Glasgow Math. J. **51** (2009) 659–667. © 2009 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089509990073. Printed in the United Kingdom

THE EXPONENTIAL DIOPHANTINE EQUATION $n^x + (n+1)^y = (n+2)^z$ REVISITED

BO HE

College of Mathematics and Software Science, Sichuan Normal University, Chengdu 610068, People's Republic of China
e-mail: bhe@live.cn

and ALAIN TOGBÉ

Mathematics Department, Purdue University North Central, 1401 S, U.S. 421, Westville, IN 46391, USA
e-mail: atogbe@pnc.edu

(Received 6 May 2008; revised 15 November 2008; accepted 1 June 2009)

Abstract. Let n be a positive integer. In this paper, we consider the diophantine equation

$$n^x + (n+1)^y = (n+2)^z, \quad n \in \mathbb{N} \quad xyz \neq 0.$$

We prove that this equation has only the positive integer solutions $(n, x, y, z) = (1, t, 1, 1),\ (1, t, 3, 2),\ (3, 2, 2, 2)$. Therefore we extend the work done by Leszczyński (*Wiadom. Mat.*, vol. 3, 1959, pp. 37–39) and Makowski (*Wiadom. Mat.*, vol. 9, 1967, pp. 221–224).

2000 *Mathematics Subject Classification.* Primary 11D61; secondary 11B39, 11J86.

1. Introduction. The Diophantine equation

$$a^x + b^y = c^z, \quad a, b, c, x, y, z \in \mathbb{N} \tag{1.1}$$

has a very rich history. Many authors have studied equation (1.1) when some of the variables a, b, c, x, y, z are fixed. In 1956, Sierpiński [**15**] proved that $(x, y, z) = (2, 2, 2)$ is the only positive integral solution of the equation $3^x + 4^y = 5^z$. The same year, Jeśmanowicz [**9**] conjectured that if a, b, c are Pythagorean triples, i.e. positive integers satisfying $a^2 + b^2 = c^2$, then Diophantine equation $a^x + b^y = c^z$ has only the positive integral solution $(x, y, z) = (2, 2, 2)$. Many special cases of this conjecture have been settled. Other conjectures related to equation (1.1) were set and discussed. One is the extension of Jeśmanowicz' conjecture due to Terai (see for example [**17**–**21**]). In fact, Terai conjectured that if $a, b, c, p, q, r \in \mathbb{N}$ are fixed and $a^p + b^q = c^r$, where $p, q, r \geq 2$, and $\gcd(a, b) = 1$, then Diophantine equation (1.1) has only the solution $(x, y, z) = (p, q, r)$. Many authors have proved or disproved that the conjecture is true in some particular cases. One can see for example [**5**–**7**]. Authors have also studied equation (1.1) when a, b, c are polynomial functions. See for example [**5**–**7**, **14**, **17**–**21**].

In this paper, we consider the following Diophantine equation:

$$n^x + (n+1)^y = (n+2)^z, \quad n \in \mathbb{N} \quad xyz \neq 0. \tag{1.2}$$

In fact, equation (1.2) is a generalisation of the equation $3^x + 4^y = 5^z$ studied by Sierpiński [**15**]. The general equation was first studied by Leszczyński [**12**] and Makowski [**14**]. In fact, Makowski extended Leszczyński's work and found the solutions when $y = 1$ for $1 \leq n \leq 48$. The equation was solved when $xyz = 0$ (see Theorem 2 in [**14**]). In this case, the solutions are $(n, x, y, z) = (n, 0, 1, 1)$, $(1, 0, 3, 2)$. Therefore, we suppose $xyz \neq 0$. Our main result is the following.

THEOREM. *Equation* (1.2) *has only the positive integer solutions*

$$(n, x, y, z) = (1, t, 1, 1),\ (1, t, 3, 2),\ (3, 2, 2, 2). \tag{1.3}$$

The organisation of this paper is as follows: In Section 2, we recall some results due to LeVeque [**13**], Siksek [**16**], Ivorra [**8**] and Laurent, Mignotte and Nesterenko [**11**]. Also we prove a result by the means of Lucas sequences. These results are useful for the proof of our main theorem that will be shown in Section 3. First, for $n = 1$, LeVeque's result helps to find the solutions $(n, x, y, z) = (1, t, 1, 1)$, $(1, t, 3, 2)$. Then we use an elementary method to prove that equation (1.2) has only the solution $(n, x, y, z) = (3, 2, 2, 2)$, for $y \geq 2$ and $n \geq 2$. Finally, by means of Baker's method, we extend Leszczyński–Makowski's result by proving that equation (1.2) has no solution when $y = 1$ and $n \geq 2$.

2. Lemmas.

First, we recall a simplified result due to LeVeque [**13**].

LEMMA 1. *For fixed integers $a > 1$ and $b > 1$, the equation*

$$a^x - b^y = 1$$

has just the two solutions $(1, 1)$ *and* $(2, 3)$ *if $a = 3$, $b = 2$. In all other cases, it has at most one solution.*

Let α, β be algebraic integers. If $\alpha + \beta$ and $\alpha\beta$ are non-zero coprime rational integers and α/β is not a root of unity, then (α, β) is called a Lucas pair. Given a Lucas pair (α, β), one defines the corresponding sequence of Lucas numbers by

$$U_n = U_n(\alpha, \beta) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n = 0, 1, 2, \ldots.$$

A prime p is called a primitive divisor of $U_n(\alpha, \beta)$ if $p | U_n$ and $p \nmid (\alpha - \beta)^2 U_1 U_2 \ldots U_{n-1}$. An important problem is the existence of primitive divisor of Lucas numbers. In 2001, Bilu, Hanrot and Voutier [**3**] solved the problem. The remaining cases were solved by Abouzaid [**1**]. The case $\alpha, \beta \in \mathbb{Z}$ was solved by Birkhoff–Vandiver [**4**] and Zsigmondy [**22**] in 1904 and 1892, independently. They proved that $U_n(\alpha, \beta)$ has a primitive divisor if $n > 6$. Early in 1886, Bang [**2**] showed the result when $\beta = 1$. The following lemma comes from Bang's result.

LEMMA 2. *For a fixed integer $m \geq 2$; then equation*

$$2^X m^Y + 1 = (2m + 1)^Z, \quad m \geq 2, \tag{2.1}$$

has only the positive integer solution $X = Y = Z = 1$, except when $m = 2^{s-2} - 1$, for some integer $s \geq 4$, in which case equation (2.1) *has the additional solution* $(X, Y, Z) = (s, 1, 2)$.

Proof. If $Z = 1$, it is clear that $X = Y = 1$ by comparing the two sides of equation (2.1). Now, we assume $Z \geq 2$.

We rewrite equation (2.1) in the form

$$U_Z = U_Z(2m+1, 1) = \frac{(2m+1)^Z - 1}{(2m+1) - 1} = 2^{X-1} m^{Y-1}. \tag{2.2}$$

Since $(2m+1) - 1 = 2m$, U_Z has no primitive divisor by Lemma 2. Thus $Z \leq 6$. So equation (2.2) implies

$$U_Z = 1 + (2m+1) + (2m+1)^2 + \cdots + (2m+1)^{Z-1} \equiv Z \pmod{2m}. \tag{2.3}$$

By (2.2) and (2.3) we have

$$Z \equiv 2^{X-1} m^{Y-1} \pmod{2m}. \tag{2.4}$$

Now suppose $Y = 1$ and $X \geq 2$; then equation (2.1) becomes

$$2^X m + 1 = (2m+1)^Z.$$

Taking modulo 4, we have $1 \equiv 1 + Z \cdot 2m + \binom{Z}{2} \cdot 2m + \cdots + (2m)^Z$. This implies Z is even. If $Z = 2$, then the above equation gives $m = 2^{X-2} - 1$. Therefore, equation (2.1) has solution $(X, Y, Z) = (s, 1, 2)$ for $m = 2^{s-2} - 1\,(s \geq 4)$. If $Z = 4$, we have $2^X m = (2m+1)^4 - 1 = 8m(m+1)(2m^2 + 2m + 1)$. It follows that $2^{X-3} = (m+1)(2m^2 + 2m + 1)$. So as $2m^2 + 2m + 1$ is an odd integer greater than 1, we get a contradiction. In the case $Z = 6$, $(2m+1)^6 - 1$ has an odd divisor $4m^2 + 2m + 1$. This is also impossible.

Finally, if $Y \geq 2$, then (2.4) gives $Z \equiv 0 \pmod{m}$. This implies $m \leq Z$. As $Z \leq 6$, a straightforward computation gives no solution (m, Z) with $m \leq Z \leq 6$. □

Now we recall the following result due to Siksek [**16**] and Ivorra [**8**].

LEMMA 3. *Let k be a positive integer. If the equation*

$$x^2 - 2^k = y^n, \quad x, y, k, n \in \mathbb{N}, \ \gcd(x, y) = 1, \ y > 1, \ n \geq 3$$

has a solution then $k = 1$.

Finally we recall the following result due to Laurent, Mignotte, and Nesterenko (see Corollaire 2, p. 288, in [**11**]) on linear forms in two logarithms. For any non-zero algebraic number γ of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a \prod_{j=1}^{d} \left(X - \gamma^{(j)}\right)$, we denote by

$$h(\gamma) = \frac{1}{d}\left(\log|a| + \sum_{j=1}^{d} \log\max(1, |\gamma^{(j)}|)\right)$$

its absolute logarithmic height.

LEMMA 4. *Let γ_1 and γ_2 be multiplicatively independent and positive algebraic numbers, b_1 and $b_2 \in \mathbb{Z}$ and*

$$\Lambda = b_1 \log\gamma_1 + b_2 \log\gamma_2.$$

https://doi.org/10.1017/S0017089509990073 Published online by Cambridge University Press

Let $D := [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}]$*; for* $i = 1, 2$ *let*

$$h_i \geq \max\left\{h(\gamma_i), \frac{|\log \gamma_i|}{D}, \frac{1}{D}\right\}$$

and

$$b' \geq \frac{|b_1|}{D h_2} + \frac{|b_2|}{D h_1}.$$

If $|\Lambda| \neq 0$*, then we have*

$$\log|\Lambda| \geq -24.34 \cdot D^4 \left(\max\left\{\log b' + 0.14, \frac{21}{D}, \frac{1}{2}\right\}\right)^2 h_1 h_2.$$

3. Proof of the main theorem. When $n = 1$, equation (1.2) becomes

$$3^z = 1 + 2^y.$$

It has only two positive solutions $(y, z) = (1, 1), (3, 2)$ by Lemma 1. Therefore we obtain the first two solutions contained in (1.3). Now we suppose that $n \geq 2$. From (1.2) we have $2 \nmid n$. Let $n = 2m - 1$ with $m \geq 2$. We rewrite equation (1.2) into the form

$$(2m-1)^x + (2m)^y = (2m+1)^z. \tag{3.1}$$

We will prove the main theorem in two steps.

3.1. The Case $y \geq 2$. From equation (3.1), one can see that $(-1)^x \equiv 1 \pmod{2m}$. This implies $2|x$. By consideration modulo $4m^2$ of (3.1), we have

$$1 - 2mx \equiv 1 + 2mz \pmod{4m^2}.$$

It follows $x + z \equiv 0 \pmod{2m}$. Therefore z is also an even integer. Let $x = 2x_1$ and $z = 2z_1$. Equation (3.1) becomes

$$(2m-1)^{2x_1} + (2m)^y = (2m+1)^{2z_1}.$$

We factor the above expression to obtain

$$\left((2m+1)^{z_1} + (2m-1)^{x_1}\right)\left((2m+1)^{z_1} - (2m-1)^{x_1}\right) = (2m)^y. \tag{3.2}$$

Now let us study (3.2) according to the parity of x_1.

- First we suppose $2|x_1$. Then we have

$$(2m+1)^{z_1} + (2m-1)^{x_1} \equiv 2 + 2m(z_1 - x_1) \pmod{4m^2}. \tag{3.3}$$

One can see that equation (3.3) implies

$$\gcd\left(((2m+1)^{z_1} + (2m-1)^{x_1})/2, m\right) = 1. \tag{3.4}$$

https://doi.org/10.1017/S0017089509990073 Published online by Cambridge University Press

Then we use equation (3.2) to have

$$(2m+1)^{z_1} + (2m-1)^{x_1} | 2^{y-1}.$$

Thus equation (3.1) implies

$$(2m)^y < (2m+1)^z = (2m+1)^{2z_1} < 2^{2y-2}.$$

So we get $4(m/2)^y < 1$. This is impossible when $m \geq 2$.

- Then we assume $2 \nmid x_1$. In this case, from equation (3.2) we have

$$(2m+1)^{z_1} + (2m-1)^{x_1} \equiv 2m(z_1 + x_1) \pmod{4m^2}, \tag{3.5}$$

and

$$(2m+1)^{z_1} - (2m-1)^{x_1} \equiv 2 + 2m(z_1 - x_1) \pmod{4m^2}. \tag{3.6}$$

One can see that equation (3.6) implies

$$\gcd\big(((2m+1)^{z_1} - (2m-1)^{x_1})/2, m\big) = 1. \tag{3.7}$$

If $2 \nmid z_1$, then both $z_1 \pm x_1$ are even; then we use equation (3.6) to deduce

$$\gcd\big(((2m+1)^{z_1} - (2m-1)^{x_1})/2, 2\big) = 1. \tag{3.8}$$

Therefore, from (3.2), (3.7) and (3.8) we obtain

$$\begin{cases} (2m+1)^{z_1} + (2m-1)^{x_1} = 2^{y-1}m^y, \\ (2m+1)^{z_1} - (2m-1)^{x_1} = 2. \end{cases} \tag{3.9}$$

Adding these two equations, the following results:

$$(2m+1)^{z_1} = 2^{y-2}m^y + 1. \tag{3.10}$$

If $y = 2$, we have

$$(2m+1)^{z_1} = m^2 + 1.$$

Combining this and $m^2 + 1 < (2m+1)^2$ we obtain $z_1 = 1$. So the only positive solution of $2m + 1 = m^2 + 1$ is $m = 2$. Therefore equation (1.2) has the positive integer solution $(x, y, z) = (2, 2, 2)$, when $n = 3$. Otherwise, we consider $y \geq 3$ and use Lemma 2 to see that equation (3.10) has no solution.

If $2|z_1$, then both $z_1 \pm x_1$ are odd. Assume m is even; then from equation (3.6) we have again (3.8). We have the same form as the system (3.9). When m is odd, we obtain from equation (3.5)

$$\gcd\big(((2m+1)^{z_1} + (2m-1)^{x_1})/2, 2\big) = 1.$$

Therefore, we use equation (3.2) to obtain

$$\begin{cases} (2m+1)^{z_1} + (2m-1)^{x_1} = 2m^y, \\ (2m+1)^{z_1} - (2m-1)^{x_1} = 2^{y-1}. \end{cases} \tag{3.11}$$

https://doi.org/10.1017/S0017089509990073 Published online by Cambridge University Press

Adding these two equations gives

$$\left((2m+1)^{z_1/2}\right)^2 - 2^{y-2} = m^y.$$

It is easy to see that when $y = 2$ there is no solution. By Lemma 3, the only possibility is $y = 3$, i.e.

$$(2m+1)^{z_1} - 2 = m^3.$$

Since $m^3 < (2m+1)^3$ and z_1 is even, we have $z_1 = 2$. But the only integer solution of $m^3 - 4m^2 - 4m + 1 = (m+1)(m^2 - 5m + 1) = 0$ is $m = -1$. This is also impossible.

3.2. The case $y = 1$. The case $y = 1$ was already considered by Makowski [**14**]. He proved that equation

$$(2m-1)^x + 2m = (2m+1)^z \tag{3.12}$$

has no positive integer solution (x, y, z) in the range $2 \le 2m - 1 \le 48$. Thus we assume $m \ge 25$.

We take (3.12) modulo $2m$ and we have $(-1)^x \equiv 1 \pmod{2m}$. This implies $2|x$. We take again (3.12) modulo $4m^2$ and we obtain

$$1 - 2mx + 2m \equiv 1 + 2mz \pmod{4m^2}.$$

It follows $x + z \equiv 1 \pmod{2m}$. Therefore z is an odd integer. If $z \ge x$, then

$$2m = (2m+1)^z - (2m-1)^x \ge (2m+1)^x - (2m-1)^x \ge (2m+1)^2 - (2m-1)^2 = 8m.$$

This is impossible. Hence $z < x$.

Let

$$\Lambda = z\log(2m+1) - x\log(2m-1).$$

From (3.12) we get $e^{\Lambda} - 1 = \frac{2m}{(2m-1)^x}$. Notice that $m \ge 25$; it follows that

$$0 < \Lambda < \frac{1}{0.98(2m-1)^{x-1}}. \tag{3.13}$$

We know that

$$\frac{x-1}{x} \ge \frac{z}{x} > \frac{\log(2m-1)}{\log(2m+1)};$$

thus we have

$$x > \frac{\log(2m+1)}{\log\left(1 + \frac{2}{2m-1}\right)} > (m - 0.5)\log(2m+1). \tag{3.14}$$

As $m \ge 25$, the above inequality implies $x \ge 99$.

Now we apply Lemma 4, and we take

$$D = 1, \quad b_1 = z, \quad b_2 = -x, \quad \gamma_1 = 2m+1, \quad \gamma_2 = 2m-1.$$

https://doi.org/10.1017/S0017089509990073 Published online by Cambridge University Press

We have $h_1 = \log\gamma_1,\ \ h_2 = \log\gamma_2$. From (3.13) we get

$$\frac{z}{\log(2m-1)} - \frac{x}{\log(2m+1)} < \frac{1}{0.98(2m-1)^{x-1}\log(2m+1)\log(2m-1)} < 0.154\cdot 10^{-166}.$$

Hence, if we take

$$b' = \frac{2x}{\log(2m+1)} + 1.54\cdot 10^{-167},$$

then

$$b' > \frac{z}{\log(2m-1)} + \frac{x}{\log(2m+1)} = \frac{|b_1|}{D\,h_2} + \frac{|b_2|}{D\,h_1}.$$

If $\log b' + 0.14 \le 21$, we have

$$\log|\Lambda| \ge -24.34\cdot 21^2\log(2m-1)\log(2m+1). \tag{3.15}$$

We use (3.13) to obtain

$$\log|\Lambda| < -(x-1)\log(2m-1) + 0.021. \tag{3.16}$$

Inequalities (3.15) and (3.16) imply

$$x - 1 < 24.34\cdot 21^2\log(2m+1). \tag{3.17}$$

Combining this and (3.14), we have

$$m < 0.5 + \frac{1}{\log(2m-1)} + 24.34\cdot 21^2\frac{\log(2m+1)}{\log(2m-1)} < 10846.$$

Now if $\log b' + 0.14 > 21$, then we obtain

$$\log|\Lambda| \ge -24.34\left(\log\left(\frac{2x}{\log(2m+1)} + 1.54\cdot 10^{-167}\right) + 0.14\right)^2 \times \log(2m+1)\log(2m-1).$$

The above inequality and (3.16) imply

$$\frac{x}{\log(2m+1)} < \frac{-0.021 + \log(2m+1)}{\log(2m+1)\log(2m-1)} + 24.34\left(\log\left(\frac{2x}{\log(2m+1)} + 1.54\cdot 10^{-167}\right) + 0.14\right)^2.$$

It follows that

$$\frac{x}{\log(2m+1)} < 0.24 + 24.34\left(\log\left(\frac{x}{\log(2m+1)} + 0.77\cdot 10^{-167}\right) + 0.834\right)^2. \tag{3.18}$$

https://doi.org/10.1017/S0017089509990073 Published online by Cambridge University Press

We get $x < 1656 \log(2m+1)$. Then $b' = 2x/\log(2m+1) + 1.54 \cdot 10^{-167} < 3313$, so that $\log b' < 8.11$. But since we were assuming $\log b' + 0.14 > 21$, this is clearly a contradiction.

From all this, we deduce upper bounds for x and m, i.e.

$$m \leq 10845 \quad \text{and} \quad x < 107187. \tag{3.19}$$

From the definition of Λ and (3.13), one can see that

$$\left| \frac{\log(2m-1)}{\log(2m+1)} - \frac{z}{x} \right| < \frac{1}{0.98x(2m-1)^{x-1}\log(2m+1)}. \tag{3.20}$$

From the fact $m \geq 25$ and (3.14), we have $x \geq 99$. It is easy to verify that

$$2x^2 < 0.98x(2m-1)^{x-1}\log(2m+1).$$

Thus the left side of (3.20) is less than $\frac{1}{2x^2}$. Therefore, $\frac{z}{x}$ is a convergent in the simple continued fraction expansion of $\frac{\log(2m-1)}{\log(2m+1)}$. On the other hand, if p_r/q_r is the rth such convergent, then

$$\left| \frac{\log(2m-1)}{\log(2m+1)} - \frac{p_r}{q_r} \right| > \frac{1}{(a_{r+1}+2)q_r^2},$$

where a_{r+1} is the $(r+1)$st partial quotient to $\frac{\log(2m-1)}{\log(2m+1)}$ (see e.g. [**10**]). When $\frac{z}{x} = \frac{p_r}{q_r}$ (notice $x \geq q_r$ and $z \geq p_r$), from $m \geq 25$ and $x \geq 99$ we obtain that

$$a_{r+1} > \frac{0.98(2m-1)^{x-1}\log(2m+1)}{x} - 2 > 10^{165}. \tag{3.21}$$

Finally, we wrote a short PARI/GP program, and we found no integer m such that $25 \leq m \leq 10845$ and satisfying (3.21) with $q_r < 107187$. This completes the proof of our main theorem.

ACKNOWLEDGEMENTS. The authors express their gratitude to Prof. Wlodzimierz Kuśmierczuk and Prof. Janusz Duzinkiewicz who helped them to translate Makowski's paper [**14**] from Polish to English. They thank the anonymous referee for constructive suggestions to improve an earlier draft of this paper. The first author is supported by the Natural Science Foundation of Education Department of Sichuan Province. The second author is partially supported by Purdue University North Central.

REFERENCES

1. M. Abouzaid, Les nombres de Lucas et Lehmer sans diviseur primitif, *J. Théor. Nombres Bordeaux* **18** (2006), 299–313.

2. A. S. Bang, Tialtheoretiske Undersøgelser, *Tidsskrift Mat.* **4** (1886), 70–80, 130–137.

3. Y. Bilu, G. Hanrot and P. Voutier (with an appendix by M. Mignotte), Existence of primitive divisors of Lucas and Lehmer numbers, *J. Reine Angew. Math.* **539** (2001), 75–122.

4. G. D. Birkhoff and H. S. Vandiver, On the integral divisors of $a^n - b^n$, *Ann. Math.* **5** (2) (1904), 173–180.

5. Z. F. Cao and X. Dong, On Terai's conjecture, *Proc. Japan Acad. A Math. Sci.* **74** (1998), 127–129.

6. Z. F. Cao, A note on the the Diophantine equation $a^x + b^y = c^z$, *Acta Arth.* **91** (1999), 85–93.

7. Z. F. Cao, X. Dong and Z. Li, A new conjecture concerning the Diophantine equation $a^x + b^y = c^z$, *Proc. Japan Acad. A Math. Sci.* **78** (2002), 199–202.

8. W. Ivorra, Sur les équations $x^p + 2^\beta y^p = z^2$ et $x^p + 2^\beta y^p = 2z^2$, *Acta Arith.* **108** (2003), 327–338.

9. L. Jeśmanowicz, Some remarks on Pythagorean numbers, *Wiadom. Mat.*, **1** (1956), 196–202.

10. A. Y. Khinchin, *Continued fractions, 3rd edition* (P. Noordhof, Groningen, The Netherlands, 1963).

11. M. Laurent, M. Mignotte and Yu. Nesterenko, Formes linéaires en deux logarithmes et déterminants d'interpolation, *J. Number Theory* **55** (1995), 285–321.

12. B. Leszczyński, On the equation $n^x + (n+1)^y = (n+2)^z$, *Wiadom. Mat.* **3** (1959), 37–39.

13. W. J. LeVeque, On the equation $a^x - b^y = 1$, *Am. J. Math.* **74** (1952), 325–331.

14. A. Makowski, On the equation $n^x + (n+1)^y = (n+2)^z$, *Wiadom. Mat.* **9** (1967), 221–224.

15. W. Sierpiński, On the equation $3^x + 4^y = 5^z$, *Wiadom. Mat.* **1** (1956), 194–195.

16. S. Siksek, On the Diophantine equation $x^2 = y^p + 2^k z^p$, *J Théor. Nombres Bordeaux* **15** (2003), 839–846.

17. N. Terai, The Diophantine equation $x^2 + q^m = p^n$, *Acta Arith.* **63** (1993), 351–358.

18. N. Terai, The Diophantine equation $a^x + b^y = c^z$, *Proc. Japan Acad. A Math. Sci.* **70** (1994), 22–26.

19. N. Terai, The Diophantine equation $a^x + b^y = c^z$ II, *Proc. Japan Acad. A Math. Sci.* **71** (1995), 109–110.

20. N. Terai, The Diophantine equation $a^x + b^y = c^z$ III, *Proc. Japan Acad. A Math. Sci.* **72** (1996), 20–22.

21. N. Terai and K. Tawakuwa, A note on the the Diophantine equation $a^x + b^y = c^z$, *Proc. Japan Acad. A Math. Sci.* **73** (1997), 161–164.

22. K. Zsigmondy, Zur Theorie der Potenzreste, *Monatsh. Math.* **3** (1892), 265–284.