INTERSECTIONS OF FINITELY GENERATED
FREE GROUPS

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A result of Howson is that two finitely generated subgroups U
and V of a free group have finitely generated intersection. Hanna
Neumann showed further that, if m, n and N are the ranks of
U, V and $U \cap V$ respectively, then $N \leq 2(m-1)(n-1) + 1$, and
Burns strengthened this, showing that $N \leq 2(m-1)(n-1) - m + 2$
(if $m \leq n$). This paper presents a new and simple proof of
Burns' result. Further, the graph-theoretical ideas used provide
still stronger bounds in certain special cases.

1. Introduction

Let U and V be subgroups of a free group F , with U and V of
finite ranks m and n respectively. Howson [4] showed that $U \cap V$ is
also finitely generated, and gave a bound on its rank N in terms of m
and n . Neumann [7, 8] subsequently sharpened Howson's bound, showing
that $N \leq 2(m-1)(n-1) + 1$. On the other hand, if U or V is of finite
index in F , then the stronger bound $N \leq (m-1)(n-1) + 1$ holds (Burns
[1]), and most writers on the subject seem to believe that this bound
should hold in all·cases. (Howson [4] has examples showing that no bound
lower than this is possible.)

The strongest bound known to hold in general, however, is due to Burns
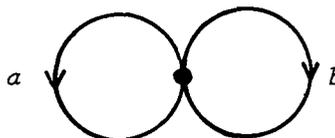

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[2], who shows that $N \leq 2(m-1)(n-1) - m + 2$ (taking $m \leq n$). Our purpose in this paper is to examine the problem again, obtaining some detailed information (§2) from which Burns' bound in particular follows.

We approach the problem graph-theoretically, as do most other writers on the subject [3]-[10]. (Servatius [9] has recently shown how Burns' argument in [2] may be rewritten graph-theoretically.) Our main argument takes the form of a simple graph-theoretical algorithm. The information obtained from this not only yields Burns' result, but also enables us to obtain better bounds in two interesting special cases. The first of these is when $m = 2$: here the presumed optimal bound $N \leq (m-1)(n-1) + 1$ reduces simply to $N \leq n$ and we are able to show that, for a given U of rank 2, this bound holds for (in some sense) most subgroups V. The second special case considered is when V is a conjugate of U : here, of course, $m = n$, and we show that $N \leq 2(m-1)^2 - 3m + 5$.

The reader will find it useful to have some familiarity with the use of graphs in combinatorial group theory, as, for example, in [5].

2. A graph-theoretical algorithm

2.1. We assume without loss of generality that U and V are subgroups of $F = F(a, b)$, the free group on generators a and b. For any finitely generated subgroup, say U, of F, a standard construction [6] yields a finite graph $\Gamma(U)$ as follows. First let Γ_U be the graph with vertices the distinct cosets Ux ($x \in F$), and edges all pairs (Ux, y), for $x \in F$ and $y \in \{a, b\}$, and where (Ux, y) runs from Ux to Uxy. (Of course, Γ_U is a quotient graph of the Cayley graph [5] of F with respect to $\{a, b\}$.) Then $\Gamma(U)$ is the smallest subgraph of Γ_U containing all cycles; that is (following [10]), the *core* of Γ_U. Since U is finitely generated, $\Gamma(U)$ is finite. (Note that $\Gamma(U)$ need not contain the vertex U.) Informally, we think of $\Gamma(U)$ as a directed, labelled graph with labels chosen from $\{a, b\}$. In particular we may represent $\Gamma(F)$ as

Clearly, all vertices of $\Gamma(U)$ are of degree 2, 3 or 4 and, as in [9], we may suppose them to be of degrees 2 and 3 only. (Isomorphically embed F into $F(\alpha, \beta)$ via the map $a \mapsto \alpha\beta^{-1}$, $b \mapsto \beta^2$, and take graphs relative to the new generators α, β.) We refer to vertices of degree 3 as *branch points*.
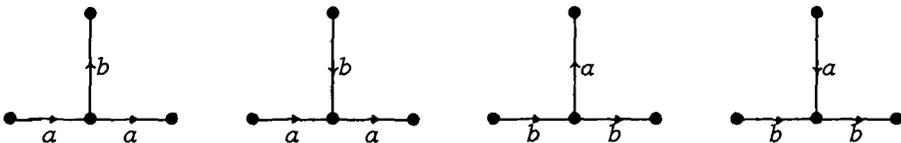
2.2. The crucial fact now is that $\Gamma(U \cap V)$ may be identified with the core of a connected component of the graph $\Gamma(U) \overset{\sim}{\times} \Gamma(V)$, obtained as a pullback in the diagram

$$
\begin{array}{ccc}
\Gamma(U) \overset{\sim}{\times} \Gamma(V) & \longrightarrow & \Gamma(V) \\
\downarrow & & \downarrow \\
\Gamma(U) & \longrightarrow & \Gamma(F)
\end{array}
$$

in the category of directed, $\{a, b\}$-labelled graphs, and where the maps into $\Gamma(F)$ are the obvious projections. (We still think of $\Gamma(F)$ in the form drawn in 2.1, though $\Gamma(U)$ and $\Gamma(V)$ now have no vertices of degree 4.)

In a graph such as $\Gamma(U)$, with vertices of degrees 2 and 3 only, we see easily that the cyclomatic number is 1 plus one-half the number of branch points. Since the rank of the subgroup is the cyclomatic number of its graph, to obtain bounds on the rank N of $U \cap V$, we need to bound the number of branch points in the core of $\Gamma(U) \overset{\sim}{\times} \Gamma(V)$.

2.3. There are four possible labellings of the edges incident with a branch point, as follows:

We refer to vertices so labelled, respectively, as *b-sources*, *b-sinks*, *a-sources*, and *a-sinks*. (Here of course a loop contributes an inward- and an outward-going edge.)

2.4. Consider any two branch points X in $\Gamma(U)$ and Y in $\Gamma(V)$. There are exactly three (reduced) paths p_1, p_2, p_3 beginning at X and

ending at the first branch point encountered; and q_1, q_2, q_3 are
similarly defined at Y . There is also a natural *pairing* of each p_i
with some q_j ; for, at least two of the labels of the edges at X are
equal to those of edges at Y ; we pair those paths beginning with
similarly labelled edges, and then, if two remain unpaired we pair them.
Suppose for convenience that p_i is paired with q_i , $i = 1, 2, 3$. Then
we say that p_i and q_i are *compatible* if there is a path p in
$\Gamma(U) \overset{\sim}{\times} \Gamma(V)$ from (X, Y) to a vertex (X', Y') , where X' and Y' are
branch points in $\Gamma(U)$ and $\Gamma(V)$ respectively, and where the projections
of p_i and q_i into $\Gamma(F)$ are both initial segments of the projection of
p . The branch points X and Y are *compatible* if p_i and q_i are
compatible for each i . (*Incompatible*, of course, will mean not
compatible.)

Now it is easy to see that whenever (X, Y) is a branch point in the
core of $\Gamma(U) \overset{\sim}{\times} \Gamma(V)$, X and Y must be compatible. Since, as mentioned
in 2.2, our concern is to bound the number of branch points in the core, it
therefore suffices to bound the number of such compatible pairs. Our main
result, which we now state, enables us to do this; its proof occupies the
remainder of §2.

THEOREM 2.5. *Given* U, $V \leq F$ *of ranks at least* 2 , *there is a*
partition of the branch points of $\Gamma(U) \cup \Gamma(V)$ (*where* $\Gamma(U)$ *and* $\Gamma(V)$
are taken as disjoint) *into two non-empty sets* P_1 *and* P_2 , *with the*
following property: all branch points of $P_1 \cap \Gamma(U)$ *are incompatible*
with those of $P_2 \cap \Gamma(V)$, *and all branch points of* $P_2 \cap \Gamma(U)$ *are*
incompatible with those of $P_1 \cap \Gamma(V)$.

2.6. If among the branch points of $\Gamma(U) \cup \Gamma(V)$ two or more distinct
labelling types occur (2.3), put all vertices of one fixed type in P_1 ,
and the others in P_2 : the conclusion of the theorem clearly holds.

2.7. Otherwise, suppose that all labelling types of branch points in
$\Gamma(U) \cup \Gamma(V)$ are the same. For concreteness, let us assume all branch
points to be b-sources (2.3). We lose no generality by doing so, since

any type of branch point can be converted into a b-source by systematic *relabelling*; that is, performing throughout $\Gamma(U)$ and $\Gamma(V)$ a suitably chosen sequence of the following steps:

   replace all a labels by b labels, and *vice versa*;

   reverse all edges labelled a ;

   reverse all edges labelled b .

   2.8. Let $\Gamma_0 = \Gamma(U)$ and $\Delta_0 = \Gamma(V)$, so that Γ_0 and Δ_0 are directed, $\{a, b\}$-labelled graphs, with vertices of degree 2 and 3 only, and with all branch points being b-sources. Additionally, because of the group-theoretical origin of Γ_0 and Δ_0 , no paths of the forms

$$\bullet \xrightarrow{a} \bullet \xleftarrow{a} \bullet \qquad \bullet \xleftarrow{a} \bullet \xrightarrow{a} \bullet \qquad \bullet \xrightarrow{b} \bullet \xleftarrow{b} \bullet \qquad \bullet \xleftarrow{b} \bullet \xrightarrow{b} \bullet$$

ever occur.

   We now describe an algorithm which, from the pair $\left(\Gamma_0, \Delta_0\right)$, successively produces pairs $\left(\Gamma_1, \Delta_1\right)$, ... of graphs of exactly the same type, except that, for some k , Γ_k and Δ_k will contain branch points of two distinct kinds, at which point the algorithm will terminate.

   2.9. The operation which transforms Γ_{i-1} and Δ_{i-1} into Γ_i and Δ_i is accomplished in the following steps, governed by the rule given below.

   (1) In Γ_{i-1} and Δ_{i-1} replace all (maximal) paths of the form

$$\bullet \xrightarrow{b} \bullet \xrightarrow{b} \bullet \xrightarrow{b} \bullet \ \ \dots \ \ \bullet \xrightarrow{b} \bullet$$

   by $\bullet \xrightarrow{b} \bullet$.

   (2) If paths of the form $\bullet \xrightarrow{b} \bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet$ appear, replace them all by $\bullet \xrightarrow{b} \bullet$, invoke the rule below and go to step (3). Otherwise, go to step (4).

   (3) Repeat step (2).

   (4) If paths of the form $\bullet \xrightarrow{b} \bullet \xrightarrow{a} \bullet$ appear, replace them

all by $\bullet\overset{b}{\longrightarrow}\bullet$, and invoke the rule.

(5) Set Γ_i and Δ_i equal to the transformed graphs.

RULE. If all branch points are b-sources, take no action.

If all branch points are of some one fixed type, but are not b-sources, relabel the graphs (2.7) and restart the algorithm at step (1).

If two distinct branch point labelling types occur, place all branch points of one type in P_1 and all the others in P_2 , set Γ_i and Δ_i equal to the transformed graphs, and stop.

2.10. We make the following assertions, giving an indication of the proofs in parentheses.

(1) The algorithm is well-defined. (Distinct paths chosen for contraction in any one step are edge-disjoint, so that the instruction to contract such paths is unambiguous; and these paths never pass through branch points (except perhaps at their ends), so that the action of replacing them by $\bullet\overset{b}{\longrightarrow}\bullet$ is well-defined.)

(2) Γ_i and Δ_i are of exactly the same type (2.8) as Γ_{i-1} and Δ_{i-1} (with the possible exception of branch point labellings). (The only point requiring checking is that paths such as $\bullet\overset{a}{\longrightarrow}\bullet\overset{a}{\longrightarrow}\bullet$ and so on cannot appear. Step (2), for example, could only produce such a path (necessarily of the form $\bullet\overset{b}{\longrightarrow}\bullet\overset{b}{\longrightarrow}\bullet$) if a contracted path $\bullet\overset{b}{\longrightarrow}\bullet\overset{a}{\longrightarrow}\bullet\overset{b}{\longrightarrow}\bullet$ were contained in a longer path $\bullet\overset{b}{\longrightarrow}\bullet\overset{a}{\longrightarrow}\bullet\overset{b}{\longrightarrow}\bullet\overset{b}{\longrightarrow}\bullet$, and this is impossible, as paths such as $\bullet\overset{b}{\longrightarrow}\bullet\overset{b}{\longrightarrow}\bullet$ have been eliminated by step (1). Similar arguments apply to the other contractions.)

(3) The numbers of branch points in Γ_{i-1} and Δ_{i-1} are exactly preserved in an application of the algorithm. (Obvious.)

(4) The number of branch points in the core of $\Gamma_i \overset{\sim}{\times} \Delta_i$ is at least

as large as the number in the core of $\Gamma_{i-1} \overset{\sim}{\times} \Delta_{i-1}$. (This follows from the obvious fact that compatibility of paired paths is preserved by the algorithm, and from the relationship between branch points and compatibility (see 2.4).)

(5) The algorithm terminates, and does so only when two types of labelling appear. (Since we strictly decrease the number of edges with each application of the algorithm, the algorithm must stop. It will do so if two types of labelling appear, or if no paths exist of the kinds contracted by the algorithm. The latter is easily seen to be impossible, given that all branch points are b-sources.)

(6) P_1 and P_2 (strictly, the corresponding sets of vertices in $\Gamma(U)$ and $\Gamma(V)$) are as claimed in the theorem. (This follows from the fact ((4) above) that compatibility is preserved.)

The theorem is now proved.

3. Applications

3.1. Using the notation of the theorem, let u_i and v_i be the cardinalities of $P_i \cap \Gamma(U)$ and $P_i \cap \Gamma(V)$ respectively, for $i = 1, 2$. Then from the theorem, and the relation (2.4) between branch points and compatibility, we have the

COROLLARY. *In the core of* $\Gamma(U) \overset{\sim}{\times} \Gamma(V)$ *there are at most* $u_1 v_1 + u_2 v_2$ *branch points.*

THEOREM 3.2 (Burns [2]). *If* $U, V \leq F$ *have ranks* m, n *respectively, and* $1 \leq m \leq n$ *, then the rank* N *of* $U \cap V$ *satisfies*

$$N \leq 2(m-1)(n-1) - m + 2 .$$

Proof. The result is trivial if $m = 1$, so let $m \geq 2$. By 2.2, $\Gamma(U)$ and $\Gamma(V)$ have $2(m-1)$ and $2(n-1)$ branch points respectively, so, in the notation of the corollary, $u_1 + u_2 = 2(m-1)$ and $v_1 + v_2 = 2(n-1)$ Therefore $u_1 v_1 + u_2 v_2 = 4(m-1)(n-1) - (u_2 v_1 + u_1 v_2)$. But from the fact that the sets P_1 and P_2 given by the main theorem are non-empty, we see easily that $u_2 v_1 + u_1 v_2 \geq 2(m-1)$, and so

$u_1 v_1 + u_2 v_2 \leq 4(m-1)(n-1) - 2(m-1)$, from which the desired result follows by 2.2 and the corollary.

3.3.   The first of the special cases mentioned in the introduction is where the subgroup U has rank $m = 2$, and $\Gamma(U)$ contains just two branch points.

First, consider the application of the algorithm to two disjoint copies of $\Gamma(U)$. Denote the branch points in the first copy of $\Gamma(U)$ by X and Y , and the corresponding branch points in the second copy by X' and Y' . Then it is easy to see that one of the sets P_1, P_2 produced by the algorithm must contain X and X' , and the other Y and Y' . From 3.1 we therefore see that the core of $\Gamma(U) \overset{\sim}{\times} \Gamma(U)$ contains at most two branch points; and since the component of the core containing (X, X') certainly has two branch points (corresponding to the fact that $U \cap U = U$), the number is exactly two.

In particular, (X, Y') and (X', Y) have degree at most 2 in the core, and hence there exist edges x_1 at X and y_1 at Y (or, strictly, Y'), belonging to paired paths (2.4), and such that no path in the core at (X, Y') commences with the edge (x_1, y_1) .

LEMMA. *There is a largest integer $l \geq 0$ for which there are reduced paths $x_1 x_2 \ldots x_l$ and $y_1 y_2 \ldots y_l$ at X and Y (respectively), having equal projections in $\Gamma(F)$.*

Proof. If no such largest l exists, there is a path $(x_1, y_1)(x_2, y_2) \ldots$ in $\Gamma(U) \overset{\sim}{\times} \Gamma(U)$ which may be extended to any length. Since $\Gamma(U) \overset{\sim}{\times} \Gamma(U)$ is finite, this path must contain a cycle. If the cycle contains (X, Y') , then there is a path in the core containing (X, Y') and commencing with (x_1, y_1) , a contradiction. If (X, Y') is not in the cycle, the cycle contains a branch point, which must clearly be (X', Y) . By symmetry, there is also a cycle containing (X, Y') and not (X', Y) , and since a path starting with (x_1, y_1) joins the two cycles, this path is in the core, also giving a contradiction. Thus the integer l exists as required.

It is now clear, given another subgroup V , that any path in $\Gamma(V)$

of length greater than l , paired with the paths p and q containing x_1 and y_1 , can be compatible with at most one of p and q . Therefore we have the

PROPOSITION. *Let* V *have rank* $n \geq 2$. *If in* $\Gamma(V)$ *all paths paired with* p *and* q *are of length greater than* l , *then* $N \leq n$.

3.4. The second special case is as follows.

PROPOSITION. *Let* U *be of rank* $m > 2$. *Then for any* $x \in F$, *the rank* N *of* $U \cap x^{-1}Ux$ *satisfies*

$$N \leq 2(m-1)^2 - 3m + 5 .$$

Proof. Writing $V = x^{-1}Ux$, we see easily that $\Gamma(U)$ and $\Gamma(V)$ are isomorphic. In an application of the algorithm to $\Gamma(U)$ and $\Gamma(V)$, therefore, any changes made to $\Gamma(U)$ are made simultaneously to $\Gamma(V)$, and *vice versa*. It follows that (in the notation of the corollary) $u_1 = v_1$ and $u_2 = v_2$. Then, from the fact that the sets P_1 and P_2 produced by the algorithm are non-empty, and since $\Gamma(U)$ and $\Gamma(V)$ have $2(m-1)$ branch points (2.2), we see easily that $u_1v_1 + u_2v_2$ can be no greater than $1.1 + (2m-3)(2m-3) = 4m^2 - 12m + 10$. Hence, by the corollary, the core of $\Gamma(U) \widetilde{\times} \Gamma(V)$ contains at most this number of branch points. However, the component of the core corresponding to the inter-section of U with itself already contains exactly $2(m-1)$ branch points, so any other component can have at most $4m^2 - 14m + 12$ branch points. The ranks of the intersections represented by such a component must therefore be bounded as stated, by 2.2, and since also $m \leq 2(m-1)^2 - 3m + 5$ when $m > 2$, the result follows.

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