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# FRACTIONAL POISSON PROCESS: LONG-RANGE DEPENDENCE AND APPLICATIONS IN RUIN THEORY

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**Abstract**

We study a renewal risk model in which the surplus process of the insurance company is modelled by a compound fractional Poisson process. We establish the long-range dependence property of this nonstationary process. Some results for ruin probabilities are presented under various assumptions on the distribution of the claim sizes.

*Keywords:* Fractional Poisson process; renewal process; long-range dependence; ruin probability

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## 1. Introduction

Increasing interest has recently been paid to anomalous diffusion, i.e. processes whose variances increase in time according to a power law $t^{\gamma}$ with $\gamma \neq 1$. This is effectively the case in seismology (see [21]) where power-law functions are used to model earthquake interarrival times. We refer the reader to [7] for other geophysical applications of such power-law interarrival times.

One of the models of random processes that demonstrates such a phenomenon was investigated by Repin and Saichev [22], who were the first to introduce the so-called fractional Poisson process. In the differential equations governing the Poisson process they replaced the time derivative with a fractional derivative (see also [5], [6], [14], and [16] for similar approaches). They also considered the characterization of the Poisson process as a sum of independent, nonnegative random variables. We assume that these random variables have the Mittag-Leffler distribution instead of the exponential distribution. This is the so-called renewal approach that has been studied in [17]. This will be the approach that we adopt in our article. Note, however, that Meerschaert *et al.* [20] proved that the fractional Poisson process (defined as a renewal process) coincides with the fractal time Poisson process which is defined as the time-changed usual Poisson process with the right-continuous inverse of a standard $h$-stable subordinator.

The power law that governs the time evolution of the variance is often related to the notion of long-range dependence. The long-range dependence is well defined for stationary processes and little seems to be known about the extension of the long-range dependence to nonstationary processes. Such an extension has been proposed in [13] and this will be our starting point for

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a new result concerning the fractional Poisson process. Indeed, we establish the long-range dependence property of the fractional Poissonian noise (see Proposition 1 below). This is the main result of this work and it is the subject of Section 3. This property clearly justifies the use of the fractional Poisson process in many concrete models and gives some precisions about the intuitive approach using the power decay of the variance. For practical purposes, we mention that simulations of the waiting times and parameter estimation for the fractional Poisson process have been carried out in [9] and [25].

Our main motivation behind studying the fractional Poisson process is its application in actuarial sciences, in particular, as mentioned above, its use in modelling extreme events, such as earthquakes or storms. The long-range dependence of the fractional Poisson process, as well as the fact that the expectations of the interarrivals are infinite, are other admissible arguments to consider the following model.

We shall work with the renewal risk model in which the surplus process of the insurance company is modelled by

$$R_t = u + ct - \sum_{i=1}^{N_h(t)} X_i, \qquad t \geq 0, \tag{1}$$

where $u$ is the initial capital, $c$ is the constant premium rate, and the sequence of independent and identically distributed random variables $(X_i)_{i\geq 1}$ models the sizes of the successive claims (hence, these random variables are supposed to be nonnegative). To the best of our knowledge, only the work of Beghin and Macci [4] deals with a fractional model for insurance. In that work, the authors established a large deviation principle for the fractional Poisson process and proved asymptotic results for the ruin probabilities of an insurance model like the one given by (1). To complete the review of the existing literature on the fractional compound Poisson process, we also mention [3] and [24].

In the second part of this work, we give an overview of the known results that apply to this context. Some are easy, but strengthened by the fact that our investigations are a first step toward the description of fractional Poisson models of surplus processes. Let us briefly describe the properties we establish.

In Section 4 we use the duality relation between our model and a compound Poisson model with arbitrary claim size distribution. This allows us to establish a closed-form formula for the density of the time to ruin when the claim sizes are exponentially distributed. The ruin probabilities in finite and infinite time are also studied.

The ruin probability in the context of heavy-tailed claim sizes is the topic of Section 5. The properties that we establish are a consequence of the light-tailed distribution of the fractional Poisson process.

Finally, a Lundberg inequality is proposed in Section 6 in which a bound for the ruin probability is proposed when the claim sizes have a light-tailed distribution.

Some preliminary results on the fractional Poisson process are gathered in Section 2 and the proof of a technical inequality is given in Appendix A. Section 3 is devoted to the long-range dependence property.

## 2. Preliminaries on the fractional Poisson process

The fractional Poisson process was first defined in [22] as a renewal process with Mittag-Leffler waiting times. This means that it has independent and identically distributed waiting

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times $(\Delta_{T_k})_{k\geq 1}$, with distribution given by

$$\mathbb{P}(\Delta_{T_k} > t) = E_h(-\lambda t^h)$$

for $\lambda > 0$ and $0 < h \leq 1$, where

$$E_h(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+hk)}$$

is the Mittag-Leffler function ($\Gamma$ denotes the Euler gamma function) which is defined for any complex number $z$. We can also characterize the distribution of the waiting times by their Laplace transform

$$L_h(\xi) = \mathbb{E}(\exp(-\xi \Delta_{T_k})) = \frac{\lambda}{\lambda + \xi^h}.$$

With $T_n = \Delta_{T_1} + \cdots + \Delta_{T_n}$ the time of the $n$th jump, the process $(N_h(t))_{t\geq 0}$ defined by

$$N_h(t) = \max\{n \geq 0\colon T_n \leq t\} = \sum_{k\geq 1} \mathbf{1}_{\{T_k \leq t\}}$$

is a renewal process with Mittag-Leffler waiting times. It is called a fractional Poisson process of parameter $h$. When necessary, we write $\Delta_{T_k} = T_k - T_{k-1}$ with the convention that $T_{k-1} = 0$ if $k = 0$ (hence, $\Delta_{T_1} = T_1$). Of course, when $h = 1$, the Mittag-Leffler function with parameter 1 is the exponential function, and the waiting times become exponential. Thus, a fractional Poisson process of parameter $h = 1$ is the usual Poisson process. As in the classical case, we do not specify the dependence on $\lambda$ in the notation of the fractional Poisson process $N_h$. Since the parameter $\lambda$ plays a minor role in the properties of the process, this eases the notation for the reader.

In [22], it was proved that there exists a constant $C$ such that

$$\mathbb{P}(\Delta_{T_k} > t) \sim Ct^{-h} \quad \text{as } t \to +\infty.$$

Consequently, the interarrivals $\Delta_{T_k}$ have heavy tails and infinite mean for $0 < h < 1$. We shall assume in the sequel that the fractional Poisson process is light tailed, that is, $\mathbb{E}(\mathrm{e}^{\xi N_h(t)}) < \infty$ for any $\xi > 0$. This is a consequence of the existence of the probability generating function, which is given by

$$\mathbb{E}(z^{N_h(t)}) = E_h(\lambda t^h (z-1)) \tag{2}$$

for any $z > 0$ (see [9], [14], and [16]). The above result has been proved using fractional differential equations and fractional calculus. We note that an alternative approach is feasible using the fractional Poisson process representation (see [20]) as the fractal time Poisson process $(N_1(E_h(t)))_{t\geq 0}$. This process is defined as the time-changed usual Poisson process $(N_1(t))_{t\geq 0}$ with $(E_h(t))_{t\geq 0}$ the right-continuous inverse of a standard $h$-stable subordinator $(D_h(t))_{t\geq 0}$. This means that $E_h(t) = \inf\{r > 0\colon D_h(r) > t\}$, where $\mathbb{E}(\mathrm{e}^{-sD_h(t)}) = \exp(-ts^h)$. In the rest of this work we will use the renewal approach. However, it is worth mentioning the fractal time approach because it is a powerful tool that will be useful for obtaining new properties of both ruin problems and Poisson process studies. For example, one may deduce a diffusion-type approximation of the risk process using $h$-stable processes. This will be the subject of future work.

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## 3. Long-range dependence

In this section we aim to prove that the fractional Poissonian noise $(X_j^h)_{j\geq 1}$ defined for $j \geq 1$ by

$$X_j^h = N_h(j) - N_h(j-1)$$

has a long-range dependence property. Long-range dependence is frequently understood in terms of the power-law decay of the correlation function, and in the theory of stochastic processes it is standard to apply the notion of long-range dependence to only stationary processes. This is also the case for renewal processes $(N_t)_{t\geq 0}$ for which the notion of long-range dependence is defined as

$$\limsup_{t\to\infty} \frac{\operatorname{var}(N_t)}{t} = \infty \tag{3}$$

when the process is assumed to be stationary. It is known that condition (3) is then equivalent to the infiniteness of the second moment of the interarrivals. This holds for the interarrivals $\Delta_{T_k}$ of the process $N_h$ and, thus, the fractional Poisson process is said to have the long-range dependence property. We refer the reader to [10], [11], and [12] for further information about long-range dependence for stationary point processes.

The stationarity assumption is not always fulfilled in certain areas of application and this is effectively the case in our study. So the above comments cannot justify the long-range dependence property. In this section we therefore aim to determine an appropriate notion of long-range dependence for nonstationary processes. Heyde and Yang [13] suggested modifying existing second-order definitions of long-range dependence so that they apply to nonstationary processes as well. Specifically, they proposed the following definition.

**Definition 1.** A second-order process $(X_m)_{m\geq 1}$ (not necessarily stationary) has the property of long-range dependence if the block mean process

$$Y_t^{(m)} = \frac{\sum_{j=tm-m+1}^{j=tm} X_j}{\sum_{j=tm-m+1}^{j=tm} \operatorname{var}(X_j)}$$

defined for an integer $t \geq 1$ satisfies

$$\lim_{m\to\infty} \Bigg( \sum_{j=tm-m+1}^{j=tm} \operatorname{var}(X_j) \Bigg) \operatorname{var}(Y_t^{(m)}) = +\infty. \tag{4}$$

**Remark 1.** If the process $(X_m)_{m\geq 1}$ is centred then it has the long-range dependence property if

$$\lim_{m\to\infty} \Bigg( \sum_{j=tm-m+1}^{j=tm} \mathbb{E}(X_j^2) \Bigg) \operatorname{var}(Y_t^{(m)}) = +\infty$$

with

$$Y_t^{(m)} = \frac{\sum_{j=tm-m+1}^{j=tm} X_j}{\sum_{j=tm-m+1}^{j=tm} \mathbb{E}(X_j^2)}.$$

This leads to the formulation proposed in [13].

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**Remark 2.** In the above definition, if the process $(X_m)_{m\geq 1}$ is centred and stationary then (4) implies that

$$\lim_{m\to\infty} \frac{\operatorname{var}(\sum_{j=1}^m X_j)}{m} = \infty. \tag{5}$$

Thus, the variance of the sample mean of $m$ consecutive observations grows more slowly asymptotically than a sequence of independent and identically distributed random variables. We recall that a sufficient condition for (5) is $\lim_{m\to\infty} \sum_{j=1}^m \operatorname{cov}(X_0, X_j) = \infty$. Consequently, formulation (4) is in accordance with the heuristic approaches and the usual definitions of long-range dependence. We refer the reader to [13] for further discussion.

Using the notion of long-range dependence stated in Definition 1, we have the following result.

**Theorem 1.** *The fractional Poissonian noise $(X_j^h)_{j\geq 1}$ has the long-range dependence property for any $h \in (0;1)$.*

Before proving this result we show that convergence (4) holds. For a fixed integer $t \geq 1$, define

$$\Delta_t^{(m)} = \frac{\operatorname{var}(\sum_{j=tm-m+1}^{j=tm} X_j^h)}{\sum_{j=tm-m+1}^{j=tm} \operatorname{var}(X_j^h)}.$$

We note that

$$\Delta_t^{(m)} = \frac{\operatorname{var}(N_h(tm) - N_h(tm-m))}{\sum_{j=tm-m+1}^{j=tm} \operatorname{var}(N_h(j) - N_h(j-1))}. \tag{6}$$

We thus need the variances of the increments of the fractional Poisson process, which do not follow easily from the expression of the moment generating functions. We therefore use the fact that the fractional Poisson process is a renewal process and a known result about the factorial moments of a renewal process.

**Lemma 1.** *Let $0 \leq s \leq t$. We have*

$$\mathbb{E}[(N_h(t) - N_h(s)) \times (N_h(t) - N_h(s) - 1)] = 2h\biggl(\frac{\lambda}{\Gamma(1+h)}\biggr)^2 \int_s^t (t-r)^h r^{h-1}\,\mathrm{d}r. \tag{7}$$

*Proof.* The proof follows from the renewal function

$$M_h(t) = \mathbb{E}(N_h(t)) = \frac{\lambda t^h}{\Gamma(1+h)} \tag{8}$$

and Proposition 1 of [15] (see also [10]).

*Proof of Theorem 1.* For a fixed integer $t \geq 1$, we investigate the asymptotic behaviour of $\Delta_t^{(m)}$ defined in (6) as $m$ goes to $\infty$. By (7) and (8), we have

$$\begin{aligned}&\mathbb{E}((N_h(j) - N_h(j-1))^2)\\ &\quad = 2h\biggl(\frac{\lambda}{\Gamma(1+h)}\biggr)^2 \int_{j-1}^j (j-r)^h r^{h-1}\,\mathrm{d}r + \frac{\lambda}{\Gamma(1+h)}(j^h - (j-1)^h).\end{aligned}$$

Since

$$\int_{j-1}^j (j-r)^h r^{h-1}\,\mathrm{d}r \leq \int_{j-1}^j r^{h-1}\,\mathrm{d}r = \frac{1}{h}(j^h - (j-1)^h),$$

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we deduce that

$$\sum_{j=tm-m+1}^{j=tm} \mathbb{E}((N_h(j)-N_h(j-1))^2) \le \frac{\lambda}{\Gamma(1+h)}\left(\frac{2\lambda}{\Gamma(1+h)}+1\right)[t^h-(t-1)^h]m^h$$
$$\le t^h \frac{\lambda}{\Gamma(1+h)}\left(\frac{2\lambda}{\Gamma(1+h)}+1\right) m^h.$$

Using similar arguments, we have

$$\sum_{j=tm-m+1}^{j=tm} (\mathbb{E}(N_h(j)-N_h(j-1)))^2 = \sum_{j=tm-m+1}^{j=tm}\left(\frac{\lambda}{\Gamma(1+h)}h\int_{j-1}^{j} r^{h-1}\,\mathrm{d}r\right)^2$$
$$\ge \left(\frac{h\,\lambda}{\Gamma(1+h)}\right)^2 \sum_{j=tm-m+1}^{j=tm} j^{2h-2}$$
$$\ge \left(\frac{h\,\lambda}{\Gamma(1+h)}\right)^2 m(tm)^{2h-2}$$
$$\ge \left(\frac{h\,\lambda\,t^{h-1}}{\Gamma(1+h)}\right)^2 m^{2h-1}.$$

Consequently, there exists a constant $C_{t,\lambda,h}$ such that the denominator of $\Delta_t^{(m)}$ satisfies

$$\sum_{j=tm-m+1}^{j=tm} \operatorname{var}(N_h(j)-N_h(j-1))$$
$$\le t^h \frac{\lambda}{\Gamma(1+h)}\left(\frac{2\lambda}{\Gamma(1+h)}+1\right)m^h(1-C_{t,\lambda,h}m^{h-1}). \tag{9}$$

In the same way, by (7) and (8), we also may write that

$$\operatorname{var}(N_h(tm)-N_h(tm-m))$$
$$= 2h\left(\frac{\lambda}{\Gamma(1+h)}\right)^2\int_{tm-m}^{tm}(tm-r)^h r^{h-1}\,\mathrm{d}r + \frac{\lambda}{\Gamma(1+h)}((tm)^h-(tm-m)^h)$$
$$-\left\{\frac{\lambda}{\Gamma(1+h)}((tm)^h-(tm-m)^h)\right\}^2.$$

Since

$$\int_{tm-m}^{tm}(tm-r)^h r^{h-1}\,\mathrm{d}r = (tm)^{2h}\int_{1-1/t}^{1}(1-u)^h u^{h-1}\,\mathrm{d}u \ge (tm)^{2h}\mathcal{B}(1+h,h),$$

where $\mathcal{B}$ denotes the beta function, defined for $a>0$ and $b>0$ by

$$\mathcal{B}(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\,\mathrm{d}u = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

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with $\Gamma$ the gamma function, we obtain

$$\begin{aligned}
&\operatorname{var}(N_h(tm) - N_h(tm-m)) \\
&\quad\geq \left(\frac{\lambda}{\Gamma(1+h)}\right)^2 [2ht^{2h}\mathcal{B}(1+h,h) - \{t^h - (t-1)^h\}^2]m^{2h} \\
&\qquad + \frac{\lambda}{\Gamma(1+h)}(t^h - (t-1)^h)m^h \\
&\quad\geq \left(\frac{\lambda}{\Gamma(1+h)}\right)^2 t^{2h}[2h\mathcal{B}(1+h,h) - 1]m^{2h} \\
&\qquad + \frac{\lambda}{\Gamma(1+h)}(t^h - (t-1)^h)m^h,
\end{aligned} \tag{10}$$

where we have used the inequality $(t^h - (t-1)^h)^2 \leq t^{2h}$. For sufficiently large $m$, substituting (9) and (10) into (6) yields

$$\begin{aligned}
\Delta_t^{(m)} &\geq \left\{t^h \frac{\lambda/\Gamma(1+h)}{2\lambda/\Gamma(1+h)+1}[2h\mathcal{B}(1+h,h)-1]m^h + \frac{1}{2\lambda/\Gamma(1+h)+1}\right\}\frac{1}{1-C_{t,\lambda,h}m^{h-1}} \\
&\geq \left\{\frac{t^h\lambda}{2\lambda+\Gamma(1+h)}[2h\mathcal{B}(1+h,h)-1]m^h + \frac{\Gamma(1+h)}{2\lambda+\Gamma(1+h)}\right\}\frac{1}{1-C_{t,\lambda,h}m^{h-1}}.
\end{aligned}$$

We employ the technical inequality

$$2h\mathcal{B}(1+h,h) - 1 > 0, \tag{11}$$

which is valid for any $h \in (0,1)$; see Appendix A for its proof. Thus, $\lim_{m\to\infty} \Delta_t^{(m)} = +\infty$ and, consequently, the long-range dependence property holds.

## 4. Probability of ruin with exponential claim sizes

In this section we restrict our attention to the case of exponential claims. To be more precise, we consider the model defined by (1), where the random variables $(X_i)_{i\geq 1}$ are assumed to be nonnegative, independent, and identically distributed as $\mathcal{E}(\mu)$ for $\mu > 0$. We again assume that the sequence of claim sizes is independent of the fractional Poisson process $N_h$. Note that the ruin problem is nontrivial in infinite time for any $c > 0$ because $\mathbb{E}(X_1 - cT_1) = -\infty$.

### 4.1. Closed-form representation for the distribution of the ruin time

We derive an explicit formula for the distribution of the ruin time $\tau$ defined by

$$\tau = \inf\{t > 0\colon R_t < 0\}.$$

This formula is a direct application of the main result stated in Borovkov and Dickson [8], who studied the ruin time distribution for a Sparre Andersen process with exponential claim sizes.

**Proposition 1.** *Under the above assumptions on the model described by (1), the distribution of the ruin time $\tau$ has a density $p_\tau$ given by*

$$p_\tau(t) = \mathrm{e}^{-\mu(u+ct)} \sum_{n=0}^{\infty} \frac{\mu^n (u+ct)^{n-1}}{n!}\left(u + \frac{ct}{n+1}\right) f_h^{*(n+1)}(t)$$

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*with $f_h^{*n}$ the n-fold convolution of the function $f_h$ defined for $t \geq 0$ by*

$$f_h(t) = ut^{h-1} E_{h,h}(-\lambda t^h), \tag{12}$$

*where*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

*is the generalized two-parameter Mittag-Leffler function.*

*Proof.* We apply Theorem 1 of [8] with $f_h$ the density of the interarrivals $\Delta_{T_n}$. For a proof of (12), see [9].

### 4.2. Ruin probability in finite time

In this subsection we are interested in the Laplace transform of the probability $\psi(u, t)$ of ruin with finite time $0 < t < \infty$, defined by

$$\psi(u, t) = \mathbb{P}(R_s < 0 \text{ for some } s \leq t).$$

Proposition 2 below is a straightforward application of Theorem 1 of [18] (see also [26]). In [18] the author used a duality between the classical risk process in which the aggregate claims up to time $t$ are modelled as a compound Poisson process and the dual risk process in which the claim sizes are exponential and the interarrival times follow another law. We refer the reader to [19] for further details about this duality.

**Proposition 2.** *For any $x > 0$, it holds that*

$$\xi \int_0^{\infty} \mathrm{e}^{-\xi t} \psi(u, t) \, \mathrm{d}t = 1 - y(\xi) \exp\{-u\mu(1 - y(\xi))\}, \qquad \xi > 0,$$

*where $y(\xi)$ is the unique solution of*

$$y(\xi) = \frac{\lambda}{\lambda + (\xi + c\mu(1 - y(\xi)))^h}, \qquad \xi > 0. \tag{13}$$

*Proof.* See Theorem 1 of [18].

**Remark 3.** It is well known that (13) has a unique solution. Indeed, since we have $\xi \mapsto L_h(\xi) = \lambda/(\lambda + \xi^h)$, the Laplace transform of the waiting times $\Delta_{T_k}$, solving (13) is equivalent to solving

$$L_h(\xi + C - Cs) - s = 0$$

for a fixed $\xi > 0$ with $C = c\mu$. Since the left-hand side is a convex function with a negative value at $s = 1$ and a positive value at $s = 0$, it follows that there exists a unique root $y(\xi)$.

Of course, for practical purposes, the above proposition requires the numerical inversions of the Laplace transform. Numerical examples of such inversions will not be given here.

### 4.3. Ruin probability in infinite time

In this subsection we are interested in the infinite-time ruin probability, defined by

$$\psi(u) = \mathbb{P}(R_s < 0 \text{ for some } s \geq 0).$$

The following result holds.

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**Proposition 3.** *Under the assumptions of this section, we have*

$$\psi(u) = \left(1 - \frac{\gamma}{\mu}\right)\mathrm{e}^{-\gamma u},$$

*where* $\gamma > 0$ *is the unique solution of*

$$\gamma^h - \mu\gamma^{h-1} + \frac{\lambda}{c^h} = 0. \tag{14}$$

*Proof.* Since the fractional Poisson process is a renewal process and $\mu\mathbb{E}(\Delta_{T_1}) = +\infty$, the result is a consequence of Theorem VI.2.2 of [2].

**Remark 4.** Equation (14) can be explicitly solved for some $h \in (0, 1]$ (e.g. $h = \frac{1}{2}, \frac{1}{3}$, or $\frac{2}{3}$) and easily worked out numerically in the general case. For $h = 1$, we retrieve the Poisson case $\gamma = \mu - \lambda/c$.

## 5. Ruin probability in the presence of heavy-tailed claim sizes

In this section we are concerned with distributions of the claim sizes $(X_i)_{i\geq 1}$ having a heavy right tail $\bar{F}(t) = 1 - F(t)$ with $F(t) = \mathbb{P}(X_1 \leq t)$. In order to be more precise, we need the following definition.

**Definition 2.** A distribution $F$ is said to be subexponential if it is concentrated on $(0,\infty)$ and if $\lim_{t\to\infty} \overline{F^{*2}}(t)/\overline{F}(t) = 2$, where $F^{*2}$ is the convolution square.

In this section we shall work with the model given by (1), where we now assume that the distribution of $X_1$ is subexponential. Since $X_1$ is now heavy tailed, its mean is not necessarily finite. Our result will state an equivalent of the probability

$$\psi(u, t) = \mathbb{P}(R_s < 0 \text{ for some } s \leq t)$$

of ruin at time $t$, as the initial capital $u$ tends to $\infty$. This will be a consequence of the behaviour of the tail of a random sum, when the random number of terms is light tailed and the heavy-tailed independent random variables in the sum are subexponential. This is stated in the next well-known lemma, which is Lemma X.2.2 of [2]. We recall it for the sake of conciseness.

**Lemma 2.** *Let* $(Y_i)_{i\geq 1}$ *be a sequence of independent and identically distributed random variables with a common subexponential distribution* $F$, *and let* $K$ *be an independent, integer-valued random variable satisfying* $\mathbb{E}(z^K) < \infty$ *for some* $z > 1$. *Then it holds that*

$$\mathbb{P}\left(\sum_{i=1}^{K} Y_i > x\right) \sim \mathbb{E}(K)\overline{F}(x) \quad \text{as } x \to \infty. \tag{15}$$

Now we can state the following proposition.

**Proposition 4.** *Let* $(R_t)_{t\geq 0}$ *be the risk process given by (1). If the distribution* $F$ *of the claim sizes is subexponential then*

$$\psi(u, t) \sim \mathbb{E}(N_h(t))\overline{F}(u) \tag{16}$$

*as* $u$ *goes to* $+\infty$.

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We note that we also have

$$\psi(u,t) \sim \frac{\lambda t^h \overline{F}(u)}{\Gamma(1+h)} \quad \text{as } u \to \infty,$$

thanks to the explicit expression on the mean of $N_h(t)$ given in (8).

*Proof of Proposition 4.* We start from the inequalities

$$\mathbb{P}\bigg(\sum_{i=1}^{N_h(t)} X_i > u + ct\bigg) \le \psi(u,t) \le \mathbb{P}\bigg(\sum_{i=1}^{N_h(t)} X_i > u\bigg) \tag{17}$$

and we apply (15) to $\mathbb{P}(\sum_{i=1}^{N_h(t)} X_i > x)$ with $x = u$ or $x = u + ct$. By (2) we know that $\mathbb{E}(z^{N_h(t)})$ is finite for any $z > 1$, so by Lemma 2 we deduce that

$$\mathbb{P}\bigg(\sum_{i=1}^{N_h(t)} X_i > x\bigg) \sim \mathbb{E}(N_h(t))\overline{F}(x) \quad \text{as } x \to \infty.$$

Moreover, it holds that $\overline{F}(u+t) \sim \overline{F}(u)$ as $u \to \infty$. Therefore, (17) yields (16).

An extension of the previous result can be achieved for a $k$-dimensional risk process

$$\boldsymbol{R}_t = \boldsymbol{u} + \boldsymbol{c}t - \sum_{i=1}^{N_h(t)} \boldsymbol{X}_i, \qquad t \ge 0. \tag{18}$$

In (18), the process $(\boldsymbol{R}_t)_{t\ge 0}$ is defined by $\boldsymbol{R}_t = (R_t^1, \dots, R_t^k)$, where the processes $R^j$ satisfy

$$R_t^j = u^j + c^j t - \sum_{i=1}^{N_h(t)} X_i^j, \qquad t \ge 0,\ 1 \le j \le k.$$

Clearly, $\boldsymbol{u} = (u^1, \dots, u^k)$ is the initial capital vector, $\boldsymbol{c} = (c^1, \dots, c^k)$ is the premium intensity vector, and the claim vectors $\boldsymbol{X}_i$ are equal to $(X_i^1, \dots, X_i^k)$ for $i \ge 1$. The sequence $(\boldsymbol{X}_n)_{n\ge 1}$ is a sequence of independent and identically distributed random vectors with a joint distribution satisfying

$$F(x_1, \dots, x_k) = \mathbb{P}(X^1 \le x_1, \dots, X^k \le x_k) = \prod_{j=1}^{k} \mathbb{P}(X^j \le x_k) := \prod_{j=1}^{k} F_j(x_j)$$

with obvious notation. Since the number of claims $N_h(t)$ in model (18) is light tailed (recall that $\mathbb{E}(z^{N_h(t)}) < \infty$ for any $z > 0$), following the same lines as the proof of Proposition 9.4 of [2], we obtain the next result.

**Proposition 5.** *Assume that the distributions of the claim sizes $F_j$ are subexponential for $1 \le j \le k$. For an initial capital vector $\boldsymbol{u}$, we denote by $\tau_{\max}(\boldsymbol{u})$ the first time all the components of $\boldsymbol{R}$ are negative:*

$$\tau_{\max}(\boldsymbol{u}) = \inf\{s > 0\colon \max\{R_s^1, \dots, R_s^k\} < 0\}.$$

*Then, for any $t > 0$, it holds that*

$$\mathbb{P}(\tau_{\max}(\boldsymbol{u}) \le t) \sim \mathbb{E}[(N_h(t))^k] \prod_{j=1}^{k} \overline{F_j}(u_j)$$

*when $u_j \to \infty$ for any $1 \le j \le k$.*

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## 6. Ruin probability with light-tailed claim sizes

Now, in model (1), we assume that the common distribution of $X_i$ is light tailed (hence, $\mathbb{E}(\mathrm{e}^{\xi X_1}) < \infty$ for any $\xi > 0$). Since the fractional Poisson process has a light-tailed distribution, we are interested in the tails of random sums of a light-tailed number of light-tailed terms. Using large deviations for the fractional process, asymptotic results for ruin probabilities of an insurance model with a fractional Poisson claim number process have been studied by Beghin and Macci [4]. Their results also apply in our situation.

In this section we aim to provide nonasymptotic results in the same spirit as the celebrated Lundberg inequality. We denote by $(S_h(t))_{t\geq 0}$ the compound fractional Poisson process, which is naturally defined as

$$S_h(t) = \sum_{i=1}^{N_h(t)} X_i.$$

The independence of the process $N_h$ and the sequence $(X_i)_{i\geq 1}$ allows us to calculate the moment generating function of $S_h$. Indeed, it was proved in [16] that, for any $\xi > 0$,

$$\mathbb{E}(\mathrm{e}^{\xi S_h(t)}) = E_h(\lambda t^h (g(\xi) - 1)), \tag{19}$$

where the function $g$ is the Laplace transform of the random variables $X_i$ defined by $g(\xi) = \mathbb{E}(\mathrm{e}^{\xi X_1})$ for any $\xi > 0$. We have the following bound on the ruin probability in finite time.

**Proposition 6.** *Let $(R_t)_{t\geq 0}$ be the risk process given by (1) under the assumption that the common distribution of $X_i$ is light tailed. Then, for any $t > 0$, there exists $\xi_0(t, h, \lambda)$ such that, for any $u \geq 0$, we have*

$$\psi(u, t) \leq 2\mathrm{e}^{-\xi_0(t,h,\lambda)u}. \tag{20}$$

**Remark 5.** The constant $\xi_0(t, h, \lambda)$ is explicitly given by (23). Of course, (20) is meaningless unless

$$u \geq u_0 := \frac{\ln(2)}{\xi_0(t, h, \lambda)}.$$

*Proof of Proposition 6.* By (17), we only have to estimate $\mathbb{P}(\sum_{i=1}^{N_h(t)} X_i > u)$. Using the Chebyshev exponential inequality and (19), we deduce that, for any $\xi > 0$,

$$\mathbb{P}\bigg(\sum_{i=1}^{N_h(t)} X_i > u\bigg) = \mathbb{P}(\exp(\xi S_h(t) > \mathrm{e}^{\xi u})) \leq \mathrm{e}^{-\xi u} E_h(\lambda t^h (g(\xi) - 1)). \tag{21}$$

Now we prove an upper bound for the Mittag-Leffler function. We recall that

$$E_h(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1 + hk)}.$$

The minimum value of $x \mapsto \Gamma(x)$ is achieved for positive values in a point denoted by $x_0$ (which is approximately equal to 1.462). We have $\Gamma(x_0) \simeq 0.8856$. Thus, if $\tilde{g}$ is the function defined for $h \in (0,1)$ by $\tilde{g}(h) = \Gamma(1 + hk)$, $\tilde{g}$ has a minimum in $h_0 = (x_0 - 1)/k$. Thus,

$$\tilde{g}(h) \geq \tilde{g}\bigg(\frac{x_0 - 1}{k}\bigg) = \Gamma\bigg(1 + \frac{x_0 - 1}{k} k\bigg) = \Gamma(x_0),$$

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and we obtain, for any $|x| < 1$,

$$E_h(x) \le \frac{1}{\Gamma(x_0)} \sum_{k=0}^{\infty} x^k \le \frac{1}{\Gamma(x_0)(1-x)}. \tag{22}$$

Now we briefly study the function $g$ defined for $\xi \ge 0$ by $g(\xi) = \mathbb{E}(\mathrm{e}^{\xi X_1})$. Since $X_1$ is a positive random variable, $g$ is strictly increasing. Hence, $g$ is one to one from $[0, +\infty)$ to $[1, +\infty)$. So we may define $\xi_0(t, h, \lambda)$ as the unique positive real such that

$$g(\xi_0) = 1 + \frac{2\Gamma(x_0) - 1}{2\lambda t^h \Gamma(x_0)}. \tag{23}$$

We start from (21) and use (22) with $0 < x = \lambda t^h (g(\xi_0(t, h, \lambda)) - 1) = (2\Gamma(x_0) - 1)/(2\Gamma(x_0)) < 1$. We obtain

$$\mathbb{P}\biggl(\sum_{i=1}^{N_h(t)} X_i > u\biggr) \le \frac{\mathrm{e}^{-\xi_0(t,h,\lambda)u}}{\Gamma(x_0)(1 - \lambda t^h (g(\xi_0(t, h, \lambda)) - 1))} = 2\mathrm{e}^{-\xi_0(t,h,\lambda)u},$$

and (20) is proved.

**Remark 6.** We note that estimation (21) is more accurate. For example, one may use it to plot the function $\xi \mapsto \mathrm{e}^{-\xi u} E_h(\lambda t^h (g(\xi) - 1))$ for small values of $\xi$ and numerically check the eventual minimum. Such a procedure is feasible since the Mittag-Leffler function is now evaluated using scientific software.

## Appendix A. Proof of inequality (11)

We denote by $f$ the function defined by $x \mapsto \ln(2x\mathcal{B}(1 + x, x))$ for $x \in (0,1)$. We follow a technical trick used in [23]. Inequality (11) will be a consequence of the positivity of $f$ on the interval $(0,1)$. Since $x\Gamma(x) = \Gamma(x + 1)$, we have

$$2x\mathcal{B}(1 + x, x) = \frac{2x\Gamma(x)\Gamma(x + 1)}{\Gamma(2x + 1)} = \frac{2\Gamma(x + 1)^2}{\Gamma(2x + 1)},$$

and, consequently,

$$f(x) = \ln(2) + 2\ln\Gamma(x + 1) - \ln\Gamma(2x + 1).$$

We denote by $\Psi$ the function $(\ln\Gamma)' = \Gamma'/\Gamma$ (usually called the digamma function). We obtain

$$f'(x) = 2(\Psi(x + 1) - \Psi(2x + 1)).$$

Since $\Psi'(x) = \sum_{k=0}^{\infty} 1/(x + k)^2$ (see [1, p. 13]), we deduce that

$$\begin{aligned} f''(x) &= 2(\Psi'(x + 1) - 2\Psi'(2x + 1)) \\ &= 2\sum_{k=0}^{\infty} \frac{1}{(x + 1 + k)^2} - \frac{1}{2(x + (k + 1)/2)^2} \\ &= \sum_{k=0}^{\infty} \frac{1}{(x + 1 + k)^2} + \sum_{k=0}^{\infty} \frac{1}{(x + 1 + k)^2} - \frac{1}{(x + (k + 1)/2)^2} \end{aligned}$$

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$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{1}{(x+1+k)^2} + \sum_{j=0}^{\infty} - \frac{1}{(x+(2j+1)/2)^2} \\
&= \sum_{k=0}^{\infty} \frac{1}{(x+1+k)^2} + \sum_{k=0}^{\infty} - \frac{1}{(x+k+1/2)^2} \\
&= \Psi'(x+1) - \Psi'\big(x+\tfrac{1}{2}\big).
\end{aligned}
$$

Moreover, $\Psi'$ is a decreasing function because $\Psi''(x) = -2\sum_{k=0}^{\infty}(x+k)^{-3}$. It follows that $f'' \le 0$ on $(0,1)$. As a decreasing function, $f'$ satisfies $f'(x) \le f'(0) = 0$ and, consequently, $f$ is itself a decreasing function. Finally, we deduce that

$$
f(x) \ge f(1) = \ln\bigg(\frac{2\Gamma(2)^2}{\Gamma(3)}\bigg) = 0,
$$

and the proof is complete.

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