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ISOPERIMETRIC INEQUALITIES FOR L_p GEOMINIMAL SURFACE AREA*

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Abstract. In this paper, we establish a number of L_p-affine isoperimetric inequalities for L_p-geominimal surface area. In particular, we obtain a Blaschke–Santaló type inequality and a cyclic inequality between different L_p-geominimal surface areas of a convex body.

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1. Introduction and main results. Geominimal surface area was introduced by Petty [**26**] more than three decades ago. Since then it has become apparent that this seminal concept and its general L_p extensions, which are due to Lutwak [**15**, **17**], serve as bridges connecting affine differential geometry, relative differential geometry and Minkowski geometry. Isoperimetric inequalities involving geominimal surface area are not only closely related to many affine isoperimetric inequalities involving affine surface area (see e.g. [**9**, **10**, **12**, **15**, **17**, **24**, **25**, **30–35**]), but, in fact, clarify the equality conditions of many of these inequalities. A comprehensive theory on L_p affine and geominimal surface area was first established in a remarkable work of Lutwak [**17**]. We note that all these isoperimetric inequalities are part of a new L_p Brunn–Minkowski theory initialized by Lutwak (see e.g. [**2**, **5**, **15–18**, **22**, **35**]) which has found a number of applications in analysis (see e.g. [**1**, **6**, **7**, **19**]). In general, all these works can be viewed as parts of the L_p valuation theory (see e.g. [**12**, **13**, **28**, **29**]). For a very recent progress towards an Orlicz Brunn–Minkowski theory developed by Lutwak, Yang and Zhang, which is more general than the L_p Brunn–Minkowski theory, we refer to [**4**, **11**, **20**, **21**].

Following an important line of research in affine geometry, this paper is devoted to the isoperimetric inequalities for L_p geominimal surface areas, which can be derived from the Blaschke–Santaló inequality, the L_p centro-affine inequality and the L_p Petty projection inequality. Moreover, a cyclic inequality between different L_p geominimal surface areas of a convex body and a Brunn–Minkowski-type inequality for L_p geominimal surface area of the L_p Blaschke-sum of convex bodies are also established.

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Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interior) in $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interior and the set of convex bodies centred at the origin (i.e. symmetric about the origin), we write $\mathcal{K}_o^n$ and $\mathcal{K}_s^n$, respectively. The unit ball in $\mathbb{R}^n$ and its surface will be denote by B and S^{n-1}, respectively. The volume of the unit ball B will be denoted by $\omega_n = \pi^{n/2}/\Gamma(1+n/2)$.

If $K \in \mathcal{K}^n$, then its support function $h_K = h(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$ is defined by $x \in \mathbb{R}^n \setminus \{0\}$

$$h(K, x) = \max\{x \cdot y : y \in K\}.$$

For real $p \geq 1, \lambda, \mu \geq 0$ (not both zero), the Firey linear combination $\lambda \cdot K +_p \mu \cdot L$ of $K, L \in \mathcal{K}_o^n$ is defined by (see [**3**])

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K, \cdot)^p + \mu h(L, \cdot)^p.$$

For $p \geq 1$, the L_p mixed volume, $V_p(K, L)$, of $K, L \in \mathcal{K}_o^n$, was defined in [**16**] by

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

For $K \in \mathcal{K}_o^n$ and $p \geq 1$, the L_p geominimal surface area, $G_p(K)$, was defined in [**17**] (the case $p = 1$ defined in [**26**]) by

$$\omega_n^{p/n} G_p(K) = \inf\left\{n V_p(K, Q) V(Q^*)^{p/n} : Q \in \mathcal{K}_o^n\right\}, \tag{1.1}$$

where Q^* is the polar body of Q defined by

$$Q^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, \forall y \in Q\}.$$

It is easily verified that $Q^{**} = Q$.

One of the most important inequalities in convex geometry is the Blaschke–Santaló inequality about polar body (see e.g. [**22**, **24**, **27**]): if $K \in \mathcal{K}_s^n$, then

$$V(K)V(K^*) \leq \omega_n^2 \tag{1.2}$$

with equality if and only if K is an ellipsoid. Recently, Haberl and Schuster [**5**] showed that there is an interesting asymmetric L_p version of (1.2).

For $p \geq 1$, define $\omega_p = \pi^{p/2}/\Gamma(1+p/2)$. The L_p projection body, $\Pi_p K$, of $K \in \mathcal{K}_o^n$ is defined [**18**] by

$$h^p_{\Pi_p K}(u) = \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |u \cdot v|^p dS_p(K, v),$$

where $c_{n,p} = \omega_{n+p}/\omega_2\omega_n\omega_{p-1}$ and $u, v \in S^{n-1}$. $S_p(K, \cdot)$ is a positive Borel measure on S^{n-1}, called the L_p surface area measure of K. We note that the constant $c_{n-2,p}$ in the definition of the L_p projection body is chosen so that for the unit ball B we have $\Pi_p B = B$.

It was shown by Lutwak [**16**] that corresponding to each $K \in \mathcal{K}_o^n$, the L_p surface area measure, $S_p(K, \cdot)$ is absolutely continuous with respect to the surface area $S(K, \cdot)$ of K, and has Radon–Nikodym derivative

$$\frac{dS_p(K, \cdot)}{dS(K, \cdot)} = h(K, \cdot)^{1-p}. \tag{1.3}$$

A convex body $K \in \mathcal{K}_o^n$ will be said to have an L_p curvature function $f_p(K,\cdot): S^{n-1} \to \mathbb{R}$, if $S_p(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure, S, and

$$\frac{dS_p(K,\cdot)}{dS} = f_p(K,\cdot).$$

We will denote by $\mathcal{F}_o^n$ the set of all bodies in $\mathcal{K}_o^n$ that have an L_p curvature function. For $K \in \mathcal{F}_o^n$, the L_p affine surface area of K, $\Omega_p(K)$, was defined by Lutwak **[17]** as

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K,u)^{n/(n+p)} dS(u). \tag{1.4}$$

The radial function $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \setminus \{0\} \to [0,\infty)$, of a compact star-shaped (about the origin) $K \subset \mathbb{R}^n$, is defined for $x \in \mathbb{R}^n \setminus \{0\}$ by

$$\rho(K,x) = \max\{\lambda \geq 0 : \lambda x \in K\}.$$

If ρ_K is positive and continuous, K will be called a star body (about the origin). Write $\mathcal{S}_o^n$ for the set of star bodies in $\mathbb{R}^n$.

Motivated by Leichtweiß **[9]**, Lutwak **[17]** introduced the L_p affine surface area, $\Omega_p(K)$, of $K \in \mathcal{K}_o^n$, by

$$n^{-p/n}\Omega_p(K)^{(n+p)/n} = \inf\left\{nV_p(K,Q^*)V(Q)^{p/n} : Q \in \mathcal{S}_o^n\right\}, \tag{1.5}$$

where

$$V_p(K,Q^*) = \frac{1}{n}\int_{S^{n-1}} \rho_Q^{-p} dS_p(K,u).$$

It was also shown in **[17]** that if $K \in \mathcal{K}_o^n$ has L_p curvature function, then the definition (1.5) gives the integral formula (1.4).

Two essential results for this general L_p affine surface area of a convex body in $\mathcal{K}_o^n$ are the following.

THEOREM A. (see **[17]**) *If $K \in \mathcal{K}_o^n$ with its centroid lying in the origin, then for $p \geq 1$*

$$\Omega_p(K)\Omega_p(K^*) \leq (n\omega_n)^2,$$

with equality if and only if K is an ellipsoid.

THEOREM B. (see **[17]**) *If $K \in \mathcal{K}_o^n$ and $1 \leq p < q < r$, then*

$$\Omega_q(K)^{(n+q)(r-p)} \leq \Omega_p(K)^{(n+p)(r-q)}\Omega_r(K)^{(n+r)(q-p)}.$$

In this paper, we establish the following results for the L_p geominial surface area of a convex body in $\mathcal{K}_o^n$. It will be shown that two of our main theorems will strengthen Theorems A and B presented above. The following are our main results:

THEOREM 1. *If $K \in \mathcal{K}_s^n$ and $p \geq 1$, then*

$$G_p(K)^n \leq n^n\omega_n^{n-p}V(\Pi_pK)^p, \tag{1.6}$$

with equality if and only if K is an ellipsoid.

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THEOREM 2. *If $K \in \mathcal{K}_s^n$ and $1 \leq p < n$, then*

$$G_p(K)G_p(K^*) \leq (n\omega_n)^2, \tag{1.7}$$

with equality if and only if K is an ellipsoid.

THEOREM 3. *If $K \in \mathcal{K}_o^n$, $1 \leq p < q < r$, then*

$$G_q(K)^{(r-p)} \leq G_p(K)^{(r-q)} G_r(K)^{(q-p)}. \tag{1.8}$$

A very important observation is the relationship between the L_p affine surface area and L_p geominimal surface areas due to Lutwak [**17**]: if $K \in \mathcal{K}_o^n$ and $p \geq 1$, then

$$\Omega_p(K)^{n+p} \leq (n\omega_n)^p G_p(K)^n \tag{1.9}$$

with equality if and only if K is an ellipsoid.

Therefore, in view of equation (1.9), Theorem 2 strengthens Theorem A for origin-symmetric convex bodies, and Theorem 3 strengthens Theorem B for each convex body with the origin in its interior. Moreover, from Theorem 1 we can get the following L_p affine projection inequality (see [**23**]).

COROLLARY 1. *If $K \in \mathcal{K}_o^n$ and $p \geq 1$, then*

$$\Omega_p(K)^{n+p} \leq n^{n+p}\omega_n^n V(\Pi_p K)^p,$$

with equality if and only if K is an ellipsoid.

2. Preliminaries. In this section, we collect some basic well-known facts that we will use in the proofs of our results. For references about the Brunn–Minkowski theory, see [**3**, **27**].

According to the definitions of the polar body, the support function and radial function, it follows for $K \in \mathcal{K}_o^n$ that

$$h_{K^*} = \frac{1}{\rho_K}, \qquad \rho_{K^*} = \frac{1}{h_K}. \tag{2.1}$$

As defined in Section 1, for $p \geq 1$, the L_p-mixed volume, $V_p(K, L)$, of $K, L \in \mathcal{K}_o^n$ is given by

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

Lutwak [**16**] proved that corresponding to each $K \in \mathcal{K}_o^n$, there is a positive Borel measure, $S_p(K, \cdot)$, on S^{n-1} such that

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p dS_p(K, u) \tag{2.2}$$

for each $L \in \mathcal{K}_o^n$. The measure $S_1(K, \cdot)$ is just the classical surface area measure, $S(K, \cdot)$, of K.

From the formula (2.2), it follows immediately that for each $K \in \mathcal{K}_o^n$,

$$V_p(K, K) = V(K). \tag{2.3}$$

It is well known that the polar coordinate formula for volume reads:

$$V(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^n(u) dS(u). \tag{2.4}$$

For $n \neq p \geq 1$ and $K, L \in \mathcal{K}_s^n$, the Blaschke L_p-combination $K \check{+}_p L \in \mathcal{K}_s^n$ is defined by (see [**17**])

$$dS_p(K \check{+}_p L, \cdot) = dS_p(K, \cdot) + dS_p(L, \cdot), \tag{2.5}$$

where $dS_p(K, \cdot)$ denotes the L_p- surface area measure of K. For $p = 1$, equation (2.5) is just the classical Blaschke combination $K \check{+} L$.

It was shown in [**17**] that if $p \geq 1$ and $K \in \mathcal{K}_o^n$, then there exists a unique body $T_p K \in \mathcal{K}_o^n$ such that

$$G_p(K) = nV_p(K, T_p K) \quad \textit{and} \quad V(T_p^* K) = \omega_n. \tag{2.6}$$

$T_p K$ is called the L_p-Petty body of K. Here $T_p^* K$ denotes the polar body of $T_p K$ (rather than $(T_p K)^*$).

Finally, we will say two star bodies K and L to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

3. Proof of the main Theorems. To prove the Theorems presented in Section 1, we still need some additional lemmas.

LEMMA 1. (see [**23**]) *If $K, L \in \mathcal{K}_o^n$, then for $p \geq 1$,*

$$V_p(K, \Pi_p L) = V_p(L, \Pi_p K). \tag{3.1}$$

*For $p = 1$, the identity of equation (3.1) was obtained in [**14**].*

LEMMA 2. (see [**17**]) *If $p \geq 1$, and $K \in \mathcal{K}_0^n$, then*

$$G_p(K)^n \leq n^n \omega_n^p V(K)^{n-p}, \tag{3.2}$$

with equality if and only if K is an ellipsoid.

LEMMA 3. (see [**18**]) *If $K \in \mathcal{K}_o^n$, then for $p \geq 1$,*

$$V(K)^{(n-p)/p} V(\Pi_p^* K) \leq \omega_n^{n/p}, \tag{3.3}$$

with equality if and only if K is an ellipsoid centred at the origin.

Proof of Theorem 1. From definition (1.1), it follows that for $Q \in \mathcal{K}_o^n$,

$$\omega_n^p G_p(K)^n \leq n^n V_p(K, Q)^n V(Q^*)^p.$$

Suppose $L \in \mathcal{K}_o^n$. Take $\Pi_p L$ for Q, to get

$$\omega_n^p G_p(K)^n \leq n^n V_p(K, \Pi_p L)^n V(\Pi_p^* L)^p.$$

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Thus, by equation (3.1), and the L_p-Petty projection inequality (3.3), we obtain

$$\omega_n^p G_p(K)^n \le n^n \omega_n^n V_p(L, \Pi_p K)^n V(L)^{-(n-p)},$$

with equality implying that L is an ellipsoid. Now take $\Pi_p K$ for L, use equation (2.3), and the result is the desired inequality (1.6). Note that equality in equation (1.6) implies that $\Pi_p K$ is an ellipsoid.

Suppose there is equality in equation (1.6). Hence,

$$G_p(K)^n = n^n \omega_n^{n-p} V(\Pi_p K)^p.$$

and $\Pi_p K$ is a centred ellipsoid. Thus $V(\Pi_p K)V(\Pi_p^* K) = \omega_n^2$. From definition (1.1), it follows that for all $Q \in \mathcal{K}_o^n$,

$$n^n \omega_n^n V(\Pi_p K)^p = \omega_n^p G_p(K)^n \le n^n V_p(K, Q)^n V(Q^*)^p.$$

Take $K \in \mathcal{K}_s^n$ for Q, using equation (2.3), to get,

$$\omega_n^n V(\Pi_p K)^p \le V(K)^n V(K^*)^p.$$

The Blaschke–Santaló inequality (1.2) now shows that

$$\omega_n^n V(\Pi_p K)^p \le \omega_n^{2p} V(K)^{n-p}.$$

But, as noted previously, $V(\Pi_p K)V(\Pi_p^* K) = \omega_n^2$. Hence the last inequality is

$$\omega_n^{n/p} \le V(K)^{(n-p)/p} V(\Pi_p^* K)^p.$$

The equality conditions of the L_p-Petty projection inequality (3.3) show that K must therefore be an ellipsoid. □

Proof of Theorem 2. By applying Lemma 2 and the Blaschke–Santaló inequality (1.2), we obtain

$$\begin{aligned} G_p(K)^n G_p(K^*)^n &\le n^{2n} \omega_n^{2p} V(K)^{n-p} V(K^*)^{n-p} \\ &\le n^{2n} \omega_n^{2p} \omega_n^{2n-2p} \\ &= (n\omega_n)^{2n}. \end{aligned}$$

Thus

$$G_p(K) G_p(K^*) \le (n\omega_n)^2.$$

This is just (1.7). By the equality conditions of the Blaschke–Santaló inequality and equation (3.2), equality in Theorem 2 holds if and only if K is an ellipsoid. □

Proof of Theorem 3. The desired inequality follows immediately from the definition of L_p geominimal surface area once the following fact is established: given $Q_1, Q_2 \in \mathcal{K}_o^n$, there exists a $Q_3 \in \mathcal{K}_o^n$ such that

$$[V_q(K, Q_3) V(Q_3^*)^{\frac{q}{n}}]^{r-p} \le [V_p(K, Q_1) V(Q_1^*)^{\frac{p}{n}}]^{r-q} [V_r(K, Q_2) V(Q_2^*)^{\frac{r}{n}}]^{q-p}.$$

To show this, define $Q_3 \in \mathcal{K}_o^n$ by

$$h_{Q_3}^{-q(r-p)} = h_{Q_1}^{-p(r-q)} h_{Q_2}^{-r(q-p)},$$

i.e.

$$\rho_{Q_3^*}^{q(r-p)} = \rho_{Q_1^*}^{p(r-q)} \rho_{Q_2^*}^{r(q-p)}.$$

Since

$$\rho_{Q_3^*}^{n} = \rho_{Q_1^*}^{np(r-q)/q(r-p)} \rho_{Q_2^*}^{nr(q-p)/q(r-p)},$$

the Hölder inequality (see [**8**]) and the polar coordinate formula for volume give

$$V(Q_3^*)^{q(r-p)} \le V(Q_1^*)^{p(r-q)} V(Q_2^*)^{r(q-p)}.$$

Since

$$h_{Q_3}^q h_K^{1-q} = [h_{Q_1}^p h_K^{1-p}]^{(r-q)/(r-p)} [h_{Q_2}^r h_K^{1-r}]^{(q-p)/(r-p)},$$

the Hölder inequality, together with equations (1.3) and (2.2), yield

$$V_q(K, Q_3)^{r-p} \le V_p(K, Q_1)^{r-q} V_r(K, Q_2)^{q-p}.$$

When the last two inequalities are combined, the desired inequality is obtained. □

4. Additional inequalities. The notion of L_p centroid body was introduced by Lutwak and Zhang (see [**22**]). For each compact star-shaped (about the origin) K in $\mathbb{R}^n$ and for real number $p \ge 1$, the polar L_p centroid body $\Gamma_p^* K$ of K is defined by

$$\rho_{\Gamma_p^* K}^{-p}(u) = \frac{1}{c_{n,p} V(K)} \int_K |u \cdot x|^p dx, \tag{4.1}$$

where the integration is with respect to Lebesgue measure. The normalization above is chosen so that for the unit ball, B, we have $\Gamma_p^* B = B$. In [**18**] the authors proved the following L_p centro-affine inequality.

THEOREM C. *If $K \in \mathcal{S}_o^n$, then for $p \ge 1$,*

$$V(K) V(\Gamma_p^* K) \le \omega_n^2, \tag{4.2}$$

with equality if and only if K is an ellipsoid centred at the origin .

By equation (2.6), we can easily obtain:

LEMMA 4. *If $K \in \mathcal{K}_o^n$, and $p \ge 1$, then*

$$\omega_n^p G_p(K)^n = n^n V_p(K, T_p K)^n V(T_p^* K)^p. \tag{4.3}$$

Next, we combine Theorem C with the definition of L_p-geominimal surface area $G_p(K)$ to obtain the following result.

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THEOREM 4. *If $K \in \mathcal{K}_o^n$, and $L \in \mathcal{S}_o^n$, $p \geq 1$, then*

$$V(L)^p G_p(K)^n \leq n^n V_p(K, \Gamma_p L)^n \omega_n^p, \tag{4.4}$$

with equality if and only if K and L are dilatable ellipsoid centred at the origin.

Proof. From definition (1.1), it follows that for $Q \in \mathcal{K}_o^n$,

$$\omega_n^p G_p(K)^n \leq n^n V_p(K, Q)^n V(Q^*)^p.$$

Suppose $L \in \mathcal{S}_o^n$. Take $\Gamma_p L$ for Q, to get

$$\omega_n^p G_p(K)^n \leq n^n V_p(K, \Gamma_p L)^n V(\Gamma_p^* L)^p, \tag{4.5}$$

with equality in equation (4.5) if and only if $T_p K$ and $\Gamma_p L$ are dilates by Lemma 4.
From an application of Theorem C, we obtain

$$\omega_n^p G_p(K)^n \leq n^n V_p(K, \Gamma_p L)^n \omega_n^{2p} V(L)^{-p}.$$

Thus,

$$V(L)^p G_p(K)^n \leq n^n V_p(K, \Gamma_p L)^n \omega_n^p.$$

By the equality conditions of the inequalities (4.2) and (4.5), equality holds in equation (4.4) if and only if K and L are dilated ellipsoids centred at the origin. □

COROLLARY 2. *If $L \in \mathcal{K}_o^n$, $p \geq 1$, then*

$$V(L)^p G_p(\Gamma_p L)^n \leq n^n V(\Gamma_p L)^n \omega_n^p, \tag{4.6}$$

with equality if and only if L is an ellipsoid centred at the origin.

Proof. Put $K = \Gamma_p L$ in inequality (4.4), and use equation (2.3) to see that inequality (4.6) immediately is obtained. By the equality conditions of equation (4.4) and Theorem 4, equality holds in equation (4.6) if and only if L is an ellipsoid centred at the origin. □

Finally, we obtain a Brunn–Minkowski-type inequality for the L_p-geominimal surface area of an L_p- Blaschke combination of two convex bodies.

THEOREM 5. *If $K, L \in \mathcal{K}_o^n$, and $p \geq 1$, then*

$$G_p(K \check{+}_p L) \geq G_p(K) + G_p(L), \tag{4.7}$$

with equality if and only if K and L are dilates.

Proof. For $K, L \in \mathcal{K}_o^n$, $p \geq 1$, by equation (2.2) and (2.5), for all $Q \in \mathcal{K}_o^n$, we have

$$V_p(K \check{+}_p L, Q) = V_p(K, Q) + V_p(L, Q). \tag{4.8}$$

From the definition (1.1), we have

$$\begin{aligned}\omega_n^{p/n} G_p(K\check{+}_pL) &= \inf\left\{nV_p(K\check{+}_pL, Q)V(Q^*)^{p/n} : Q \in \mathcal{K}_o^n\right\}\\ &= \inf\left\{n[V_p(K, Q) + V_p(L, Q)]V(Q^*)^{p/n} : Q \in \mathcal{K}_o^n\right\}\\ &\geq \inf\left\{nV_p(K, Q)V(Q^*)^{p/n} : Q \in \mathcal{K}_o^n\right\}\\ &\quad + \inf\left\{nV_p(L, Q)V(Q^*)^{p/n} : Q \in \mathcal{K}_o^n\right\}\\ &= \omega_n^{p/n} G_p(K) + \omega_n^{p/n} G_p(L).\end{aligned}$$

This shows that

$$G_p(K\check{+}_pL) \geq G_p(K) + G_p(L).$$

The equality of equation (4.7) holds if and only if $K\check{+}_pL$ are dilates with K and L, respectively. That is K and L are dilates. $\square$

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