A CONFIGURATION OF LINES IN THREE DIMENSIONS

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1. Introduction

In 1849, Cayley and Salmon discovered that a general cubic surface in projective space of three dimensions over the complex numbers has twenty-seven lines on it. They remarked that all the properties of the twenty-seven lines would not become apparent until a better notation than they had given was found. This notation was discovered by Schläfli in 1858 in the double-six theorem (henceforth referred to as ($\mathscr{D}$)): *given five skew lines $a_1, \ldots, a_5$ with a single transversal b_6 such that no four of the a_i lie in a regulus, the four a_i excluding a_j have a second transversal b_j and the five lines $b_1, \ldots, b_5$ thus obtained have a transversal a_6—the completing line of the double-six.* The other fifteen lines of the cubic surface are then $c_{ij} = a_ib_j \cap a_jb_i$, where a_ib_j is the plane containing a_i and b_j.

In 1898, Grace gave the following extension of the double-six theorem: *given six skew lines $c_1, \ldots, c_6$ with a transversal α such that no four of the c_i lie in a regulus and no five have a further transversal, then α and the set of five c_i excluding c_j determine a double-six with completing line d_j and the six lines $d_1, \ldots, d_6$ have a transversal β—the Grace line.*

It is natural to ask whether Grace's theorem may be similarly extended and whether there is an infinite sequence of such theorems. Thus the first question is: if, from seven skew lines $A_1, \ldots, A_7$ with a transversal Δ, the set of six A_i excluding A_j determine the Grace line Γ_j, do the seven Grace lines $\Gamma_1, \ldots, \Gamma_7$ have a transversal?

In (9), this question was answered in the negative by the following method. If the theorem is true over the complex field, say, the lines $\Gamma_1, \ldots, \Gamma_7$ can be obtained from the lines $A_1, \ldots, A_7$ by solving certain sets of linear equations. If the line coordinates of A_i are $A_i^{(1)}, \ldots, A_i^{(6)}$, then the line coordinates of $\Gamma_1, \ldots, \Gamma_7$ will lie in the polynomial ring $\boldsymbol{Z}[\{A_i^{(j)}\}]$, where $\boldsymbol{Z}$ is the ring of integers. Let $\Gamma_1, \ldots, \Gamma_5$ lie in the linear complex W. Then $\Gamma_1, \ldots, \Gamma_7$ have a transversal if W is special and if both Γ_6 and Γ_7 lie in W. Thus three identities must be satisfied in the ring $\boldsymbol{Z}[\{A_i^{(j)}\}]$. Therefore, if the theorem is true over the complex field, the theorem is true over any field for which the lines $\Gamma_1, \ldots, \Gamma_7$ exist and which is a homomorphic image of $\boldsymbol{Z}$. In (9), the field $GF(31)$ was chosen and, with the aid of a computer, a set of $A_1, \ldots, A_7$ found which produced the lines $\Gamma_1, \ldots, \Gamma_7$. They did not have a transversal. So the question

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was negatively answered. The same result has since been found independently by Longuet-Higgins (11), who gave two examples over the real field. In both cases, the seven lines $\Gamma_1, \ldots, \Gamma_7$ belonged to a (non-special) linear complex, confirming in these cases a conjecture due to Babbage.

Two questions now arise. Firstly, can this result be proved without using a computer? Secondly, what is the complete configuration obtained from Δ and $A_1, \ldots, A_7$? This paper answers the second question.

Grace (6) originally proved his theorem by considering six hyperspheres through a point in four dimensions. Every four have a second point in common. Thus each set of five hyperspheres produces five points. These five points lie on a hypersphere. So, from six hyperspheres, six sets of five can be formed and from each set of five a new hypersphere is obtained. It was shown that the six new hyperspheres have a point in common.

This result was transformed to one on lines and linear complexes in three dimensions, and this result in turn was specialised by using special linear complexes and by involving $(\mathscr{D})$ to give the required result.

Brown (3) considered an extension of Grace's theorem in four dimensions by commencing with seven hyperplanes through a point. This transforms into a theorem in three dimensions about seven linear complexes with a line in common, but says nothing about seven lines with a transversal. Nevertheless, we shall show that the analogy does hold. The review by Coxeter (4) of Brown's paper was extremely helpful in suggesting a revised notation similar to that below, as well as in identifying the group of the configuration as a familiar one.

2. Notation and preliminaries

The geometry throughout takes place in a projective space of three dimensions over an arbitrary field K with the single condition that K is large enough for the configuration to exist: it was shown in (7) and (8) that the double-six exists for all K except $GF(q)$ with $q = 2$, 3 or 5 and that, for Grace's figure to exist, $q \geqq 9$.

Indices written on the same level will be interchangeable; e.g. $\Gamma^i_{jkl} = \Gamma^i_{pqr}$ where $p\,q\,r$ is any permutation of $j\,k\,l$.

$\mathscr{R}(l_1 l_2 \ldots l_n)$ indicates that the lines $l_1, l_2, \ldots, l_n$ lie in a regulus.

To establish the configuration, two theorems will be required—$(\mathscr{D})$ and Kubota's theorem (henceforth referred to as $(\mathscr{K})$). Grace's theorem was proved by Wren (14) in the following way: beginning with the lines $c_1, \ldots, c_6$ with transversal α, the four c_i excluding c_j and c_k have a further transversal α_{jk} and the five lines α_{ij} with $j = 1, \ldots, 6, j \neq i$, have a transversal d_i by $(\mathscr{D})$. It was then shown that the six lines $c_1, c_2, d_3, d_4, d_5, d_6$ have a transversal β_{12}. By constructing double-sixes from the lines d_i and β_{ij} similar to those with the lines c_i and α_{ij}, it was shown that the six lines d_i have a transversal β. The existence of the lines β_{ij} was shown differently by Kubota (10) as follows. In the above situation, *the four reguli* $(\alpha_{12}\alpha_{13}\alpha_{14})$, $(\alpha_{12}\alpha_{23}\alpha_{24})$, $(\alpha_{13}\alpha_{23}\alpha_{34})$, $(\alpha_{14}, \alpha_{24}, \alpha_{34})$ *have a common line* β_{56}: this theorem is $(\mathscr{K})$. By mapping the lines on to the Klein quadric in five dimensions and projecting on to a plane, it is equivalent to the result that the four circumcircles of the four triangles obtained by omitting in turn each of four lines in the plane have a common point (8).

The configuration to be constructed consists of two types of lines, which will be called Greek (g-lines) and Latin (l-lines) and denoted accordingly. The first table below gives in stages the list of results either assumed or to be proved in § 3. The second table gives the intersections between the g-lines and the l-lines as they are obtained. Here, the right-hand column lists the number of l-lines that the given g-line meets. The third table lists the numbers of l-lines and g-lines. In all, the g-lines and l-lines form a tactical configuration (576, 7; 56, 72), in which each of the 576 g-lines meets seven l-lines and each of the 56 l-lines meets 72 g-lines.

Stages of the Configuration

S_1: $A_1, A_2, A_3, A_4, A_5, A_6, A_7$ have the transversal Δ

S_2: A_1, A_2, A_3, A_4 also have the transversal Δ_{567}

S_3: $\Delta_{123}, \Delta_{124}, \Delta_{125}, \Delta_{126}, \Delta_{127}$ have the transversal A_{12}

S_4: $A_{12}, A_{13}, A_{14}, A_{15}, A_6, A_7$ have the transversal Γ^1_{67}

S_5: $A_{12}, A_{13}, A_{14}, A_{15}, A_{16}, A_{17}$ have the transversal Γ_1

S_6: $A_1, A_{56}, A_{57}, A_{67}$ have the transversals Δ_{567} and Δ^1_{567}

S_7: $\Delta^1_{ijk}, \Delta^2_{ijk}$ $(\forall\ i, j, k \neq 1, 2)$ have the transversal B_{12}

S_8: $A_{12}, A_{13}, A_{14}, B_{56}, B_{57}, B_{67}$ have the transversal Γ^1_{567}

S_9: $A_1, B_{12}, B_{13}, B_{14}, B_{15}, B_{16}, B_{17}$ have the transversal Δ_1

S_{10}: $A_{23}, B_{14}, B_{15}, B_{16}, B_{17}$ have the transversal Δ^1_{23}

S_{11}: Γ^1_{ijk} $(\forall\ i, j, k \neq 1)$ has the transversal B_1

S_{12}: B_1 meets Γ_1

S_{13}: $B_1, B_2, B_3, B_4, B_{56}, B_{57}, B_{67}$ have the transversal Γ_{567}

S_{14}: $B_1, B_2, B_3, B_4, B_5, B_6, B_7$ have the transversal Γ

Intersections of the Configuration

S_1:	Δ	meets	A_i	7
S_2, S_3:	Δ_{ijk}	meets	A_l, A_{ij}	7
S_4:	Γ^i_{jk}	meets	A_j, A_{il}	6
S_5:	Γ_i	meets	A_{ij}	6
S_6:	Δ^i_{jkl}	meets	A_i, A_{jk}	4
S_7:	Γ^i_{jk}	meets	B_{jk}	1
S_7:	Δ^i_{jkl}	meets	B_{im}	3
S_8:	Γ^i_{jkl}	meets	A_{im}, B_{jk}	6
S_9:	Δ_i	meets	A_i, B_{ij}	7
S_{10}:	Δ^i_{jk}	meets	A_{jk}, B_{il}	5
S_{11}:	Γ^i_{jkl}	meets	B_i	1
S_{11}:	Δ^i_{jk}	meets	B_j	2
S_{12}:	Γ_i	meets	B_i	1
S_{13}:	Γ_{ijk}	meets	B_l, B_{ij}	7
S_{14}:	Γ	meets	B_i	7

Lines of the Configuration

Latin		Greek	
		Δ	1
7	A_i		
		Δ_{ijk}	35
21	A_{ij}		
		Γ^i_{jk}	105
		Γ_i	7
		Δ^i_{jkl}	140
21	B_{ij}		
		Γ^i_{jkl}	140
		Δ_i	7
		Δ^i_{jk}	105
7	B_i		
		Γ_{ijk}	35
		Γ	1
56			576

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To set the picture, there follow some elementary results on line geometry.

(1) Four skew lines lie in a regulus if and only if they are linearly dependent.

(2) Four skew lines have only one transversal if and only if the four skew lines are linearly independent and the five lines are linearly dependent.

(3) Five skew lines, four of which have just two transversals, have two transversals if and only if they are linearly dependent.

(4) Six skew lines lie in a linear complex if and only if they are linearly dependent.

(5) Five skew lines have exactly one transversal if and only if the six lines are linearly dependent but the five are not.

(6) Five skew lines have exactly one transversal and are part of a double-six if and only if the six are linearly dependent but no five of the six are.

(7) A necessary condition that six skew lines with a transversal lead to Grace's theorem is that every five of the seven lines are linearly independent (this condition is not quite sufficient (**8**), p. 357).

(8) A necessary condition that seven skew lines with a transversal lead to Brown's configuration is that every five of the eight lines are linearly independent.

3. Construction of the configuration

S_1. Let us begin with the seven lines A_i with transversal Δ satisfying condition (8) above.

S_2. A_1, A_2, A_3, A_4 have the further transversal Δ_{567}.

S_3. By ($\mathscr{D}$), $\Delta_{123}, \Delta_{124}, \Delta_{125}, \Delta_{126}, \Delta_{127}$ have a transversal A_{12}. Thus there are 21 double-sixes like

$$\begin{matrix} A_{12} & A_3 & A_4 & A_5 & A_6 & A_7 \\ \Delta & \Delta_{123} & \Delta_{124} & \Delta_{125} & \Delta_{126} & \Delta_{127}. \end{matrix}$$

S_4. Apply ($\mathscr{K}$) to the case $c_1 = A_2$, $c_2 = A_3$, $c_3 = A_4$, $c_4 = A_5$, $c_5 = A_6$, $c_6 = A_7$. Then $\alpha_{12} = \Delta_{123}, \alpha_{13} = \Delta_{124}, \alpha_{14} = \Delta_{125}, \alpha_{23} = \Delta_{134}, \alpha_{24} = \Delta_{135}$, $\alpha_{34} = \Delta_{145}$. So the reguli in the left-hand column below have a common line: their complementary reguli are written on the right.

$$\begin{matrix} \Delta_{123} & \Delta_{124} & \Delta_{125} & \qquad & A_6 & A_7 & A_{12} \\ \Delta_{132} & \Delta_{134} & \Delta_{135} & & A_6 & A_7 & A_{13} \\ \Delta_{142} & \Delta_{143} & \Delta_{145} & & A_6 & A_7 & A_{14} \\ \Delta_{152} & \Delta_{153} & \Delta_{154} & & A_6 & A_7 & A_{15} \end{matrix}$$

Let this common line be Γ^1_{67}; it therefore meets $A_{12}, A_{13}, A_{14}, A_{15}, A_6, A_7$.

S_5. **Theorem.** *All fifteen lines Γ^1_{ij} are distinct.*

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Proof. If $\Gamma_{67}^1 = \Gamma_{45}^1$, it meets A_{12}, A_4, A_5, A_6, A_7, which, by ($\mathscr{D}$), have only the one transversal Δ_{123}; so $\Gamma_{67}^1 = \Delta_{123}$. Then, by ($\mathscr{K}$), we have

$$\mathscr{R}(\Delta_{123}\Delta_{142}\Delta_{143}\Delta_{145})$$

which implies $\mathscr{R}(A_{14}A_5A_6A_7)$, contradicting the existence of a double-six such as the above. So $\Gamma_{67}^1 \neq \Gamma_{45}^1$.

If $\Gamma_{67}^1 = \Gamma_{57}^1$, it meets A_5, A_6, A_7, A_{12}, A_{13}, A_{14}. But the three sets of four $\{A_5, A_6, A_7, A_{12}\}$, $\{A_5, A_6, A_7, A_{13}\}$, $\{A_5, A_6, A_7, A_{14}\}$ have just the respective pairs of transversals $\{\Delta_{123}, \Delta_{124}\}$, $\{\Delta_{123}, \Delta_{134}\}$, $\{\Delta_{124}, \Delta_{134}\}$. As all Δ_{ijk} are distinct, the six lines have no transversal. So $\Gamma_{67}^1 \neq \Gamma_{57}^1$. So all fifteen lines Γ_{ij}^1 are distinct.

At this stage, in order that the figure does not degenerate, it must be assumed that any set of six lines like A_{12}, A_{13}, A_{14}, A_{15}, A_{16}, A_{17} are skew: in fact, the six lines are all skew or all concurrent, (8), p. 354.

It was previously shown that the four lines A_5, A_6, A_7, A_{12} do not lie in a regulus.

Theorem. *No four of the lines A_6, A_7, A_{12}, A_{13}, A_{14}, A_{15} lie in a regulus.*

Proof.

$$\begin{aligned}
\mathscr{R}(A_6, A_7, A_{12}, A_{13}) &\Rightarrow \mathscr{R}(\Gamma_{67}^1, \Delta_{123}, \Delta_{124}, \Delta_{134}) \\
&\Rightarrow \mathscr{R}(A_5, A_6, A_7, A_{12}). \\
\mathscr{R}(A_7, A_{12}, A_{13}, A_{14}) &\Rightarrow \mathscr{R}(\Gamma_{67}^1, \Delta_{123}, \Delta_{124}, \Delta_{134}). \\
\mathscr{R}(A_{12}, A_{13}, A_{14}, A_{15}) &\Rightarrow \mathscr{R}(\Gamma_{26}^1, \Gamma_{27}^1, \Gamma_{36}^1, \Gamma_{67}^1) \\
&\Rightarrow \mathscr{R}(A_6, A_7, A_{12}, A_{13}).
\end{aligned}$$

So, in all cases, there is a contradiction.

There are $7 \times \binom{6}{3} = 140$ double-sixes of the type

$$\begin{matrix} A_2 & A_3 & A_4 & A_{15} & A_{16} & A_{17} \\ \Gamma_{34}^1 & \Gamma_{24}^1 & \Gamma_{23}^1 & \Delta_{167} & \Delta_{157} & \Delta_{156} \end{matrix}$$

and double-sixes such as

$$\begin{matrix} A_2 & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} \\ \Gamma_1 & \Gamma_{23}^1 & \Gamma_{24}^1 & \Gamma_{25}^1 & \Gamma_{26}^1 & \Gamma_{27}^1 \end{matrix}$$

and

$$\begin{matrix} A_3 & A_{12} & A_{14} & A_{15} & A_{16} & A_{17} \\ \Gamma_1^* & \Gamma_{23}^1 & \Gamma_{34}^1 & \Gamma_{35}^1 & \Gamma_{36}^1 & \Gamma_{37}^1. \end{matrix}$$

Theorem. $\Gamma_1^* = \Gamma_1$.

Proof. As A_{14}, A_{15}, A_{16}, A_{17} lie in one half of a double-six and meet Γ_{23}^1, they have just one further transversal. So $\Gamma_1^* = \Gamma_1$.

There are then 42 double-sixes of the above type.

The lines Γ_i, $i = 1, \ldots, 7$, are the Grace lines.

S_6. A_1, A_{56}, A_{57}, A_{67} have the transversal Δ_{567}. The four lines do not lie in a regulus, since

$$\mathscr{R}(A_1 A_{56} A_{57} A_{67}) \Rightarrow \mathscr{R}(\Gamma^5_{12}\Gamma^5_{13}\Gamma^5_{14}\Gamma^6_{12}\Gamma^6_{13}\Gamma^6_{14}\Gamma^7_{12}\Gamma^7_{13}\Gamma^7_{14}\Delta_{567})$$
$$\Rightarrow \mathscr{R}(A_1 A_2 A_{56} A_{57}),$$

contradicting a theorem in S_5. So A_1, A_{56}, A_{57}, A_{67} have a second transversal Δ^1_{567}. (It is here assumed that the degeneracy $\Gamma^6_{12} = \Gamma^7_{12}$ has not occurred: if $\Gamma^6_{12} = \Gamma^7_{12}$, then this line meets A_1, A_2, A_{36}, A_{37} and is Γ^3_{12} or Δ_{367}. If $\Gamma^6_{12} = \Delta_{367}$, then Δ_{367} meets A_{46}, which is impossible as they are polar in a double-six. Hence $\Gamma^6_{12} = \Gamma^3_{12}$ and all Γ^i_{12} are the same. In general, as will be shown below, the five lines Γ^i_{12} lie in a regulus.)

S_7. Now consider Δ_{567} meeting the five lines A_1, A_2, A_{56}, A_{57}, A_{67}. Every four have a unique second transversal. So, by $(\mathscr{D})$, we have the double-six

$$\begin{matrix} B^{12}_{567} & A_1 & A_2 & A_{67} & A_{57} & A_{56} \\ \Delta_{567} & \Delta^2_{567} & \Delta^1_{567} & \Gamma^5_{12} & \Gamma^6_{12} & \Gamma^7_{12}. \end{matrix}$$

Theorem. *The ten lines B^{12}_{ijk} are identical.*

Proof. Apply $(\mathscr{K})$ to the case $\alpha = \Delta_{567}$, $c_1 = A_1$, $c_2 = A_3$, $c_3 = A_{56}$, $c_4 = A_{57}$, $c_5 = A_2$, $c_6 = A_{67}$. Then

$$\alpha_{12} = \Delta^2_{567}, \alpha_{13} = \Gamma^7_{23}, \alpha_{14} = \Gamma^6_{23}, \alpha_{23} = \Gamma^7_{12}, \alpha_{24} = \Gamma^6_{12}, \alpha_{34} = \Delta_{467}.$$

Thus we have

$$\begin{matrix} \Delta^2_{567} & \Gamma^7_{23} & \Gamma^6_{23} & \qquad & A_2 & A_{67} & B^{23}_{567} \\ \Delta^2_{567} & \Gamma^7_{12} & \Gamma^6_{12} & & A_2 & A_{67} & B^{12}_{567} \\ \Gamma^7_{23} & \Gamma^7_{12} & \Delta_{467} & & A_2 & A_{67} & A_{47} \\ \Gamma^6_{23} & \Gamma^6_{12} & \Delta_{467} & & A_2 & A_{67} & A_{46}. \end{matrix}$$

The left-hand reguli have a common line. The other transversal to A_2, A_{67}, A_{47}, A_{46} apart from Δ_{467} is Δ^2_{467}: this is then the required line. So Δ^2_{467} meets B^{12}_{567}; similarly Δ^1_{467} meets B^{12}_{567}. So B^{12}_{567} meets Δ^1_{467}, Δ^2_{467}, Γ^6_{12}, Γ^7_{12}.

Consider now the double-six

$$\begin{matrix} B^{12}_{467} & A_1 & A_2 & A_{67} & A_{47} & A_{46} \\ \Delta_{467} & \Delta^2_{467} & \Delta^1_{467} & \Gamma^4_{12} & \Gamma^6_{12} & \Gamma^7_{12}. \end{matrix}$$

Therefore Δ^1_{467}, Δ^2_{467}, Γ^6_{12}, Γ^7_{12} have the three transversals B^{12}_{567}, B^{12}_{467}, A_{67}, where $A_{67} \neq B^{12}_{467}$ and $A_{67} \neq B^{12}_{567}$. As the above four lines do not belong to a regulus, $B^{12}_{467} = B^{12}_{567}$. Thus all ten lines B^{12}_{ijk} are identical.

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Write B_{12} for B_{ijk}^{12}. There are $\binom{7}{2} \times \binom{5}{3} = 210$ double-sixes of the form

$$\begin{matrix} B_{12} & A_1 & A_2 & A_{45} & A_{35} & A_{34} \\ \Delta_{345} & \Delta_{345}^2 & \Delta_{345}^1 & \Gamma_{12}^3 & \Gamma_{12}^4 & \Gamma_{12}^5 \end{matrix}$$

and all twenty lines Δ_{ijk}^1, Δ_{ijk}^2 $(i, j, k \neq 1, 2)$ meet B_{12}. Further $\mathscr{R}(A_1A_2B_{12})$ implies $\mathscr{R}(\Gamma_{12}^3\Gamma_{12}^4\Gamma_{12}^5\Gamma_{12}^6\Gamma_{12}^7)$.

S_8. Now consider A_5 meeting Δ_{134}^5, Δ_{124}^5, Δ_{123}^5, Γ_{56}^1, Γ_{57}^1. No four lie in a regulus since $\mathscr{R}(\Delta_{123}^5\Delta_{124}^5\Gamma_{56}^1\Gamma_{57}^1) \Rightarrow \mathscr{R}(A_5A_{12}A_{13}A_{14})$ and

$$\mathscr{R}(\Delta_{123}^5\Delta_{124}^5\Delta_{134}^5\Gamma_{56}^1) \Rightarrow \mathscr{R}(A_5A_{12}A_{13}A_{14})$$

also.

Therefore, by $(\mathscr{D})$, we have the double-sixes

$$\begin{matrix} A_{12} & A_{13} & A_{14} & B_{56} & B_{57} & A_5 \\ \Delta_{134}^5 & \Delta_{124}^5 & \Delta_{123}^5 & \Gamma_{57}^1 & \Gamma_{56}^1 & \Gamma_{567}^1 \end{matrix}$$

and

$$\begin{matrix} A_{12} & A_{13} & A_{14} & B_{56} & B_{67} & A_6 \\ \Delta_{134}^6 & \Delta_{124}^6 & \Delta_{123}^6 & \Gamma_{67}^1 & \Gamma_{56}^1 & \Gamma_{567}^{1*}. \end{matrix}$$

Theorem. $\Gamma_{567}^{1*} = \Gamma_{567}^1$.

Proof. As A_{12}, A_{13}, A_{14}, B_{56} lie in one half of a double-six and meet Γ_{56}^1, they have just one further transversal. So $\Gamma_{567}^{1*} = \Gamma_{567}^1$.

Γ_{567}^1 meets A_{12}, A_{13}, A_{14}, B_{56}, B_{57}, B_{67} and there are $7 \times \binom{6}{3} \times 3 = 420$ double-sixes of the above type.

S_9. Consider the six lines A_1, A_{23}, B_{14}, B_{15}, B_{16}, B_{17}. It will ultimately be shown that they form one half of a double-six.

Theorem. *No four of A_1, A_{23}, B_{14}, B_{15}, B_{16}, B_{17} lie in a regulus.*

Proof. Similarly to the one above, the following six lines form one-half of a double-six and so no four lie in a regulus: A_{23}, A_{26}, A_{27}, B_{14}, B_{15}, A_1.

$$\mathscr{R}(A_{23}B_{14}B_{15}B_{16}) \text{ implies that } \Gamma_{156}^2 \text{ meets } B_{14}.$$

Hence

$$\mathscr{R}(\Gamma_{145}^2\Gamma_{156}^2\Delta_{237}^1) \Rightarrow \mathscr{R}(A_{23}A_{27}B_{14}B_{15}).$$

$$\mathscr{R}(B_{14}B_{15}B_{16}B_{17}) \Rightarrow \mathscr{R}(\Delta_{234}^1\Delta_{235}^1\Delta_{236}^1\Delta_{237}^1)$$

$$\Rightarrow \mathscr{R}(A_1A_{23}B_{14}B_{15}).$$

$$\mathscr{R}(A_1B_{14}B_{15}B_{16}) \Rightarrow \mathscr{R}(\Delta_{234}^1\Delta_{235}^1\Delta_{236}^1\Delta_{237}^1).$$

In each case, there is a contradiction. So no four of the given six lines lie in a regulus.

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Δ^1_{234} meets A_1, A_{23}, A_{24}, A_{34}, B_{15}, B_{16}, B_{17}. Let the second transversal of A_1, B_{15}, B_{16}, B_{17} be Δ_1.

Theorem. Δ_1 *meets* B_{12}, B_{13}, B_{14}.

Proof. Apply ($\mathscr{K}$) to the case $\alpha = \Delta^1_{234}$, $c_1 = A_{24}$, $c_2 = A_{34}$, $c_3 = B_{15}$, $c_4 = B_{16}$, $c_5 = B_{17}$, $c_6 = A_1$. Then

$$\alpha_{12} = \Delta_1, \alpha_{13} = \Delta^1_{345}, \alpha_{14} = \Delta^1_{346}, \alpha_{23} = \Delta^1_{245}, \alpha_{24} = \Delta^1_{246}, \alpha_{34} = \Gamma^4_{17}.$$

Thus we have

$$\begin{array}{ccc} \Delta_1 & \Delta^1_{345} & \Delta^1_{346} \\ \Delta_1 & \Delta^1_{245} & \Delta^1_{246} \\ \Delta^1_{345} & \Delta^1_{245} & \Gamma^4_{17} \\ \Delta^1_{346} & \Delta^1_{246} & \Gamma^4_{17} \end{array} \qquad \begin{array}{ccc} A_1 & B_{17} & X \\ A_1 & B_{17} & Y \\ A_1 & B_{17} & A_{45} \\ A_1 & B_{17} & A_{46}. \end{array}$$

The reguli on the left all have a line Z in common, which meets all the lines on the right. But the only transversal of A_1, B_{17}, A_{45}, A_{46} other than Γ^4_{17} is Δ^1_{456}. So Z is Δ^1_{456}. Hence $\mathscr{R}(\Delta_1\Delta^1_{345}\Delta^1_{346}\Delta^1_{456})$. Hence B_{12}, which meets the last three lines meets Δ_1. Also $\mathscr{R}(\Delta_1\Delta^1_{245}\Delta^1_{246}\Delta^1_{456})$; B_{13} meets the last three lines and therefore Δ_1. Similarly, by putting $c_1 = A_{23}$ above, it may be shown that B_{14} meets Δ_1.

Therefore it has been shown that A_1, B_{12}, B_{13}, B_{14}, B_{15}, B_{16}, B_{17} have the transversal Δ_1.

S_{10}. Now consider Δ_1 meeting A_1, B_{14}, B_{15}, B_{16}, B_{17}. No four of the five lines lie in a regulus. So, by ($\mathscr{D}$), the double-six may be completed:

$$\begin{array}{cccccc} A_{23} & A_1 & B_{14} & B_{15} & B_{16} & B_{17} \\ \Delta_1 & \Delta^1_{23} & \Delta^1_{234} & \Delta^1_{235} & \Delta^1_{236} & \Delta^1_{237}. \end{array}$$

There are $7 \times \binom{6}{2} = 105$ double-sixes of this type and 105 lines Δ^i_{jk}.

S_{11}. Consider the construction of Grace's figure on Δ_1 and B_{12}, B_{13}, B_{14}, B_{15}, B_{16}, B_{17}. This gives double-sixes like

$$\begin{array}{cccccc} B^2_1 & B_{13} & B_{14} & B_{15} & B_{16} & B_{17} \\ \Delta_1 & \Delta^1_{23} & \Delta^1_{24} & \Delta^1_{25} & \Delta^1_{26} & \Delta^1_{27}. \end{array}$$

Theorem. *All six lines* B^7_i $(i \neq 7)$ *are the same.*

Proof. Apply ($\mathscr{K}$) to the case $\alpha = \Delta_1$, $c_1 = A_1$, $c_2 = B_{12}$, $c_3 = B_{13}$, $c_4 = B_{14}$, $c_5 = B_{15}$, $c_6 = B_{16}$. Then

$$\alpha_{12} = \Delta^1_{27}, \alpha_{13} = \Delta^1_{37}, \alpha_{14} = \Delta^1_{47}, \alpha_{23} = \Delta^1_{237}, \alpha_{24} = \Delta^1_{247}, \alpha_{34} = \Delta^1_{347}.$$

Then we have

$$\begin{array}{ccccccc} \Delta^1_{27} & \Delta^1_{37} & \Delta^1_{47} & \qquad & B_{15} & B_{16} & B^7_1 \\ \Delta^1_{27} & \Delta^1_{237} & \Delta^1_{247} & & B_{15} & B_{16} & A_{27} \\ \Delta^1_{37} & \Delta^1_{237} & \Delta^1_{347} & & B_{15} & B_{16} & A_{37} \\ \Delta^1_{47} & \Delta^1_{247} & \Delta^1_{347} & & B_{15} & B_{16} & A_{47}. \end{array}$$

The line common to the four reguli on the left is the second transversal other than Δ^1_{347} of B_{15}, B_{16}, A_{37}, A_{47} and is therefore Γ^7_{156}. So Γ^7_{156} meets B^7_1 and similarly B^7_5, B^7_6. In the same way, B^7_1 and B^7_6 both meet Γ_{126}, Γ_{136}, Γ_{146}. If $B^7_1 \neq B^7_6$, then

$$\mathscr{R}(B^7_1 B^7_6 B_{16}) \Rightarrow \mathscr{R}(\Gamma^7_{126}\Gamma^7_{136}\Gamma^7_{146}\Gamma^7_{156}) \Rightarrow \mathscr{R}(A_{27}A_{37}A_{47}A_{57}),$$

which is a contradiction. So $B^7_1 = B^7_6$ and similarly all six lines B^7_i are the same.

Write B_7 for B^7_i. Thus there are seven lines B_i and each B_i meets the twenty lines Γ^i_{jkl}. Further, Δ^i_{jk} meets B_j. There are 42 double-sixes like

$$\begin{array}{cccccc} B_2 & B_{13} & B_{14} & B_{15} & B_{16} & B_{17} \\ \Delta_1 & \Delta^1_{23} & \Delta^1_{24} & \Delta^1_{25} & \Delta^1_{26} & \Delta^1_{27}. \end{array}$$

S_{12}. Now, constructing the double-six on Γ^1_{23} and B_{23}, A_{14}, A_{15}, A_{16}, A_{17}, we obtain

$$\begin{array}{cccccc} B_{23} & A_{14} & A_{15} & A_{16} & A_{17} & B_1 \\ \Gamma_1 & \Gamma^1_{234} & \Gamma^1_{235} & \Gamma^1_{236} & \Gamma^1_{237} & \Gamma^1_{23}. \end{array}$$

So B_1 meets Γ_1 giving the seventh line apart from the six lines A_{1i} $(i = 2, \ldots, 7)$ to meet Γ_1.

S_{13}. Continuing the construction of Grace's figure on Δ_1 from S_{11}, apply $(\mathscr{K})$ to the case $\alpha = \Delta_1$, $c_1 = B_{12}$, $c_2 = B_{13}$, $c_3 = B_{14}$, $c_4 = B_{15}$, $c_5 = B_{16}$, $c_6 = B_{17}$. Then

$$\alpha_{12} = \Delta^1_{23},\ \alpha_{13} = \Delta^1_{24},\ \alpha_{14} = \Delta^1_{25},\ \alpha_{23} = \Delta^1_{34},\ \alpha_{24} = \Delta^1_{35},\ \alpha_{34} = \Delta^1_{45}.$$

Therefore, we have

$$\begin{array}{ccccccc} \Delta^1_{23} & \Delta^1_{24} & \Delta^1_{25} & \qquad & B_{16} & B_{17} & B_2 \\ \Delta^1_{23} & \Delta^1_{34} & \Delta^1_{35} & & B_{16} & B_{17} & B_3 \\ \Delta^1_{24} & \Delta^1_{34} & \Delta^1_{45} & & B_{16} & B_{17} & B_4 \\ \Delta^1_{25} & \Delta^1_{35} & \Delta^1_{45} & & B_{16} & B_{17} & B_5. \end{array}$$

The reguli on the left have a common line γ^1_{67}, which meets B_{16}, B_{17}, B_2, B_3, B_4, B_5.

No five of the lines B_1, B_2, B_3, B_4, B_{67} lie in a regulus, since

$$\mathscr{R}(B_1B_2B_3B_4) \Rightarrow \mathscr{R}(\gamma^7_{26}\gamma^7_{36}\gamma^7_{46}\gamma^7_{56}) \Rightarrow \mathscr{R}(B_1B_2B_3B_{67})$$

and since

$$\mathscr{R}(B_1B_2B_3B_{67}) \Rightarrow \mathscr{R}(\Delta^6_{12}\Delta^6_{13}\Delta^6_{23}\Delta^7_{12}\Delta^7_{13}\Delta^7_{23}) \Rightarrow \mathscr{R}(B_1B_{47}B_{57}B_{67}),$$

contradicting the existence of a double-six like that in S_{11}.

Therefore, using the lines γ^i_{jk}, we have the double-sixes

$$D: \quad \begin{matrix} B_1 & B_2 & B_3 & B_4 & B_5 & B_{67} \\ \gamma^7_{16} & \gamma^7_{26} & \gamma^7_{36} & \gamma^7_{46} & \gamma^7_{56} & \Gamma^7_6, \end{matrix}$$

and

$$D': \quad \begin{matrix} B_1 & B_2 & B_3 & B_4 & B_6 & B_{57} \\ \gamma^7_{15} & \gamma^7_{25} & \gamma^7_{35} & \gamma^7_{45} & \gamma^7_{56} & \Gamma^7_5. \end{matrix}$$

Theorem. *All six lines Γ^7_i are the same.*

Proof. $\Gamma^7_6 \neq \Gamma^7_5 \Rightarrow \mathscr{R}(\gamma^7_{56}\Gamma^7_6\Gamma^7_5) \Rightarrow \mathscr{R}(B_1B_2B_3B_4)$, contradicting the existence of D. So all six Γ^7_i are the same and will be written Γ^7.

Theorem. *All three lines γ^5_{67}, γ^6_{57}, γ^7_{56} are the same.*

Proof. $\Gamma^7 \neq \gamma^6_{57}$ as Γ^7 meets B_5 and γ^6_{57} does not. So

$$\gamma^7_{56} \neq \gamma^6_{57} \Rightarrow \mathscr{R}(\Gamma^7\gamma^6_{57}\gamma^7_{56}) \Rightarrow \mathscr{R}(B_1B_2B_3B_4),$$

again a contradiction. So $\gamma^7_{56} = \gamma^6_{57} = \gamma^5_{67}$ and will be written Γ_{567}. Γ_{567} meets B_1, B_2, B_3, B_4, B_{56}, B_{57}, B_{67}.

S_{14}. **Theorem.** *All seven lines Γ^i are the same.*

Proof. $\Gamma^7 \neq \Gamma_{567}$ and $\Gamma^6 \neq \Gamma_{567}$ since both pairs occur in a double-six. So $\Gamma^7 \neq \Gamma^6 \Rightarrow \mathscr{R}(\Gamma^7\Gamma^6\Gamma_{567}) \Rightarrow \mathscr{R}(B_1B_2B_3B_4)$, a contradiction. So $\Gamma^7 = \Gamma^6$ and similarly all seven Γ^i are the same.

Write $\Gamma = \Gamma^i$. Then Γ meets B_1, B_2, B_3, B_4, B_5, B_6, B_7.

4. Description of the configuration

We now have the complete figure of 576 g-lines and 56 l-lines, where each g-line meets seven l-lines and each l-line meets 72 g-lines. Further, the figure can be constructed from any g-line and the seven l-lines which meet it.

A change in notation now leads to a great simplification as below:

Old	Δ	Δ_{ijk}	Γ^i_{jk}	Δ^i_{jk}	Γ_{ijk}	Γ	A_i	B_i
New	Δ_8	Δ^8_{ijk}	Γ^i_{jk8}	Δ^i_{jk8}	Γ^8_{ijk}	Γ_8	B_{i8}	A_{i8}

Δ_i, Γ_i, Δ^i_{jkl}, Γ^i_{jkl}, A_{ij}, B_{ij} remain the same.

Now there are

$$\begin{array}{rll} 8 & \text{lines} & \Delta_i \\ 8 & \text{lines} & \Gamma_i \\ 280 & \text{lines} & \Delta^i_{jkl} \\ 280 & \text{lines} & \Gamma^i_{jkl} \\ 28 & \text{lines} & A_{ij} \\ 28 & \text{lines} & B_{ij} \end{array}$$

where the indices vary from 1 to 8 with no repetitions.

The intersections of the lines are as follows:

$$\begin{array}{lll} \Delta_i & \text{meets} & B_{ij} \\ \Delta^i_{jkl} & \text{meets} & A_{jk}, B_{im} \\ \Gamma^i_{jkl} & \text{meets} & A_{im}, B_{jk} \\ \Gamma_i & \text{meets} & A_{ij}. \end{array}$$

The 576 g-lines fall naturally into 36 double-eights, one like

$$\begin{array}{cccccccc} \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 & \Delta_5 & \Delta_6 & \Delta_7 & \Delta_8 \\ \Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 & \Gamma_5 & \Gamma_6 & \Gamma_7 & \Gamma_8 \end{array}$$

and $\binom{8}{4}/2 = 35$ like

$$\begin{array}{cccccccc} \Delta^1_{234} & \Delta^2_{341} & \Delta^3_{412} & \Delta^4_{123} & \Delta^5_{678} & \Delta^6_{785} & \Delta^7_{856} & \Delta^8_{567} \\ \Gamma^1_{234} & \Gamma^2_{341} & \Gamma^3_{412} & \Gamma^4_{123} & \Gamma^5_{678} & \Gamma^6_{785} & \Gamma^7_{856} & \Gamma^8_{567} \end{array}$$

where the significance is quite unlike that of the double-six, but, for example, if the construction is begun with the line Δ_8 and the seven l-lines which meet it, then $\Gamma_1, \ldots, \Gamma_7$ are the seven Grace lines obtained and Γ_8 is the completing line of the configuration.

Each g-line occurs in exactly one double-eight. The eight g-lines forming one-half of a double-eight have, as transversals, 28 of the l-lines: the eight g-lines forming the other half have the other 28 l-lines as transversals.

Since each g-lines meets seven l-lines, there are $576 \times 21/6 = 2016$ double-sixes. They are as follows:

$$\begin{array}{lccccccl} m^1_{23} & A_{23} & B_{14} & B_{15} & B_{16} & B_{17} & B_{18} & 8\binom{7}{2} = 168 \\ & \Delta_1 & \Delta^1_{234} & \Delta^1_{235} & \Delta^1_{236} & \Delta^1_{237} & \Delta^1_{238} & \\ & & & & & & & \\ n^1_{23} & B_{23} & A_{14} & A_{15} & A_{16} & A_{17} & A_{18} & 168 \\ & \Gamma_1 & \Gamma^1_{234} & \Gamma^1_{235} & \Gamma^1_{236} & \Gamma^1_{237} & \Gamma^1_{238} & \end{array}$$

p^{11234}_{55678}	A_{12}	A_{13}	A_{14}	B_{56}	B_{57}	B_{58}	$8.7.\binom{6}{3} = 1120$
	Δ^5_{134}	Δ^5_{124}	Δ^5_{123}	Γ^1_{578}	Γ^1_{568}	Γ^1_{567}	
q^{123}_{456}	A_{23}	A_{31}	A_{12}	B_{56}	B_{64}	B_{45}	$\binom{8}{3}\binom{5}{3} = 560$
	Γ^1_{456}	Γ^2_{456}	Γ^3_{456}	Δ^4_{123}	Δ^5_{123}	Δ^6_{123}	

Each g-line lies in 21 double-sixes and each l-line lies in $2016 \cdot 6/56 = 216$ double-sixes.

There is a key theorem for this configuration which deals with five, six and seven skew lines having a transversal. *Given a line b with n transversals $a_1, \ldots, a_n$, the locus of points P such that the $n+1$ planes $Pb, Pa_1, \ldots, Pa_n$ touch a quadric is a cubic surface, a twisted cubic or a single point P according as n is five, six or seven*, Baker (1), p. 195. *Dually, the locus of planes π such that the $n+1$ points $\pi.b, \pi.a_1, \ldots, \pi.a_n$ lie on a conic is a cubic envelope, a cubic developable or a single plane according as n is five, six or seven.* Passing from five to six, six cubic surfaces all containing the twisted cubic are obtained: passing from six to seven, seven twisted cubics are obtained all containing the point.

For $n = 7$, there are other naturally associated algebraic varieties. A quartic surface is determined by 34 conditions. For b and $a_1, \ldots, a_7$ to lie on a quartic surface, there are $5+7 \cdot 4 = 33$ conditions. So there is a pencil of such surfaces through the eight lines. Two of these are special: firstly, one may require that the point P obtained above lies on the surface; secondly, one may require that b is a double line of the surface. Either condition gives a unique quartic surface. The former surface also contains the seven twisted cubics, each of which has six of the a_i as chords. Both surfaces are described in (9).

Beginning with Γ_1 meeting $A_{12}, A_{13}, A_{14}, A_{15}, A_{16}, A_{17}$, six double-sixes are obtained with the completing lines $B_{28}, B_{38}, B_{48}, B_{58}, B_{68}, B_{78}$. These twelve l-lines are chords of a twisted cubic t^8_1. Thus there are 56 cubics t^j_i with chords A_{ik}, B_{jk} $(k \neq i, j)$. From Γ_1 and the seven lines $A_{12}, \ldots, A_{18}$, the seven cubics $t^i_1 (i \neq 1)$ are obtained, which are concurrent at the point P_1. Thus there are eight points P_i and eight points Q_i, where Q_1, for example, is the meet of $t^1_2, \ldots, t^1_8$.

Beginning with Δ^1_{234} meeting $A_{23}, A_{24}, A_{34}, B_{15}, B_{16}, B_{17}$, six double-sixes with completing lines $A_{28}, A_{38}, A_{48}, B_{56}, B_{57}, B_{67}$ are obtained. These twelve lines are chords of a twisted cubic t_{2348}. Thus there are $\binom{8}{4} = 70$ cubics t_{ijkl} with chords A_{ij}, B_{mn} $(m, n \neq i, j, k, l)$. This gives 126 cubics in all. From sixes of the seven l-lines meeting Δ^1_{234}, the seven cubics

$$t_{2345}, t_{2346}, t_{2347}, t_{2348}, t^1_2, t^1_3, t^1_4$$

are obtained and these meet at Q_1. Similarly, from the seven l-lines meeting

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Γ^1_{234}, the seven cubics t_{1567}, t_{1568}, t_{1578}, t_{1678}, t_1^2, t_1^3, t_1^4, are obtained and they meet at P_1.

Theorem. *All* 126 *cubics* t_i^j, t_{ijkl} *have a point P in common.*

Proof. From above, t_i^j contains P_i, Q_j; t_{ijkl} contains P_i, Q_m. The two cubics t_{1567} and t_{1568} have the six chords A_{15}, A_{16}, A_{56}, B_{23}, B_{24}, B_{34} in common and so at most one point in common. Thus

$$t_{1567}.t_{1568} = P_1 = P_5 = P_6 = Q_2 = Q_3 = Q_4.$$

Hence all P_i and Q_j are the same point, which will be called P. So all 126 cubics have the point P in common.

Dually there are 126 cubic developables, all of which have a plane π in common. π meets each g-line and its seven transversals in the eight points of a conic. The eight planes joining P to a g-line and its seven transversals touch a quadric cone with vertex P.

If a cubic surface contains five chords of a twisted cubic, then the surface contains the curve. P lies on each of the 2016 cubic surfaces (containing the double-sixes), each of which contains two of the 126 twisted cubics. For example, m^1_{23} contains the l-lines A_{23}, B_{14}, B_{15}, B_{16}, B_{17}, B_{18} and therefore just the two cubics t_2^1, t_3^1. So each cubic curve lies on $2016 \times 2/126 = 32$ surfaces. For example, t_1^2 lies on six surfaces m^2_{1i}, six surfaces n^1_{2i} and twenty surfaces p^{11ijk}_{22lmn}.

If a twisted cubic has thirteen points in common with a quartic surface, the curve lies on the surface. So, for any g-line, there exists a unique quartic surface containing it, the seven l-lines meeting it and the seven twisted cubics with sixes of the seven lines as chords. Thus the number of such surfaces through one of the 126 cubic curves is $576 \times 7/126 = 32$.

One final figure is the number of twisted cubics which have a given l-line as chord, viz. $126 \times 12/56 = 27$.

Therefore, to summarise the numerical properties of the figure, there are the following tactical configurations:

g-lines, l-lines	(576, 7; 56, 72)
cubic curves, l-lines	(126, 12; 56, 27)
cubic surfaces, g-lines	(2016, 6; 576, 21)
cubic surfaces, l-lines	(2016, 6; 56, 216)
cubic surfaces, cubic curves	(2016, 2; 126, 32)
quartic surfaces, g-lines	(576, 1; 576, 1)
quartic surfaces, l-lines	(576, 7; 56, 72)
quartic surfaces, cubic curves	(576, 7; 126, 32)
quartic surfaces, cubic surfaces	(576, 21; 2016, 6).

In the case of the last tactical configuration, for each quartic surface there are 21 cubic surfaces, which meet it in a degenerate curve of order 12 consisting of a g-line, five l-lines and two twisted cubics; for each cubic surface, there are 6 quartic surfaces meeting it in such a curve.

If, in the construction of the configuration, A_7 is taken to be a chord of the unique twisted cubic t with $A_1, \ldots, A_6$ as chords, then all 126 twisted cubics t_i^j, t_{ijkl} are the same, viz. t, and all 56 l-lines A_{ij}, B_{ij} are chords of t. Also, there is no special point P.

5. Groups of the configuration

From five skew lines with a transversal, the 27 lines of a cubic surface are obtained: they form 36 double-sixes. The substitutions which preserve the configuration are as follows: any two lines of one half of a double-six may be interchanged; the two halves of a double-six may be interchanged; any double-six may be transformed into any other. Thus the group, G_5, of substitutions of the twenty-seven lines has order $6! \times 2 \times 36 = 51{,}840$ and is well-known.

From six skew lines with a transversal, the 44 lines of Grace's extension are derived. They form 32 double-sixes with halves as follows:

$$\begin{array}{rlcccccc} 6 & \text{like} & d_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ 6 & \text{like} & c_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ 20 & \text{like} & c_1 & c_2 & c_3 & d_4 & d_5 & d_6. \end{array}$$

The generators of the group, G_6, of the configuration are

$$(c_ic_j)(d_id_j), \quad (c_id_i)(c_jd_j), \quad i, j = 1, \ldots, 6;$$

i.e. these generate all permutations of the lines of any half of a double-six and all interchanges of any two halves. So G_6 has order $32 \times 6! = 23{,}040$.

Let M and N be the following subgroups of G_6: $M = \langle\{(c_ic_j)(d_id_j)\}\rangle$ and $N = \langle\{(c_id_i)(c_jd_j)\}\rangle$. M is isomorphic to S_6. N is abelian and has order 32; so N is an elementary abelian group of order 32 and isomorphic to

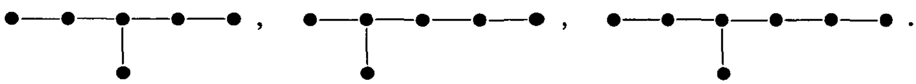
$$C_2 \times C_2 \times C_2 \times C_2 \times C_2,$$

where C_2 is cyclic of order 2. Further N is a normal subgroup of G_6. Therefore $G_6 = MN$ and is a split extension (or semi-direct product) of N by M.

The order of the group, G_7, of the configuration that has been derived from seven skew lines with a transversal is best seen from the double-eights. Any two lines in the same row of one may be interchanged; the two rows of a double-eight may be interchanged; and any double-eight may be changed into any other. So the order of G_7 is

$$8! \times 2 \times 36 = 2{,}903{,}040.$$

The groups G_5, G_6, G_7 are related to the semi-simple Lie algebras E_6, D_6, E_7, Bourbaki (2), pp. 256-266, and to the reflexion groups whose fundamental regions are the spherical simplexes E_6, B_6, E_7, Coxeter (**5**), ch. 11. The corresponding Dynkin graphs are



G_5 and G_6 are subgroups of G_7 of indices 56 and 126 respectively.

The group G_7 has an element γ of order two, where

$$\gamma = \prod_{i,j,k,l,m,n,p} (A_{ij}B_{ij})(\Delta_k\Gamma_k)(\Delta^p_{lmn}\Gamma^p_{lmn});$$

i.e. γ interchanges A_{ij} and B_{ij}, Δ_k and Γ_k, Δ^p_{lmn} and Γ^p_{lmn} for all values of the indices.

There exists a polarity which interchanges the lines a_i and b_i ($i = 1, \ldots, 6$) of a double-six. Further, there exists a polarity which interchanges the lines α and β, c_i and d_i, α_{jk} and β_{jk} ($i, j, k = 1, \ldots, 6$) of Grace's extension of the double-six. So it seems natural to postulate the following.

Conjecture. *There exists a polarity which induces γ and interchanges the point P and the plane π.*

The group G_7 has a subgroup of index two which is isomorphic to the group of the 28 bitangents of a non-singular plane quartic curve, (**4**). In fact, it will be shown that the configuration obtained has—modulo the involution γ—corresponding geometrical properties to those of the 28 bitangents. Thus the group of the bitangents is isomorphic to $G_7/\langle\gamma\rangle$.

6. The correspondence between the configuration and the bitangents of a non-singular plane quartic curve

Firstly, it is necessary to give a brief review of the properties of the 28 bitangents of a non-singular plane quartic curve Q; cf. Salmon (**12**), p. 223.

Let the bitangents be denoted by T_{ij} ($i, j = 1, \ldots, 8$, $i<j$). Then, given any pair of bitangents, five other pairs are uniquely determined so that any two of the six pairs have their eight points of contact with Q on a conic. Such a set of 12 lines is called a *Steiner set*. There are 63 Steiner sets—28 like $\{T_{1i}, T_{2i} \mid i = 3, \ldots, 8\}$ and 35 like

$$\{T_{12}, T_{34};\ T_{13}, T_{24};\ T_{14}, T_{23};\ T_{56}, T_{78};\ T_{57}, T_{68};\ T_{58}, T_{67}\}.$$

There exist sets of seven bitangents, called *Aronhold sets*, such that no three of the seven have their six points of contact on a conic. There are 288 Aronhold sets—8 like $\{T_{1i} \mid i = 2, \ldots, 8\}$ and 280 like $\{T_{12}, T_{13}, T_{14}, T_{15}, T_{67}, T_{68}, T_{78}\}$

There are 36 ways in which the 28 bitangents can be arranged (each appearing twice) as the elements of an 8×8 symmetric matrix excluding the main diagonal such that each row and column is an Aronhold set. This is a *Hesse arrangement*. It is, in fact, the basis for the notation T_{ij} for the bitangents.

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Thus the 288 Aronhold sets fall naturally by eights into the 36 Hesse arrangements.

Any five bitangents of an Aronhold set determine a sixth such that their twelve points of contact lie on a cubic curve: the six lines also touch a conic. There are 1008 of these hexads—168 like $\{T_{23}, T_{14}, T_{15}, T_{16}, T_{17}, T_{18}\}$, 560 like $\{T_{12}, T_{13}, T_{14}, T_{56}, T_{57}, T_{58}\}$ and 280 like $\{T_{12}, T_{13}, T_{23}, T_{45}, T_{46}, T_{56}\}$.

Finally, given seven general lines in the plane, there exists a unique quartic curve having these lines as bitangents. The remaining 21 bitangents can be constructed linearly from the initial seven.

The correspondence between the bitangents of Q and our configuration is then as follows.

Number	Plane object	Space object
36	Hesse arrangement of the 28 bitangents	Double-eight of g-lines
63	Steiner set of 12 lines	Pair of twisted cubics each with 12 chords
288	Aronhold set of 7 lines	Pair of g-lines each with 7 transversal l-lines
1008	Cubic curve and set of 6 lines	Pair of cubic surfaces each containing 6 l-lines

The pairs of space objects which are the same modulo γ are as follows:

$$\text{cubics} \begin{cases} t_i^j & \text{with chords } A_{ik}, B_{jk} \quad (k \neq i, j) \\ t_j^i & \text{with chords } A_{jk}, B_{ik} \quad (k \neq i, j) \end{cases}$$

$$\text{cubics} \begin{cases} t_{ijkl} & \text{with chords } A_{ij}, B_{mn} \quad (m, n \neq i, j, k, l) \\ t_{mnpq} & \text{with chords } A_{mn}, B_{ij} \quad (i, j \neq m, n, p, q) \end{cases}$$

$$\text{g-lines} \begin{cases} \Delta_i & \text{with transversals } B_{ij} \quad (j \neq i) \\ \Gamma_i & \text{with transversals } A_{ij} \quad (j \neq i) \end{cases}$$

$$\text{g-lines} \begin{cases} \Delta^i_{jkl} & \text{with transversals } A_{jk}, B_{im} \quad (m \neq i, j, k, l) \\ \Gamma^i_{jkl} & \text{with transversals } A_{im}, B_{jk} \quad (m \neq i, j, k, l) \end{cases}$$

$$\text{cubic surfaces} \begin{cases} m^i_{jk} & \text{containing l-lines } A_{jk}, B_{il} \quad (l \neq i, j, k) \\ n^i_{jk} & \text{containing l-lines } A_{il}, B_{jk} \quad (l \neq i, j, k) \end{cases}$$

$$\text{cubic surfaces} \begin{cases} p^{iijkl}_{mmnpq} & \text{containing l-lines } A_{ij}, B_{mn} \\ p^{mmnpq}_{iijkl} & \text{containing l-lines } A_{mn}, B_{ij} \end{cases}$$

$$\text{cubic surfaces} \begin{cases} q^{ijk}_{lmn} & \text{containing l-lines } A_{ij}, B_{lm} \\ q^{lmn}_{ijk} & \text{containing l-lines } A_{lm}, B_{ij}. \end{cases}$$

It seems possible that the correspondence could be made more precise. If the 56 l-lines are projected from P on to π, the 56 lines $Pl \cap \pi$ are obtained, which touch 576 conics in sets of seven. Dually, the 56 points of intersection $l \cap \pi$ lie in sevens on 576 conics. It seems doubtful that these 56 points could be the points of contact of 28 bitangents of a quartic curve, since they do not lie suitably in eights on conics.

7. Conclusion

The complete configuration of 632 lines derived from seven skew lines with a transversal has been obtained. Although we know that the seven Grace lines $\Gamma_1, \ldots, \Gamma_7$ do not in general have a transversal, § 1, it has not been shown without using a computed example. If one supposed that the seven Grace lines did have a transversal, then the eight g-lines of any row of a double-eight would have a transversal. It is therefore possible that, to understand the above configuration completely, it is necessary to investigate the configuration obtainable from eight skew lines with a transversal. For this case, however, there does not seem to be an associated irreducible variety, as was the case for five, six and seven lines.

From Longuet-Higgins's examples (**11**), it seems highly probable that the seven Grace lines always belong to a linear complex. This has two consequences. Firstly, it means that the eight g-lines of any row of a double-eight belong to a linear complex, thus giving 72 linear complexes in all. Secondly, by a theorem of Todd (**13**), p. 63, it means that any seven g-lines belonging to one half of a double-eight lie on a quartic surface, thus associating another set of 576 quartic surfaces with our figure.

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