OLD AND NEW RESULTS ON KNOTS

HERBERT SEIFERT AND WILLIAM THRELFALL

THE theory of knots undertakes the task of giving a complete survey of all existing knots. A solid mathematical foundation was not laid to this theory until our century. A mathematician of the rank of Felix Klein thought it to be nearly hopeless to treat knot problems with the same exactness as we are accustomed to from classical mathematics. We want to give here a short summary of the modern topological methods enabling us to approach the knot problem in a mathematical way.

In order to exclude pathological knots, as for instance knots being entangled an infinite number of times, we will define a knot as a polygon lying in the space. In other words: a knot is a closed sequence of segments without double points. In Figure 1 some examples of knots are given in plane projection.

Now the question arises when two knots are to be called equivalent. One might be induced to call them so if one of them can be transformed into the other by a deformation without self-intersection. But this definition needs a restriction, otherwise every two knots would be equivalent. For one could transform both of them into an unknotted curve, a process shown for the trefoil knot by Figure 2. We therefore permit only more special transformations which do not allow such a tightening. We will call two knots *equivalent*, or of the same type, if they can be transformed into one another by a finite number of operations of the following kind: Let Δ be a triangle having one or two sides (and no other points) in common with the knot; then we add the boundary of Δ modulo 2 to the knot. This means the sides of Δ are to be added to the segments of the knot, and the segments, then occurring twice, are to be dropped. The two possible cases of such transformations are illustrated by Figure 3 and Figure 4. One may picture the knot as being pulled over the surface of the triangle.

The following definition has the same meaning, as can be proved: Two knots are equivalent if and only if the one can be transformed into the other by a topological, simplicial, sense-preserving mapping of the whole space onto itself [6]. These so-called semilinear self-transformations of space forming a group, the theory of knots may be regarded as part of the geometry belonging to this group.

We usually represent a knot in the drawing plane by parallel projection. The type of the knot is uniquely determined by its projection. It is always possible to choose the direction of the projecting lines in such a way that only "ordinary double points" occur, which means that in a double point one

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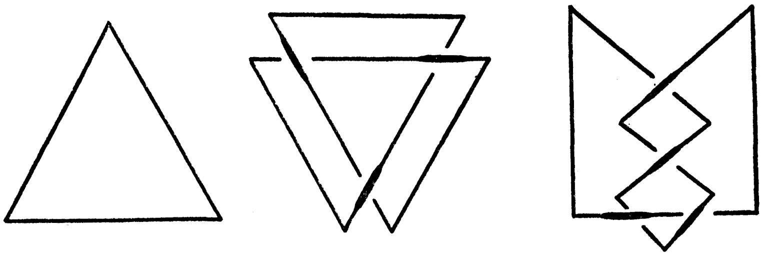
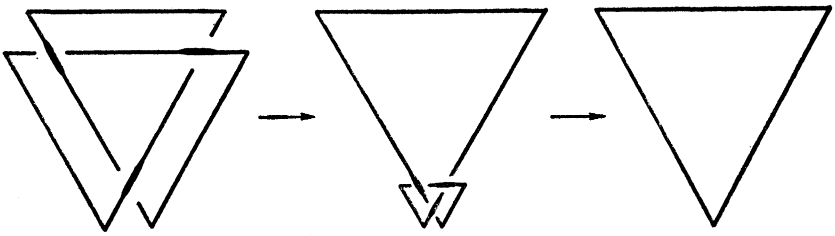


Fig. 1



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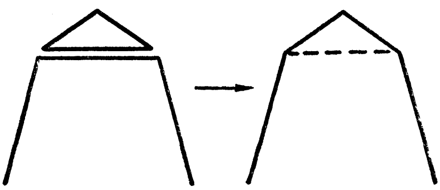
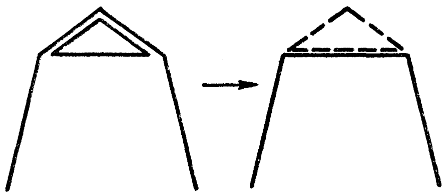


Fig. 3



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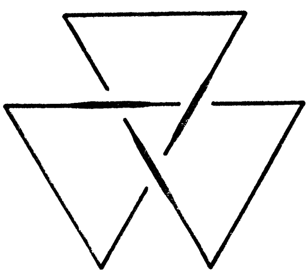

Fig. 5

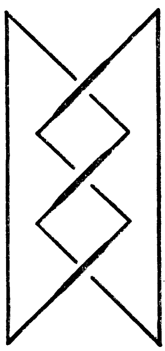

Fig. 6

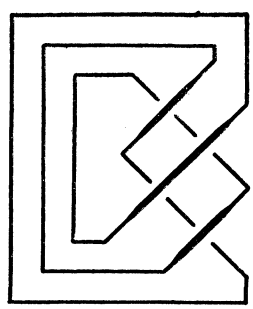

Fig. 7

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segment of the knot is crossed by one other segment only. The lower segment is drawn interrupted in the figures. One and the same knot and its equivalents are capable of an infinity of different projections. For instance Figures 5-7 are all projections of the trefoil knot.

There is one way near at hand for getting a survey of all possible knots. One has to construct systematically all projections with 2, 3, 4, . . . double points, and one has to find out by trying which of these projections represent equivalent knots. Of all the possible projections of the same type of knot, one distinguished by having the least number of double points or other simple qualities, will be picked out as a representative and be admitted to an inventory of knots. This is how the index of knots of Alexander and Briggs [2] was constructed. It contains 84 knots with up to 9 double points.

It remains to be seen whether different knots of the index are really not equivalent. This question, the true knot problem, cannot be decided by trying. For there is an infinity of possibilities of transforming a knot, and the reason why we may fail in transforming one knot of the index into another may be lack of skill or perseverance.

In order to prove two such knots to be really not equivalent, deeper methods are wanted. They offer themselves in the topological invariants of the complement of the knot, i.e., the space from which the knot has been taken away: if the knot k can be transformed into the knot k' by a semilinear transformation of the space R then the complements $R-k$ and $R-k'$ must be homeomorphic. Therefore the *topological invariants of* $R-k$ *are knot-invariants*. Thus the theory of knots is closely connected with the topology of three-dimensional manifolds, and every new topological invariant of three-dimensional manifolds is at the same time a new knot invariant. The only problem left to the knot theory is then to develop a method of calculating these invariants out of a given projection of the knot. Let us review the main results which have been attained in this direction.

One of the most important knot invariants is the *group of the knot* [4]. It is the fundamental group of the complement $R-k$. The fundamental group is defined for every connected complex L of any dimension. One has to choose a point O of L and to draw all closed oriented paths starting from O and running on L. Two such paths W and W' are called homotopic and are considered to be in the same class of paths, if W can be deformed on L into W', O remaining fixed, yet self-intersections being allowed throughout the deformation. The classes of paths are considered as the elements of a group; this is the fundamental group of L. The product W_1W_2 of two paths W_1 and W_2 is obtained by passing first along W_1 and then along W_2.

In the case of the group of a knot k, the complex L is the complement $R-k$. Thus the knot group of a "circle" is the free group of one generator; for every closed path that does not meet k can be deformed without intersecting k into a definite power of the path S entangling the circle once (Figure 8). The knot group therefore consists of all the powers of the class of the path S. The path

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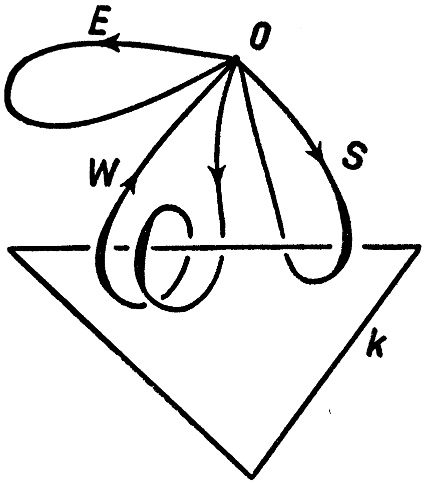

Fig. 8

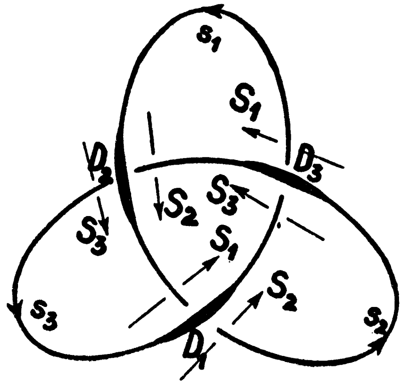

Fig. 9

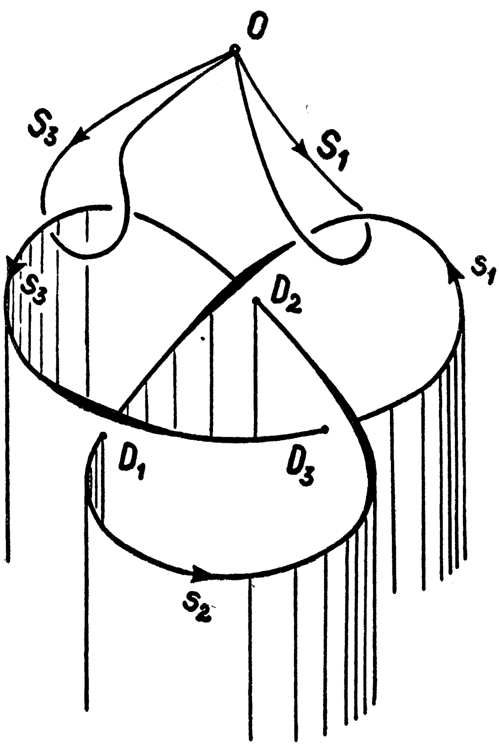

Fig. 10

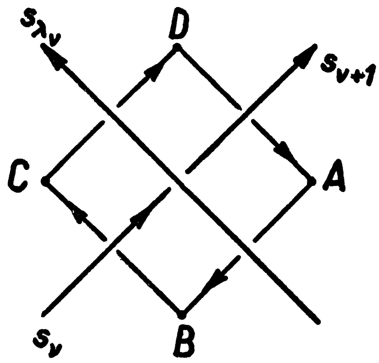

Fig. 11

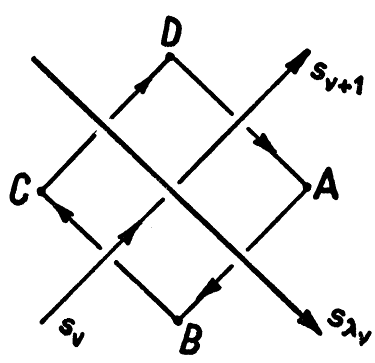

Fig. 12

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W, for instance, shown in Figure 8, can be deformed into S^{-2}; the path E is homotopic to zero and represents the unity of the knot group.

Let k be an arbitrary knot given by its projection. We may represent its group by generators and defining relations in the following way. To every double point of the projection there correspond two points of the knot, one lying on the crossing, the other below, on the crossed branch. The latter may be called a crossed point. If there are n double points in the projection, then there will be n crossed points on the knot. We shall denote them by $D_1, D_2, \ldots, D_n$ in such an order as is given by a certain orientation of the knot. The knot is divided by the crossed points into n oriented segments $s_1, s_2, \ldots, s_n$, the segment s_ν running from $D_{\nu-1}$ to D_ν (indices are to be reduced modulo n so that D_0 means the same as D_n). We choose the origin O of the closed paths above the drawing plane, and we adjoin to the n segments $s_1, s_2, \ldots, s_n$ n elements $S_1, S_2, \ldots, S_n$ of the knot group which will generate the whole group. They are classes of paths of $R-k$. The class S_ν is represented by a path starting from O, entangling s_ν once in the positive sense and returning to O. To entangle in the positive sense means that the sense of the rotation of the oriented path around s_ν together with the orientation of s_ν determines a right-handed screw. We indicate such a way in the knot projection simply by an arrow, representing the part of the path which is overcrossed by the knot. Without changing the class of a path one may draw the path along the crossing segment over a double point. Figure 9 shows an example. Every class S_ν of paths is represented by two different arrows marked with the same letter. Figure 10 gives a picture of the same knot. In order to improve the view we added a cylinder underneath, with its generating lines in the direction of the projecting parallels.

To every double point there corresponds a certain defining relation of the knot group. To realize this let us consider Figures 11 and 12. They show the two possible kinds of crossing (overcrossing from right to left and from left to right). The closed path $ABCD$ represented in the projection is obviously homotopic to zero as it lies wholly beneath the knot. Now let the origin O of the closed paths be situated above the drawing-plane. If we then move the points A, B, C, D towards O, drawing the path $ABCD$ behind, this closed path will become a product of our generating elements S_μ. In the case of Figure 11 this product is

$$S_{\lambda_\nu} S_\nu S_{\lambda_\nu}^{-1} S_{\nu+1}^{-1},$$

and in the case of Figure 12 it is

$$S_{\lambda_\nu}^{-1} S_\nu S_{\lambda_\nu} S_{\nu+1}^{-1}.$$

We therefore have in the two cases the respective relations

$$\text{(1a)} \qquad R_\nu \equiv S_{\lambda_\nu} S_\nu S_{\lambda_\nu}^{-1} S_{\nu+1}^{-1} = 1,$$

$$\text{(1b)} \qquad R_\nu \equiv S_{\lambda_\nu}^{-1} S_\nu S_{\lambda_\nu} S_{\nu+1}^{-1} = 1.$$

These relations, formed for $\nu = 1, 2 \ldots, n$, constitute (as can be proved) a complete system of defining relations for the knot group. In the case of the trefoil knot (Figure 9) it is

$$
\begin{aligned}
R_1 &\equiv S_3 S_1 S_3^{-1} S_2^{-1} = 1 \\
R_2 &\equiv S_1 S_2 S_1^{-1} S_3^{-1} = 1 \qquad (2) \\
R_3 &\equiv S_2 S_3 S_2^{-1} S_1^{-1} = 1.
\end{aligned}
$$

It may be noticed, however, that the relations (1) are not all essential; one of them, being a consequence of the others, may be dropped.

We have so far constructed the knot group out of a given projection of the knot. But our result is still insufficient for distinguishing given knots. For the generating elements and the defining relations depend upon the choice of the projection, and no method is known for determining whether two groups given by generating elements and defining relations are identical or not. The problem of isomorphism of groups is as unsolved as the problem of equivalence of knots. For instance, the knot group of the trefoil knot for which we have found the defining relations (2) may as well be given by two generating elements A and B and the one defining relation $A^2 = B^3$. It is not at all obvious how the generating elements A and B may be expressed by S_1, S_2, S_3. Nevertheless the knot group is one of the most important knot invariants as it is the starting point for other and calculable invariants.

Besides of the fundamental group F of a complex L there is another well known invariant, the *homology group* H. One may define it as the abelianised fundamental group, that is the quotient group of F by its commutator group F_0:

$$H = F/F_0.$$

We see that the homology group can be derived from the fundamental group, and therefore it is in general a weaker invariant than the fundamental group. On the other hand it has the advantage of being determined by a finite system of numerical invariants which can be calculated by rational methods. For, every abelian group having a finite number of generating elements is the direct product of p cyclic groups of infinite order and certain cyclic groups of finite order, as is shown in the theory of elementary divisors. If the abelian group is the homology group H, the orders of the finite groups are called coefficients of torsion and the number p is the Betti number of the complex L.

Calculating the homology group H of the complement $L = R - k$ of a knot k one will find always the same result. H is the free group of one generator; in other words: $p = 1$ and there are no coefficients of torsion. This is a consequence of the relations (1). They reduce to $\bar{S}_\nu = \bar{S}_{\nu+1}$ by abelianising (the bar indicates the corresponding element of the abelianised group). The generators of the abelian group $\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_n$ are therefore all equal to one another, say $= S$, and the abelianised knot group is the free cyclic group of one generator, S. It therefore cannot be utilized for the classification of knots.

But new invariants may be deduced from the *covering manifolds* of $R - k$. These covering manifolds are likewise connected with k in an invariant manner.

Alexander discovered that the homology groups of the covering manifolds are in general different for different knots. They may be therefore utilized for distinguishing knots. The so-called *cyclic* covering manifolds are of special importance. In order to construct them let us remark that an orientable surface without singularities may be framed in any knot k in such a way that k is the only boundary curve of the surface. In the case of the circle one may take an element (a 2-cell) (Figure 13); in the case of the trefoil knot the surface given by Figure 14 will do. This figure shows the surface bounded by the knot in plane projection. In the three-dimensional space the two hatched parts of the projection cohere along three segments, which are double points in the projection. This surface is a perforated torus as may be shown by calculating its Euler characteristic. The surface of Figure 15, however, would not be fit for our purpose, as it is non-orientable (a Möbius strip).

We now cut the space R along the surface. We get a three-dimensional "sheet," and we attach g replicas of this sheet to one another in cyclic order around the knot. The analogous process, one dimension lower, is the construction of a Riemann surface on the sphere: the sphere is cut along an arc; it becomes a two-dimensional sheet, and g of these sheets are to be attached around the two branch points. In this way we get a g-fold covering manifold R_g of R, the knot k being the branch-line. By taking k out of R_g we obtain a covering manifold $R_g - k$ "*without* ramification." It is to be noticed that R_g depends only on the number g and the knot k, but not on the surface used for the construction. (However, there exist in general still other non-cyclic covering manifolds which are not to be considered here.)

In order to calculate the fundamental group F_g and the homology group H_g of $R_g - k$ we proceed from the theorem that the fundamental group of a covering manifold is isomorphic with a subgroup of the fundamental group F of the basic manifold. We obtain the subgroup by copying through into the basic manifold all oriented closed paths of the covering manifold starting from a fixed point. We therefore have to find in $R - k$ those closed paths W starting from O to which correspond in $R_g - k$ closed paths starting from one and the same point O. These are exactly the paths whose intersection number with the surface bounded by k is a multiple of g. An intersection point is to be counted positive or negative according as W pierces the surface from right to left or from left to right. (We may speak of left and right as our surface is orientable and therefore two-sided in the space). The intersection number is called also the *looping coefficient* of W with k. It is independent of the surface. We thus have found: the fundamental group F_g of $R_g - k$ is isomorphic with the subgroup of the (classes of) paths, the looping coefficients of which with k are multiples of g.

Reidemeister [7] has shown how to derive generators and defining relations of a subgroup from generators and defining relations of the whole group. By applying this method to the knot group one arrives after some calculations [13] at the following simple result.

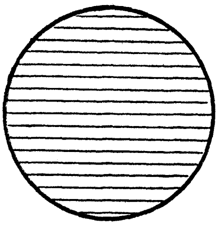

Fig. 13

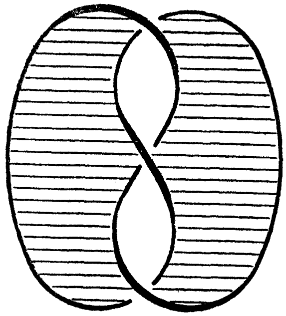

Fig. 14

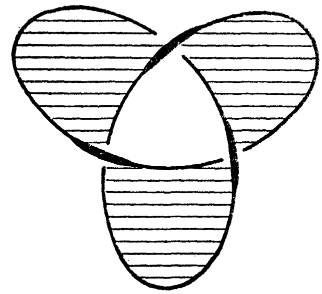

Fig. 15

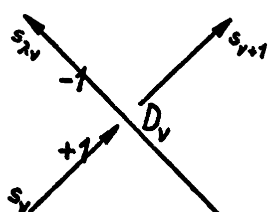

Fig. 16

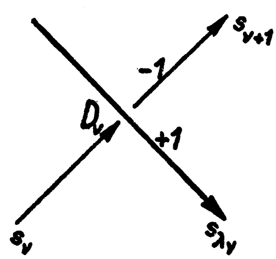

Fig. 17

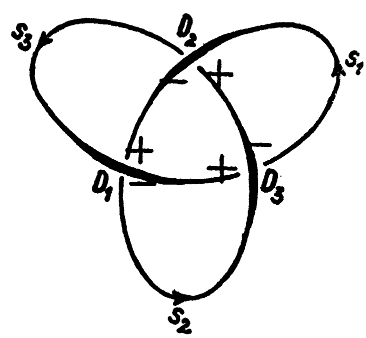

Fig. 18

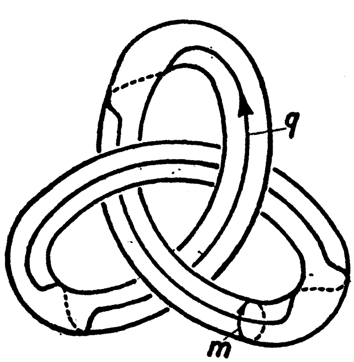

Fig. 19

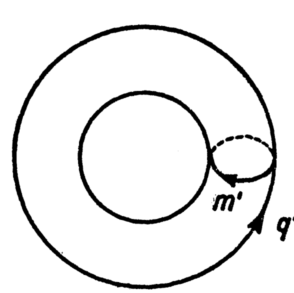

Fig. 20

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In order to determine the homology group H_g of the g-fold cyclic covering manifold $R_g - k$, the double points of the knot projection are to be denoted seriatim by $D_1, D_2, \ldots, D_n$, and the corresponding segment running from one crossed point to the next by $s_1, s_2, \ldots, s_n$ as has been done above. Let s_{λ_ν} be the crossing segment in the double point D_ν. Then we will write on this segment at the point D_ν the number -1 or $+1$ according as the crossing takes place from right to left or from left to right. We attach to the segment lying on the left side of s_{λ_ν} and coinciding with D_ν the opposite number, $+1$ or -1 as shown in Figures 16 and 17. From these numbers we form the matrix A, the rows corresponding to the double points $D_1, D_2, \ldots, D_n$ and the columns to the segments $s_1, s_2, \ldots, s_n$. (It may happen that in the case of a crossing from right to left s_ν is the same as s_{λ_ν}. Then s_ν will have two numbers $+1$ and -1 as suffixes of D_ν. In this case we put the sum of these two numbers, i.e., 0, in the position (D_ν, s_ν) of the matrix A; similarly in the case of a crossing from left to right.) We obtain from this matrix by suppressing the last row and the last column a matrix $\bar{A}$, and from this by adding the first column to the second, then the second to the third, etc., a matrix Γ of $n-1$ rows and $n-1$ columns. The torsion coefficients of $R_g - k$ are then the elementary divisors (other than 1) of the matrix

$$\Gamma^g - (\Gamma - E)^g,$$

E being the unit matrix of $n-1$ rows, whereas the rank defect of this matrix, augmented by one, gives the Betti number of $R_g - k$.

This theorem allows us to calculate the homology groups of all cyclic covering manifolds by means of one and the same matrix Γ. As Alexander has shown, one may thus verify that the 84 knots of the Alexander-Briggs index are distinct with a few exceptions.

Let us take as an example the trefoil knot. According to our prescription we number the double points and the segments, and we determine the indices ± 1. In Figure 18 we have written the signs $+$ and $-$ of the indices on the segments. The matrices A, $\bar{A}$ and Γ become

$$A = \begin{array}{c|ccc} & s_1 & s_2 & s_3 \\ \hline D_1 & 1 & 0 & -1 \\ D_2 & -1 & 1 & 0 \\ D_3 & 0 & -1 & 1 \end{array}, \quad \bar{A} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

We therefore get

$$\Gamma^2 - (\Gamma - E)^2 = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}.$$

By suitable transformations of the rows and columns the normal form $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ results. It has one elementary divisor 3, and the rank defect is 0. Thus the

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twofold cyclic covering manifold $R_2 - k$ has one torsion coefficient of value 3 and the Betti number 1. In the same way one may calculate

$$\Gamma^3 - (\Gamma - \mathrm{E})^3 = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix},$$

that is to say, the threefold covering manifold $R_3 - k$ has two torsion coefficients of value 2 each, and the Betti number 1. For $R_6 - k$ we find

$$\Gamma^6 - (\Gamma - \mathrm{E})^6 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This covering manifold has therefore no torsion coefficient, and the Betti number is 3.

Next to the covering manifolds $R_g - k$ of a finite number of sheets, the infinite cyclic covering manifold $R_0 - k$ is important. Its fundamental group is isomorphic with the subgroup F_0 of those elements of the knot group, whose looping coefficients with k are 0. (F_0 is the commutator group of F.) Because of the infinite number of sheets, the homology group H_0 of $R_0 - k$ can in general be given only by an infinite number of generators and relations. But a finite representation of H_0 may be obtained by interpreting H_0 as a group with operators in the following sense. There is a symmetry operation (Deckbewegung) of $R_0 - k$ transferring every sheet into the next one. It induces in H_0 an automorphism x. By making use of the operator x one obtains instead of the infinite number of generators and relations of H_0 a finite number of relations, the domain of coefficients being the integral domain of the polynomials in x and x^{-1} with integral coefficients.

It can be shown that the polynomial

$$\Delta = \left| \Gamma - x(\mathrm{E} - \Gamma) \right|,$$

Γ being the above-mentioned matrix, is a knot invariant, if one does not count a factor $\pm x^p$ which remains undetermined. This is the "*L-polynomial*" introduced by Alexander [1], see also Seifert [9]. In the case of the trefoil knot it is

$$\Delta = 1 + x + x^2,$$

whereas the circle has the L-polynomial

$$\Delta = 1.$$

An interesting application of the L-polynomials is the following relation between the L-polynomials of a special class of knots. Let k be an (oriented) knot lying in the space R, V a tubular neighbourhood of k. The boundary of V is a (two-dimensional) torus T. Let $W = R - V + T$ be the closed complement of V in R. A closed oriented curve on T without double points and non-bounding on T is called a "meridian" of V if it bounds on V, and a "parallel" of V if it bounds on W.

V can be mapped by a topological representation ϕ on an unknotted solid tube V', bounded by a two-dimensional torus T', in such a way that the parallel q of V becomes a parallel q' of V'. Figures 19 and 20 illustrate the case of k being the trefoil knot.

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Now let l be an arbitrary knot lying in V. l being a closed 1-chain of V, it is homologous on V to a multiple of k, say

$$l \sim nk \text{ on } V.$$

We may assume $n \geqq 0$ by orienting l properly. The topological representation ϕ of V on V' maps l into a knot l' of V'. Let $\Delta_k(x)$, $\Delta_l(x)$, $\Delta_{l'}(x)$ be the L-polynomials of the knots k, l, l' respectively. Then our theorem is (Seifert [12])

$$\Delta_l(x) = \Delta_{l'}(x)\Delta_k(x^n).$$

This theorem contains a result of Burau [3] concerning the special case where l is a "tube knot," the carrier knot of which is k.

An example of our theorem is given in Figure 21. Here the knot k is the trefoil knot, and $n = 0$. The case $n = 0$ has a remarkable consequence. Then we have $\Delta_k(x^n) = \Delta_k(x^0) = \Delta_k(1)$. But $\Delta_k(1) = 1$ for every knot. Therefore for $n = 0$ our theorem is $\Delta_l(x) = \Delta_{l'}(x)$. In other words the L-polynomial of l does not depend on the knot k. The doubled knots in the sense of Whitehead [14] are a special case hereof. We shall treat them soon again.

Besides the homology groups there are still other invariants of three-dimensional manifolds which play a role in distinguishing knots. In three-dimensional space, two closed oriented curves without common points have a certain looping coefficient. In the same way a looping coefficient of two closed oriented curves may be defined in other orientable three-dimensional manifolds, provided that the curves themselves or multiples of them be homologous to zero. These looping coefficients, however, will be in general *fractions*. For instance the looping coefficient of two projective lines of the projective space is 1/2. The looping coefficients possess the important property of changing their sign when the orientation of the space is reversed. From them may be deduced the so-called *looping invariants*, which are invariants of a three-dimensional manifold including a certain orientation. They may change when the orientation changes. There are examples where they do more for distinguishing knots than the knot group. For instance, it can be shown with them that the two knots of Figures 22 and 23 are not equivalent in spite of their having the same knot group. They further allow us in many cases to distinguish a knot from its symmetric (its image in a mirror). This is, for instance, the case with the trefoil knot, and the result is not to be had by the homology groups, H_0, H_2, H_3, ... It may be mentioned that these invariants are connected with the so-called quadratic form of the knot ([5] and [10]).

We have hitherto enumerated the most important knot invariants so far as they can be defined for *arbitrary* knots. What is the significance of these invariants?

The problem of isomorphism of the knot group is unsolved. Setting aside the knot group, we have the homology groups and the looping invariants of the infinite number of covering manifolds $R_g - k$. Let us call them the *homology invariants* of the knot. They suffice for distinguishing all the 84 knots of

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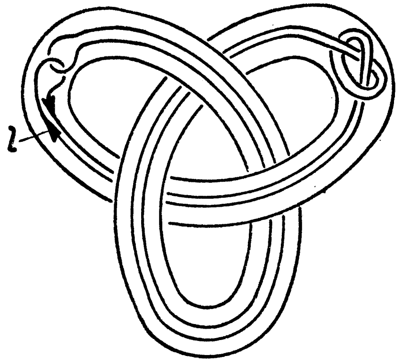

Fig. 21

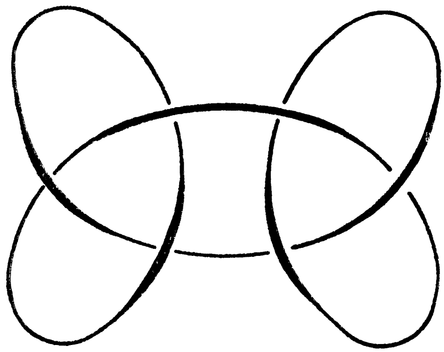

Fig. 22

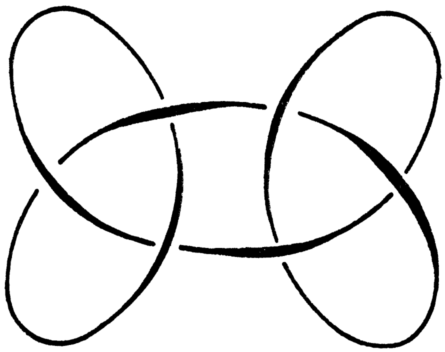

Fig. 23

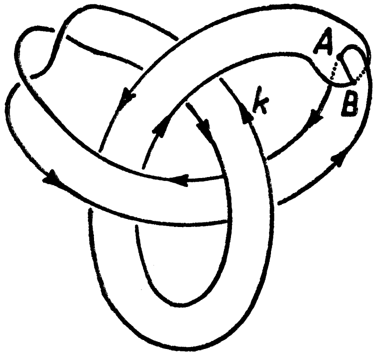

Fig. 24

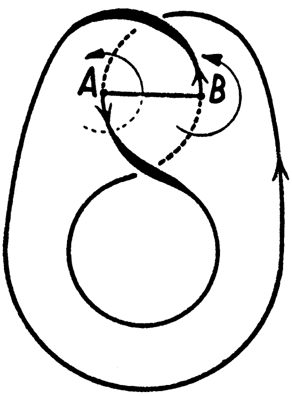

Fig. 25

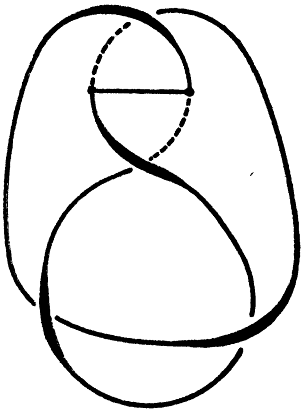

Fig. 26

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the Alexander-Briggs index. But the knot problem is still far from being solved with them. For there exists an infinite number of knots all having the same homology invariants.

There is even an infinite number of knots having the same homology invariants as the circle. For instance, certain *doubled knots* of J. H. C. Whitehead [14] are of this kind. They are defined as follows. Take a narrow ribbon, knot it in an arbitrary way and after that put the ends one into the other, so that they will penetrate each other along a segment AB, as is shown in Figure 24. Then the boundary of the ribbon will be a doubled knot. By making the ribbon more and more narrow it will finally be reduced to a knotted line $\bar{k}$, which we call the carrier knot. The carrier knot of the doubled knot shown in Figure 24 is the trefoil knot. To every carrier knot $\bar{k}$ there corresponds an infinite number of doubled knots. To construct them we only have to knot the ribbon according to $\bar{k}$ and then to twist it an arbitrary number of times. By choosing this number suitably the homology invariants of the doubled knot will be the same as those of the circle.

How is it possible to distinguish such knots? Whitehead has proved that two doubled knots can be equivalent only if their carrier knots have the same knot group. We can prove in addition that not only the knot group of $\bar{k}$ but the type of $\bar{k}$ itself is an invariant of the doubled knot. Two doubled knots, if constructed by means of inequivalent carrier knots, are therefore certainly not equivalent.

In contrast to the above-mentioned homology invariants, which may be called *algebraic* topological invariants, the invariance of the carrier knot is of a purely topological nature. Accordingly, other methods have to be used to prove it. One may proceed as follows. By definition a doubled knot is the boundary of an "element with self-penetration." This means a 2-cell of the three-dimensional space, the only singularity of which is a self-penetration along a segment AB (Figure 24). Yet we have to pay attention to the fact that this element with self-penetration is not determined by the doubled knot. For instance, one may deform it into another one without changing the boundary.

The question therefore arises how many essentially different elements with self-penetration may be bounded by the same doubled knot k. Let us explain this question as applied to the simplest example imaginable. The circle itself may be interpreted as a doubled knot, as is seen by Figure 25. The figure shows the ribbon bounded by the circle. We may provide the ribbon with a certain "index." It is defined as follows. We give the boundary k of the ribbon a certain orientation. This induces a certain orientation of the ribbon itself. A and B being the points of penetration between k and the ribbon, the orientation of k together with the orientation of the ribbon determines a certain screw (space orientation) in A. In Figure 25 this is a right-handed screw. The same right-handed screw is determined in B.

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If we reverse the orientation of k, the orientation of the ribbon is reversed at the same time. Therefore the sense of the screws does not change. Now we assign to the intersection point A the index $+1$ if the screw of A is a right-handed screw, and -1 if it is a left-handed screw, and we do the same to the other point B. Then the index of the ribbon is the sum of the two indices of A and B. In the case of Figure 25 the index is $+2$. We therefore see that a circle may bound a ribbon of index $+2$. But then it may bound a ribbon of index -2 just as well. We only have to reflect Figure 25. This process reverses the orientation of space and therefore right-handed screws become left-handed screws. The reflected knot is again a circle, but now it bounds an element with self-penetration of index -2. The result is, that a circle can bound two essentially different elements with self-penetration, which cannot be transformed into one another by a semilinear mapping of the space, for such a mapping would not change the index of the element.

We find the same situation in the case of the "four-knot." The ribbon shown in Figure 26 has the index $+2$. By reflecting it we get a ribbon of index -2. Now it is known that the reflected image of the four-knot is equivalent to the original knot. Therefore the four-knot also admits two essentially different elements with self-penetration of which it is the boundary.

We believe the circle and the four-knot to be the only two knots having this quality. We can prove the following result: *If a doubled knot k can be constructed by means of a carrier knot $\bar{k}$, $\bar{k}$ not being the circle, then there exists, apart from semilinear mappings of the space, only one element with self-penetration bounded by k.* The proof of this theorem is rather complicated. The only way is to construct the semilinear mapping of one of two elements with self-penetration bounded by k into the other. This can be done by considering the lines and points of mutual penetration of the two elements and splitting them off one by one by appropriate methods [11].

The invariance of the carrier knot is an immediate consequence of this theorem. Given the element with self-penetration, the carrier knot is obtained by joining the ends A and B of the penetrating segment by a curve running along the ribbon. The resulting closed curve is itself the carrier knot $\bar{k}$. Now let k and k' be two doubled knots derived from the carrier knots $\bar{k}$ and $\bar{k}'$. Then the equivalence of $\bar{k}$ and $\bar{k}'$ follows from the equivalence of k and k' provided that one at least of the two knots $\bar{k}$ and $\bar{k}'$ is not the circle. In the case when both of them are circles, they are obviously equivalent. In any case, therefore, the carrier knot is an invariant of the doubled knot.

Furthermore it follows from our theorem of the uniqueness of the bounded element that a doubled knot, the carrier knot of which is not a circle, will never be equivalent to its mirrored image. For the index of the bounded element would change by reflection. If the doubled knot were equivalent to its image it would bound therefore two essentially different elements of index $+2$ and -2.

The preceding theorems are concerned with special classes of knots. Let us conclude with a result relating to the decomposition of an arbitrary knot. We shall call a knot a prime knot if it is impossible to cut it after a suitable deformation by a plane having only two points P and Q in common with k, into two knots (both distinct from the circle) consisting of the two parts of k and the closing segment PQ. For instance the knot shown in Figure 22 can be decomposed into two trefoil knots. Now our theorem is, that every knot k can be decomposed into prime knots and that the series of these prime knots is unique but for the order of them [8].

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