = \int_{-\infty}^{+\infty} e^{sz} \left\{ \int_0^\infty e^{-pt} g^*(t, z) dt \right\} dz.$$

As the integral converges for all positive values of $p$, and all values of $s$ such that Re $(s) > 0$, the last equation holds for all $s$ such that $\text{Re}(s) > 0$.

Now

$$\lim_{R\to\infty} -\frac{1}{2\pi i} \int_{c-iR}^{c+iR} \frac{[1-\alpha'(s)]e^{-sz}}{p-s+\alpha(s)} ds = e^{-\theta z}, \qquad z > 0,$$

if $\sigma_0 < c < \mu p/(1-\mu)$, where $\sigma_0$ and $\mu$ are as defined in the theorem of section 3, $p > (1-\mu)\sigma_0/(1-2\mu)$, and $\theta$ is the unique root of $p-s+\alpha(s) = 0$ in Re $(s) > \sigma_0$. This follows immediately from the fact that in Re $(s) > \sigma_0$, $|1-\alpha'(s)| \leqq 1+\alpha'(0)$ and is therefore bounded, and $|p-s+\alpha(s)| > |s|-p -\mu|s| > \frac{1}{2}|s|-p$, so that the integral along the semi-circle of radius $R$ with

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centre at $s = c$ which lies to the right of the line $\text{Re}(s) = c$ tends to zero when $R \to \infty$. It now follows from a theorem of Widder [7], p. 241, on the bilateral Laplace transform, that

$$\int_0^\infty e^{-pt} g^*(t, z) dt = e^{-\theta z}, \qquad z > 0,$$

for all sufficiently large real positive $p$.

Finally, it follows from the uniqueness theorem for Laplace transform (see Widder [6], p. 63), that

$$g^*(t, z) = g(t, z) \text{ for almost all } t.$$

This completes the proof of the theorem.

## 5. The inversion of $\Gamma(p, z)$ in the case of a compound Poisson input

The Lagrange expansion technique used in this section is similar to that which is used in the derivation of the Borel-Tanner distribution in queueing theory, which is a special case.

Let the points of increase of $\xi(t)$ follow a Poisson law with parameter $\lambda$, and let the distribution function of the jumps be $B(x)$. Then

$$K(t, x) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} B_n(x), \qquad t \geqq 0,$$

where $B_n(x)$ is the $n$-th convolution of $B(x)$ with itself. Expanding $e^{-\lambda t}$ in powers of $t$ and multiplying out the two series, we find that $K(t, x)$ admits the expansion

$$K(t, x) = U(x) \left[ 1 + tK_{10}(0, x) + \frac{t^2}{2!} K_{20}(0, x) + \cdots \right] \tag{14}$$

where $K_{n0}(0, x)$ represents the $n$-th derivative of $K(t, x)$ with respect to the first argument, $t$, for $t = 0$, and $U(x)$ is the Heaviside unit function, defined previously. The $K_{n0}(0, x)$ are given by

$$K_{n0}(0, x) = (-1)^n \lambda^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} B_k(x).$$

It follows that

$$|K_{n0}(0, x)| \leqq \lambda^n \sum_{k=0}^{n} \binom{n}{k} = (2\lambda)^n,$$

so that

$$\left| \sum_{n=0}^{N} \frac{t^n}{n!} K_{n0}(0, x) \right| \leqq \sum_{n=0}^{N} \frac{(2\lambda t)^n}{n!} \leqq e^{2\lambda t}, \qquad t \geqq 0. \tag{15}$$

Thus the partial sums of the expansion (14) are uniformly dominated by $e^{2\lambda t}$.

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THEOREM 4. *If $\xi(t)$ is a Compound Poisson Process, the distribution function, $G(t, z)$, of $\tau(z)$ is given by the formula*

$$G(t,z) = \begin{cases} \dfrac{z}{t}K(t,t-z) - \displaystyle\int_z^t \frac{z}{u^2}\left[uK_{10}(u,u-z) - K(u,u-z)\right]du & \text{if} \quad t \geqq z, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Taking the Laplace-Stieltjes transform of (14) term by term, and equating the coefficients of the powers of $t$, we find that

$$p\int_0^\infty e^{-px}K_{n0}(0,x)U(x)dx = (-1)^n[\alpha(p)]^n,\ n = 0, 1, 2, \cdots.$$

From this, we deduce, using the usual rules for change of variable in Laplace transforms,

$$e^{-pz}[\alpha(p)]^n = (-1)^n p\int_0^\infty e^{-pt}K_{n0}(0,t-z)U(t-z)dt,$$

and, denoting the Laplace-Stieltjes transform of $f(t)$, (which can be written in the two equivalent forms $\int_{0-}^\infty e^{-pt}df(t)$, $p\int_0^\infty e^{-pt}f(t)dt$,) by $\mathscr{L}[f(t)]$, we can write

$$\begin{aligned} \frac{1}{n!}\frac{d^{n-1}}{dp^{n-1}}\left[e^{-pz}\{\alpha(p)\}^n\right] &= -\frac{p}{n!}\int_0^\infty e^{-pt}t^{n-1}K_{n0}(0,t-z)U(t-z)dt \\ &\quad + \frac{n-1}{n!}\int_0^\infty e^{-pt}t^{n-2}K_{n0}(0,t-z)U(t-z)dt, \\ &= -\frac{p}{n!}\int_0^\infty e^{-pt}t^{n-1}K_{n0}(0,t-z)U(t-z)dt \\ &\quad + \frac{n-1}{n!}p\int_0^\infty e^{-pt}dt\int_0^\infty u^{n-2}K_{n0}(0,u-z)U(u-z)du, \\ &= \mathscr{L}\left[\frac{t^{n-1}}{n!}K_{n0}(0,t-z)U(t-z)\right. \\ &\quad \left. + \frac{n-1}{n!}\int_0^t u^{n-2}K_{n0}(0,u-z)U(u-z)du\right]. \end{aligned} \tag{16}$$

We now use the inequalities

$$\left|\sum_{n=0}^N \frac{t^{n-1}}{n!}K_{n0}(0,t-z)\right| \leqq \frac{1}{t}e^{2\lambda t}U(t-z),$$

$$\left|\sum_{n=0}^N \frac{n-1}{n!}\int_0^t u^{n-2}K_{n0}(0,u-z)du\right| \leqq \frac{1}{t}e^{2\lambda t}U(t-z),$$

which follow easily from (15). It follows that the sums involved in the inequalities are uniformly dominated by $(1/t)e^{2\lambda t}U(t-z)$, and this function in turn has a convergent Laplace-Stieltjes transform for all $p > 2\lambda$, $z > 0$.

Using now Lebesgue's dominated convergence theorem, (see Loève [6], p. 125) we can sum equation (16) from $n = 1$ to $n = +\infty$, and we obtain

$$-z\sum_{n=1}^{\infty}\frac{1}{n!}\frac{d^{n-1}}{dp^{n-1}}[e^{-pz}\{\alpha(p)\}^n] = \mathscr{L}\Big[\frac{z}{t}\{K(t, t-z)-1\}U(t-z) - \int_0^t \frac{z}{u}K_{10}(u, u-z)U(u-z)du + \int_0^t \frac{z}{u^2}\{K(u, u-z)-1\}U(u-z)du\Big].$$

Replacing in (9), and using

$$e^{-pz} = p\int_0^\infty e^{-pt}U(t-z)dt,$$

$$\int_0^t \frac{z}{u^2}U(u-z)du = \left(1-\frac{z}{t}\right)U(t-z),$$

we finally find

$$\Gamma(p, z) = \mathscr{L}\left[\frac{z}{t}K(t, t-z)U(t-z) - \int_0^t \frac{z}{u}K_{10}(u, u-z)U(u-z)du + \int_0^t \frac{z}{u^2}K(u, u-z)U(u-z)du\right].$$

But as the Lagrange expansion (9) holds for all $p$ such that Re $(p) > 0$, it follows from the uniqueness property of the Laplace-Stieltjes transform (see Widder [7], p. 63) that if $G(t, z)$ is the distribution function of $\tau(z)$, we have

$$G(t, z) = \frac{z}{t}K(t, t-z)U(t-z) - \int_0^t \frac{z}{u}K_{10}(u, u-z)U(u-z)du + \int_0^t \frac{z}{u^2}K(u, u-z)U(u-z)du.$$

This can be rewritten more simply

$$(17)\quad G(t, z)\begin{cases}\dfrac{z}{t}K(t, t-z) - \displaystyle\int_z^t \frac{z}{u^2}[uK_{10}(u, u-z) - K(u, u-z)]du & \text{if } t \geqq z\\ 0 & \text{otherwise.}\end{cases}$$

COROLLARY. *If, for fixed* $z$, $K(t, x)$ *has continuous derivatives in both* $t$ *and* $x$ *at the point* $(t, t-z)$, *and if we write*

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$$\frac{\partial}{\partial x} K(t, x) = k(t, x),$$

*then at the point* $(t, z)$, $G(t, z)$ *has a continuous partial derivative in* $t$, *given by*

$$\frac{\partial}{\partial t} G(t, z) = g(t, z) = \frac{z}{t} k(t, t-z).$$

PROOF. Differentiating both sides of (17), we obtain

$$\begin{aligned} g(t, z) = \frac{\partial}{\partial t} G(t, z) &= -\frac{z}{t^2} K(t, t-z) + \frac{z}{t} K_{10}(t, t-z) + \frac{z}{t} k(t, t-z) \\ &\quad - \frac{z}{t} K_{10}(t, t-z) + \frac{z}{t^2} K(t, t-z), \\ &= \frac{z}{t} k(t, t-z). \end{aligned}$$

This is Kendall's formula.

## 6. The case of a discrete input

Let us now assume that the input $\xi(t)$ takes only integral values. It is then clear that emptiness can occur only at times $z+n$, where $n = 0, 1\ 2, \cdots$. We shall write

$$P\{\xi(t) = n\} = p_n(t),$$
$$P\{\tau(z) = z+n\} = q_n(z),$$

and we shall assume that the $p_n(t)$ have continuous derivatives. We then have

$$K(t, x) = \sum_{k=0}^{[x]} p_k(t),$$
$$G(t, z) = \sum_{k=0}^{[t-z]} q_k(z).$$

Equation (17) now takes the form

$$\sum_{k=0}^{n} q_k(z) = \frac{z}{z+n} \sum_{k=0}^{n} p_k(z+n) - \int_z^{z+n} \frac{z}{u^2} \left[ u \sum_{k=0}^{[u-z]} p'_k(u) - \sum_{k=0}^{[u-z]} p_k(u) \right] du.$$

Write $n-1$ for $n$ and subtract. We find

$$\begin{aligned} q_n(z) = \frac{z}{z+n} p_n(z+n) + \sum_{k=0}^{n-1} z \left[ \frac{p_k(z+n)}{z+n} - \frac{p_k(z+n-1)}{z+n-1} \right] \\ - \int_{z+n-1}^{z+n} \frac{z}{u^2} \left[ u \sum_{k=0}^{[u-z]} p'_k(u) - \sum_{k=0}^{[u-z]} p_k(u) \right] du. \end{aligned}$$

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It is easily checked that the last two terms of the right-hand side of this equation cancel out, and we are left with

$$q_n(z) = \frac{z}{z+n} p_n(z+n),$$

which is precisely Lloyd's formula.

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