AN ALGEBRAICALLY CLOSED FIELD

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1. Introduction. Let k be any algebraically closed field, and denote by $k((t))$ the field of formal power series in one indeterminate t over k. Let

$$K = \bigcup_{n=1}^{\infty} k((t^{1/n!})),$$

so that K is the field of Puiseux expansions with coefficients in k (each element of K is a formal power series in $t^{1/r}$ for some positive integer r). It is well-known that K is algebraically closed if and only if k is of characteristic zero [**1**, p. 61]. For examples relating to ramified extensions of fields with valuation [**9**, §6] it is useful to have a field analogous to K which is algebraically closed when k has non-zero characteristic p. In this paper, I prove that the set L of all formal power series of the form $\sum a_i t^{e_i}$ (where (e_i) is well-ordered, $e_i = m_i/np^{r_i}$, $n \in \mathbf{Z}$, $m_i \in \mathbf{Z}$, $a_i \in k$, $r_i \in \mathbf{N}$) forms an algebraically closed field.

It is convenient to discuss the field L in connexion with a new class of relatively complete fields, which arise by modification of the construction described by B. H. Neumann in [**6**]. Relatively complete fields which are not complete have previously been pointed out by Ostrowski [**7**] (the absolutely algebraic p-adic numbers, which have obvious generalisations) and by Moriya [**5**] (infinite algebraic extensions of complete fields).

2. Field-families. Let E be a field, Γ an ordered abelian group written additively, and $\mathscr{A}$ a family of subsets of Γ. Let E^{Γ} denote the set of maps from the set of elements of Γ to the set of elements of E, and, for $x \in E^{\Gamma}$, let $S(x) = \{\gamma \in \Gamma : x(\gamma) \neq 0\}$ denote the support of x. Let $E^{\Gamma}(\mathscr{A})$ denote $\{x \in E^{\Gamma} : S(x) \in \mathscr{A}\}$. We define addition and multiplication in $E^{\Gamma}(\mathscr{A})$ by

$$(x+y)(\gamma) = x(\gamma) + y(\gamma) \tag{1}$$

and

$$xy(\gamma) = \sum_{\alpha+\beta=\gamma} x(\alpha)\, y(\beta) \tag{2}$$

wherever the operations are meaningful; thus $E^{\Gamma}(\mathscr{A})$ is a ringoid. Let $\mathscr{W}(\Gamma\}$ be the family of all well-ordered subsets of Γ. Then Hahn [**2**] has shown that $E^{\Gamma}(\mathscr{W}(\Gamma))$ is always a field. Clearly, if $\mathscr{A}$ strictly contains $\mathscr{W}(\Gamma)$, definition (2) will not always make sense. On the other hand, however, we shall see that $\mathscr{A}$ can be a proper subset of $\mathscr{W}(\Gamma)$ and still lead to a field.

Definition. $\mathscr{A}$ is said to be a *field-family* with respect to Γ if

(i) $\mathscr{A} \subset \mathscr{W}(\Gamma)$,

(ii) the elements of the members of $\mathscr{A}$ generate the group Γ,

(iii) $A \in \mathscr{A}, B \in \mathscr{A} \Rightarrow A \cup B \in \mathscr{A}$,

(iv) $A \in \mathscr{A}, B \subset A \Rightarrow B \in \mathscr{A}$,

(v) $A \in \mathscr{A}, \gamma \in \Gamma \Rightarrow \{a+\gamma : a \in A\} \in \mathscr{A}$,

and

(vi) $A \in \mathscr{A}, A \subset \Gamma^{+} \Rightarrow \langle A \rangle \in \mathscr{A}$,

where, in (vi), $\Gamma^{+} = \{\gamma \in \Gamma : \gamma \geqq 0\}$ and $\langle A \rangle$ is the set of elements of the semigroup generated by the elements of A under addition in Γ.

As a special case of [**6**, p. 206, Theorem 3.4], we have the following:

LEMMA 1. $A \in \mathscr{W}(\Gamma), A \subset \Gamma^{+} \Rightarrow \langle A \rangle \in \mathscr{W}(\Gamma)$.

It is clear from this that $\mathscr{W}(\Gamma)$ is itself a field-family. Other examples of field-families are given in §3 below.

THEOREM 1. *If $\mathscr{A}$ is a field-family with respect to Γ, then $E^{\Gamma}(\mathscr{A})$ is a field.*

Proof. From (i), (iii) and (iv), it follows that $E^{\Gamma}(\mathscr{A})$ is an additive abelian group under the addition (1). From (i), (iii), (iv), (v) and (vi), multiplication is everywhere defined, and from (1) and (2) it is distributive over addition. Commutativity, associativity and the existence of a unit may readily be checked. It remains to show that each nonzero element of $E^{\Gamma}(\mathscr{A})$ has a multiplicative inverse in $E^{\Gamma}(\mathscr{A})$. Suppose $x \neq 0$, $x \in E^{\Gamma}(\mathscr{A})$, and, to begin with, suppose that the first element of $S(x)$ is $0 \in \Gamma$. From [**6**, Theorem 4.9] it follows that $x^{-1} \in E^{\Gamma}(\mathscr{W}(\Gamma))$, and from the proof of [**6**, Theorem 4.7], we have $S(x^{-1}) \subset \langle S(x) \rangle$. Thus $S(x^{-1}) \in \mathscr{A}$, and so $x^{-1} \in E^{\Gamma}(\mathscr{A})$. Now to deal with the general case, let $S(x)$ have first element γ. By (ii) and (iv), $\mathscr{A}$ contains a set with one element, so by (v), $\{-\gamma\} \in \mathscr{A}$. Let $y \in E^{\Gamma}(\mathscr{A})$ be defined by $y(-\gamma) = 1$, $y(\delta) = 0$ for $\delta \neq -\gamma$. Now 0 is the first element of $S(xy)$, and so

$$x^{-1} = y(xy)^{-1} \in E^{\Gamma}(\mathscr{A}).$$

This completes the proof of Theorem 1.

3. Some special cases. For any ordered abelian group Γ, let $\Delta(\Gamma)$ be the divisible envelope of Γ to which the order of Γ has been extended in the natural way. The family $\mathscr{W}(\Gamma)$ of well-ordered subsets of Γ is a field-family with respect to Γ, and $E^{\Delta(\Gamma)}(\mathscr{W}(\Gamma))$ is a subfield of $E^{\Delta(\Gamma)}(\mathscr{W}(\Delta(\Gamma)))$ isomorphic to $E^{\Gamma}(\mathscr{W}(\Gamma))$.

Let $\mathscr{X}(\Gamma)$ be the family of subsets of $\Delta(\Gamma)$ of the form $(1/d)A$ for all integers $d > 0$ and all $A \in \mathscr{W}(\Gamma)$. It is clear that $\mathscr{X}(\Gamma)$ is a field-family with respect to $\Delta(\Gamma)$. For any positive integer $p(\neq 0)$, let $\mathscr{Y}_p(\Gamma)$ be the family of all subsets of $\Delta(\Gamma)$ of the form

$$\bigcup_{i=0}^{\infty} p^{-i}B$$

for all $B \in \mathscr{X}(\Gamma)$. Let $\mathscr{Z}_p(\Gamma) = \mathscr{Y}_p(\Gamma) \cap \mathscr{W}(\Delta(\Gamma))$. Then $\mathscr{Z}_p(\Gamma)$ is a field-family with respect to Δ (although $\mathscr{Y}_p(\Gamma)$ is not).

We have the chain of fields

$$E^{\Gamma}(\mathscr{W}(\Gamma)) \cong E^{\Delta(\Gamma)}(\mathscr{W}(\Gamma)) \subset E^{\Delta(\Gamma)}(\mathscr{X}(\Gamma)) \subset E^{\Delta(\Gamma)}(\mathscr{Z}_p(\Gamma)) \subset E^{\Delta(\Gamma)}(\mathscr{W}(\Delta(\Gamma)));$$

In the particular case in which $E = k$ and $\Gamma = \mathbf{Z}$, we have the fields of §1. Thus

$$k((t)) \cong k^{\mathbf{Q}}(\mathscr{W}(\mathbf{Z})), \qquad K \cong k^{\mathbf{Q}}(\mathscr{X}(\mathbf{Z})) \qquad \text{and} \qquad L \cong k^{\mathbf{Q}}(\mathscr{X}_p(\mathbf{Z})),$$

where p is the characteristic of k.

4. Relative completeness. With the notation of §2, let $\mathscr{A}$ be a field-family with respect to Γ, and define a function $v: E^{\Gamma}(\mathscr{A}) \to \Gamma \cup \{\infty\}$ by setting $v(x)$ equal to the first element of $S(x)$ for $x \neq 0$, and by setting $v(0) = \infty$. Under the conventions that $\infty = \infty + \infty = \infty + \gamma > \gamma$ for all $\gamma \in \Gamma$, v is a valuation of the field $E^{\Gamma}(\mathscr{A})$, which we refer to as the *natural valuation*. We have the properties $v(xy) = v(x) + v(y)$, $v(x \pm y) \geqq \min\{v(x), v(y)\}$. The residue class field of $E^{\Gamma}(\mathscr{A})$ under v is E.

When a field with a valuation has the property that for each algebraic extension of the field there is just one way of extending the valuation, the field is said to have the *unique extension property*. If a field with a valuation has the property that Hensel's lemma (Lemma 3, below) is true over its valuation ring, then the field is said to be *relatively complete*. A field is relatively complete if and only if it has the unique extension property (see [**10**] and [**11**]).

It is well-known that $E^{\Gamma}(\mathscr{W}(\Gamma))$ is relatively complete for any Γ and E [**4**, Theorems 26, 27 and 12]. Since $k^{\mathbf{Q}}(\mathscr{X}(\mathbf{Z}))$ is algebraic over $k^{\mathbf{Q}}(\mathscr{W}(\mathbf{Z}))$, $k^{\mathbf{Q}}(\mathscr{X}(\mathbf{Z}))$ has the unique extension property. More generally, we have the following result.

THEOREM 2. *Let E be any field, Γ any ordered abelian group, and $\mathscr{A}$ any field-family with respect to Γ. Under the natural valuation, $E^{\Gamma}(\mathscr{A})$ is relatively complete.*

Proof. We prove that Hensel's lemma holds for $E^{\Gamma}(\mathscr{A})$. Let $B = \{x \in E^{\Gamma}(\mathscr{A}) : v(x) \geqq 0\}$ extend v to be a function defined on the polynomial ring $B[X]$ by putting

$$v\left(\sum_{i=0}^{n} a_i X^i\right) = \min_{i=0,\dots,n} v(a_i).$$

The extended function v has the properties

$$v(fg) \geqq v(f) + v(g) \tag{3}$$

and

$$v(f \pm g) \geqq \min\{v(f), v(g)\}. \tag{4}$$

Extend S to be a function from $B[X]$ to $\mathscr{A}$ by putting

$$S\left(\sum_{i=0}^{n} a_i X^i\right) = \bigcup_{i=0}^{n} S(a_i).$$

We now follow through the usual proof of Hensel's lemma [**8**], paying attention to properties of the function S. First we have a lemma which enables the induction to be carried through.

LEMMA 2. *Let $f, g, h, l, m \in B[X]$ be such that g is monic, $v(f) = v(g) = v(h) = 0$, $\partial^\circ g < \partial^\circ f$ $v(f - gh) = \gamma > 0$, $v(1 - lg - mh) = \delta > 0$. Let $Q = S(f) \cup S(g) \cup S(h) \cup S(l) \cup S(m)$. Then there exist $g', h' \in B[X]$ such that g' is monic, $v(g') = v(h') = 0$, $\partial^\circ g = \partial^\circ g'$, $v(g - g') \geqq \gamma > 0$, $v(h - h') \geqq \gamma > 0$, $v(f - g'h') \geqq \gamma + \min\{\gamma, \delta\}$, $v(1 - lg' - mh') \geqq \min\{\gamma, \delta\}$, and $Sg' \subset Q$, $Sh' \subset Q$.*

Proof. Since g is monic, we can divide $(f-gh)m$ by g and obtain $q, r \in B[X]$ with $\partial^\circ r < \partial^\circ g$ such that $(f-gh)m = gq+r$. From the way in which the coefficients of q arise, we see that $v(q) \geqq v(f-gh)+v(m) \geqq \gamma$. We can deduce from (4) that $v(r) \geqq \gamma$. Moreover $S(q) \subset Q$, and hence $S(r) \subset Q$. Take $g' = g+r$ and $h' = h+(1+q)l(f-gh)+qh$. From the equation

$$f-g'h' = (f-gh)(1-lg-mh)-rq(h+l(f-gh))-lm(f-gh)^2$$

it follows that g', h' have all the required properties. This completes the proof of the lemma.

LEMMA 3 (Hensel's lemma). *Assume the hypotheses of Lemma* 2. *Then there exist* g_0, h_0 *in* $B[X]$ *such that* $f = g_0 h_0$.

Proof. Let $\mathscr{T}$ be the family of triples (g_i, h_i, γ_i) such that g_i is monic, $\partial^\circ g_i = \partial^\circ g$ $v(g-g_i) > 0$, $v(h-h_i) > 0$, $v(1-lg_i-mh_i) > 0$, $v(f-g_i h_i) = \gamma_i$, $S(g_i) \subset Q$ and $S(h_i) \subset Q$. Define a partial order on $\mathscr{T}$, $(g_i, h_i, \gamma_i) < (g_j, h_j, \gamma_j)$, to mean $\gamma_i < \gamma_j$, $v(g_i-g_j) \geqq \gamma_i$, $v(h_i-h_j) \geqq \gamma_i$. $\mathscr{T}$ is non-empty. Any chain in $\mathscr{T}$ has an upper bound in $\mathscr{T}$, for if $\{(g_i, h_i, \gamma_i)\}$ is a chain (a totally ordered set) then we define $\bar{g}$ by $\bar{g}(X) = \sum \bar{a}_j X^j$, where $\bar{a}_j(\alpha) = a_{ij}(\alpha)$ provided $\alpha < \gamma_i$ for some γ_i in the chain (where $g_i(X) = \sum a_{ij} X^j$), and otherwise $\bar{a}_j(\alpha) = 0$. Define $\bar{h}$ similarly. With $\bar{\gamma} = v(f-\bar{g}\bar{h})$, it is clear that $\bar{\gamma} \geqq \gamma_i$ for each i. Then $(\bar{g}, \bar{h}, \bar{\gamma}) \in \mathscr{T}$. It is important that $S(\bar{g}) \subset Q$ and $S(\bar{h}) \subset Q$ by construction, and that $Q \in \mathscr{A}$.

From Zorn's lemma, $\mathscr{T}$ has a maximal element, say (g_0, h_0, γ_0). By Lemma 2, if $\gamma_0 \neq \infty$, then (g_0, h_0, γ_0) is not maximal. Hence $\gamma_0 = \infty$ and $f = g_0 h_0$. Lemma 3 is now proved.

Theorem 2 follows at once from Lemma 3.

5. Algebraically closed fields.

LEMMA 4. *Let k be an algebraically closed field of characteristic $p \neq 0$, let Δ be a divisible ordered abelian group, let $\mathscr{A}$ be a field-family with respect to Δ. Then any finite normal extension M of $k^\Delta(\mathscr{A})$ is of degree p^n (for some $n \in \mathbf{N}$), and contains a normal subextension of degree p over $k^\Delta(\mathscr{A})$ generated by a zero of the polynomial $X^p - X - a$, where $a \in k^\Delta(\mathscr{A})$ and $v(a) < 0$.*

Proof. Since $k^\Delta(\mathscr{A})$ has the unique extension property, the ramification theory for valuations (expounded in [**12**]) shows that the decomposition field of $M/k^\Delta(\mathscr{A})$ is $k^\Delta(\mathscr{A})$. Because k is algebraically closed, the inertia field is $k^\Delta(\mathscr{A})$, and because Δ is divisible the ramification field is also $k^\Delta(\mathscr{A})$. Hence $[M : k^\Delta(\mathscr{A})]$ is the degree of M over the ramification field, and is therefore a power of p. The existence of a normal subextension of degree p follows trivially from Sylow theory. By a well-known result [**3**, p. 98, ex. 4] this extension is generated by a zero of $X^p - X - a$ for some $a \in k$. For $v(a) \geqq 0$ it follows from Hensel's lemma that $X^p - X - a$ is reducible (in fact, has a linear factor). Hence $v(a) < 0$, and the lemma is proved.

THEOREM 3. *Let k be an algebraically closed field of characteristic $p \neq 0$, let Γ be an ordered abelian group, and let Δ be the divisible envelope of Γ. Then $k^\Delta(\mathscr{Z}_p(\Gamma))$ is algebraically closed.*

Proof. Suppose not. Then there exists $a \in k^\Delta(\mathscr{Z}_p(\Gamma))$ such that $v(a) < 0$ and $X^p - X - a$ is irreducible. We may write $a = a_1 + a_2$, where

$$S(a_1) = \{\gamma \in S(a) : \gamma < 0\}, \qquad S(a_2) = \{\gamma \in S(a) : \gamma \geqq 0\}.$$

As in the proof of Lemma 4, $X^p - X - a_2$ has a linear factor, so there exists c in $k^\Delta(\mathscr{X}_p(\Gamma))$ such that $c^p - c = a_2$. We shall next show that there exists b in $k^\Delta(\mathscr{X}_p(\Gamma))$ such that $b^p - b = a_1$. Let $b \in k^\Delta$ be defined by

$$b(\alpha) = \sum_{i=1}^{\infty} (a_1(p^i\alpha))^{p^{-i}}$$

for each $\alpha \in \Delta$. The sum is finite for each $\alpha < 0$, because $(p^i\alpha)$ is an infinite strictly descending sequence, and the support of a_1 is well-ordered. For $\alpha \geqq 0$, $b(\alpha) = 0$ since $S(a_1) \cap \Gamma^+ = \phi$.

Since $S(a_1)$ is well-ordered and $S(a_1) \subset \{\gamma : \gamma < 0\}$,

$$\bigcup_{i=1}^{\infty} p^{-i} S(a_1)$$

is also well-ordered. This may be seen as follows. Let, if possible,

$$p^{-r_1}\gamma_1 > p^{-r_2}\gamma_2 > \ldots > p^{-r_i}\gamma_i > \ldots$$

be an infinite strictly descending sequence in

$$\bigcup_{i=1}^{\infty} p^{-i} S(a_1),$$

where each $\gamma_i \in S(a_1)$. By the well-ordering of $S(a_1)$, there is an infinite subsequence of $(p^{-r_i}\gamma_i)$ in which (γ_i) is nondecreasing. Because each $\gamma_i < 0$ on this infinite subsequence, the corresponding infinite sequence (p^{-r_i}) must be strictly increasing. As the r_i's are all positive integers, this is a contradiction. Hence

$$\bigcup_{i=1}^{\infty} p^{-i} S(a_1)$$

is well-ordered.

Because, by the definition of b,

$$S(b) \subset \bigcup_{i=1}^{\infty} p^{-i} S(a_1),$$

it follows that $b \in k^\Delta(\mathscr{X}_p(\Gamma))$. Moreover $b^p - b = a_1$, since

$$(b^p - b)(\alpha) = \sum_{i=1}^{\infty} a_1(p^{i-1}\alpha)^{p^{-(i-1)}} - \sum_{i=1}^{\infty} a_1(p^i\alpha)^{p^{-i}} = a_1(\alpha).$$

We now have $(b+c)^p - (b+c) = a_1 + a_2 = a$, so that $X^p - X - a$ is reducible over $k^\Delta(\mathscr{X}_p(\Gamma))$. It follows that $k^\Delta(\mathscr{X}_p(\Gamma))$ has no proper algebraic extension.

Corollary. *The field L described in* §1 *is algebraically closed.*

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