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MULTIVALENT AND MEROMORPHIC FUNCTIONS OF BOUNDED BOUNDARY ROTATION

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1. The class $V_k(p)$. We generalize the class V_k of analytic functions of bounded boundary rotation [**8**] by allowing critical points in the unit disc U.

Definition. Let $f(z) = a_q z^q + \ldots$ $(q \geqq 1)$ be analytic in U. Then $f(z)$ belongs to the class $V_k(p)$ if for r sufficiently close to 1,

$$(1.1)\quad \int_0^{2\pi} \operatorname{Re}\left\{1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right\} d\theta = 2p\pi$$

and

$$(1.2)\quad \limsup_{r\to 1^-} \int_0^{2\pi} \left|\operatorname{Re}\left\{1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right\}\right| d\theta \leqq pk\pi.$$

We note that (1.1) implies that f has precisely $p - 1$ critical points in U. Also, if $f(z) \in V_k(p)$, then $\operatorname{Re}\{1 + zf''(z)/f'(z)\} > 0$ for $r_0 < |z| < 1$ if and only if $k = 2$. Hence $V_2(p) = C(p)$, where $C(p)$ is the class of p-valent convex functions defined by Goodman [**4**].

If $p = 1$, then except for normalization, $V_k(p)$ reduces to the class V_k. It is well-known that the class $V_k(1)$ consists only of univalent functions if $2 \leqq k \leqq 4$. To determine the largest value of k such that each function in $V_k(p)$ is at most p-valent, we will need the following

LEMMA 1.1. *Let $f(z) \in V_k(p)$. Then*

$$\lim_{r\to 1^-} \int_0^{2\pi} \left|\operatorname{Re}\left\{1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right\}\right| d\theta \textit{ exists.}$$

Proof. Let $f(z)$ have non-zero critical points $a_1, \ldots a_{p-q}$ $(q \geqq 1)$, counting multiplicities, and let $r_0 = \max |a_j|$. Then for $r_0 < |z| < 1$,

$$\left|\operatorname{Re}\ 1 + \frac{zf''(z)}{f'(z)}\right|$$

is subharmonic. Consequently for $\rho < |z| < 1$,

$$\int_0^{2\pi} \left|\operatorname{Re}\left\{1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right\}\right| d\theta$$

is a convex function of $\log r$ and hence the limit exists.

THEOREM 1.2. *Let $f(z) \in V_k(p)$. Then $f(z)$ is at most $\max [p, \{pk/2 - 1\}]$ valent, where $\{pk/2 - 1\}$ denotes the smallest integer greater than $pk/2 - 1$.*

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Proof. By Lemma 1.1, given $\epsilon > 0$ we may choose $r_0 < 1$ so that if $r_0 < r < 1$,

$$\int_0^{2\pi} \left| \operatorname{Re}\left\{1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right\}\right| d\theta < 2\left(\frac{pk}{2} + \epsilon\right)\pi.$$

Since $f(z)$ is analytic for $|z| \leqq r$, it follows by a result due to Umezawa [**17**] that f is at most max $[p, \{pk/2 - 1\}]$ valent in $|z| \leqq r$. The result follows by letting $r \to 1$.

Note. This was proved by Brannan [**1**] in the case $p = 1$.

COROLLARY 1.3. *Let $f(z) \in V(p)$ with $k < 2 + 2/p$. Then f is at most p-valent in U.*

Our next goal is to obtain representation formulas for $V_k(p)$. We will need to use the functions

$$(1.3) \quad \psi(z, z_j) = \frac{(z - z_j)(1 - \bar{z}_j z)}{z}$$

which have been employed by Hummel [**6**] and others.

LEMMA 1.4. *Let $f(z) = a_q z^q + \ldots$ $(q \geqq 1)$ belong to $V_k(p)$ and have non-zero critical points $z_1, \ldots z_{p-q}$, counting multiplicities. Let*

$$g(z) = \int_0^z \prod_{j=1}^{p-q} \psi(z, z_j)^{-1} f'(z) dz.$$

Then $g(z)$ has $p - 1$ critical points all at $z = 0$ and $g(z) \in V_k(p)$.

Proof. It follows from the definition of $g(z)$ that

$$1 + \frac{zg''(z)}{g'(z)} = 1 + \frac{zf''(z)}{f'(z)} - \sum_{j=1}^{p-q}\left(\frac{z_j}{z - z_j} - \frac{z\,\bar{z}_j}{1 - \bar{z}_j z}\right).$$

Let $\epsilon > 0$ be given. Since for $|z| = 1$

$$\operatorname{Re}\left\{\frac{z_j}{z - z_j} - \frac{z\,\bar{z}_j}{1 - \bar{z}_j z}\right\} = 0,$$

there is an $r_0 < 1$ such that if $r_0 < r < 1$, then

$$\int_0^{2\pi}\left|\operatorname{Re}\left\{1 + \frac{re^{i\theta}g''(re^{i\theta})}{g'(re^{i\theta})}\right\}\right| d\theta \leqq \int_0^{2\pi}\left|\operatorname{Re}\left\{1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right\}\right| d\theta + \epsilon.$$

Consequently

$$\limsup_{r \to 1^-} \int_0^{2\pi}\left|\operatorname{Re}\left\{1 + \frac{re^{i\theta}g''(re^{i\theta})}{g'(re^{i\theta})}\right\}\right| d\theta \leqq pk\pi + \epsilon.$$

Since $\epsilon > 0$ is arbitrary and g has precisely $p - 1$ critical points, $g(z) \in V_k(p)$.

THEOREM 1.5. *Let $f(z) = a_q z^q + \ldots \in V_k(p)$ and suppose $f(z)$ has non-zero critical points $z_1, \ldots z_{p-q}$, counting multiplicities. Then:*

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(i) *there is a function* $\mu(t)$ *of bounded variation on* $[0, 2\pi]$ *with*

$$\int_0^{2\pi} d\mu(t) = 2p \text{ and } \int_0^{2\pi} |d\mu(t)| \leqq pk$$

and a constant A *such that*

$$(1.4) \quad f'(z) = Az^{p-1} \prod_{j=1}^{p-q} \psi(z, z_j) \exp\left[-\int_0^{2\pi} \log(1 - ze^{-it}) d\mu(t)\right];$$

(ii) *there is a function* $h(z) \in V_k$ *and a constant* A *such that*

$$(1.5) \quad f'(z) = Az^{p-1} \prod_{j=1}^{p-q} \psi(z, z_j)\, [h'(z)]^p;$$

(iii) *there are two normalized univalent starlike functions* $s_1(z)$ *and* $s_2(z)$ *and a constant* A *such that*

$$(1.6) \quad f'(z) = Az^{p-1} \prod_{j=1}^{p-q} \psi(z, z_j) \left[\frac{s_1(z)}{z}\right]^{\frac{1}{4}p(k+2)} \left[\frac{s_2(z)}{z}\right]^{-\frac{1}{4}p(k-2)}.$$

Proof. An application of Plessner's Theorem [**3**, p. 38] to the function $g(z)$ related to $f(z)$ as in Lemma 1.4 yields (1.4), and (1.6) follows by decomposing $\mu(t)$ into the difference of two increasing functions.

The following distortion theorem is an easy consequence of Theorem 1.5 and thus we omit the proof. Hummel [**6**] has similar results for the class $S(p)$.

THEOREM 1.6. *Let* $f(z) = a_q z^q + \ldots \in V_k(p)$ *have non-zero critical points* $z_1, \ldots z_{p-q}$, *counting multiplicities. Let* $R_1 = \max |z_j|$, $R_2 = \min |z_j|$. *Then with* $z = re^{i\theta}$,

$$|f'(z)| \leqq \frac{(1+r)^{\frac{1}{2}p(k-2)}}{(1-r)^{\frac{1}{2}p(k+2)}} \frac{q|a_q|}{\Pi|z_j|} r^{q-1} \prod_{j=1}^{p-q} (r + |z_j|)(1 + r|z_j|)$$

$$|f'(z)| \geqq \frac{(1-r)^{\frac{1}{2}p(k-2)}}{(1+r)^{\frac{1}{2}p(k+2)}} \frac{q|a_q|}{\Pi|z_j|} r^{q-1} \prod_{j=1}^{p-q} (r - |z_j|)(1 - |z_j|r), R_1 < r < 1$$

$$|f'(z)| \geqq \frac{(1-r)^{\frac{1}{2}p(k-2)}}{(1+r)^{\frac{1}{2}p(k+2)}} \frac{q|a_q|}{\Pi|z_j|} r^{q-1} \prod_{j=1}^{p-q} (|z_j| - r)(1 - |z_j|r), 0 < r < R_2.$$

THEOREM 1.7. *Let* $f(z) = a_q z^q + \ldots \in V_k(p)$ *have* $p - q$ *non-zero critical points* $z_1, \ldots z_{p-q}$, *counting multiplicities. The* $f(z)$ *is* q*-valently convex for* $|z| < r_q$, *where* rq *is the least positive root of*

$$\frac{p}{2}\left[\left(1 + \frac{k}{2}\right)\left(\frac{1-r}{1+r}\right) + \left(1 - \frac{k}{2}\right)\left(\frac{1+r}{1-r}\right)\right] - \sum_{j=1}^{p-q} \frac{|z_j|(1-r^2)}{(|z_j| - r)(1 - |z_j|r)} = 0.$$

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Proof. Let $\mu(t)$ be the function in (1.4) such that

$$f'(z) = qa_q z^{p-1} \prod_{j=1}^{p-q} \psi(z, z_j) \exp\left[-\int_0^{2\pi} \log (1 - ze^{-it}) d\mu(t) \right].$$

Then we compute

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} = \text{Re} \sum_{j=1}^{p-q} \frac{z\,\psi'(z, z_j)}{\psi(z, z_j)} + \text{Re} \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t).$$

Hummel [**6**] has obtained the bonds

$$(1.7) \quad \text{Re} \frac{z\psi'(z, z_j)}{\psi(z, z_j)} \geqq \frac{-|z_j|\,(1 - r^2)}{(|z_j| - r)(1 - |z_j|r)}.$$

Since $\mu(t)$ has positive variation $<p(1 + k/2)$ and negative variation $<p(k/2 - 1)$ it follows that

$$(1.8) \quad \text{Re} \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t) \geqq p\left(1 + \frac{k}{2}\right)\left(\frac{1 - r}{1 + r}\right) + p\left(1 - \frac{k}{2}\right)\left(\frac{1 + r}{1 - r}\right).$$

The result now follows by combining (1.7) and (1.8).

The following corollary is immediate.

COROLLARY 1.8. *Let* $f(z) = a_p z^p + \ldots \in V_k(p)$. *Then* $f(z)$ *is* p*-valently convex for* $|z| < \frac{1}{2}(k - (k^2 - 4)^{1/2})$ *and this result is sharp.*

2. Coefficient Bounds for $V_k(p)$. Goodman [**4**] has conjectured that if $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is at most p-valent in U, then for $n \geqq p + 1$,

$$(2.1) \quad |a_n| \leqq \sum_{j=1}^{p} \frac{2j(n + p)!}{(n^2 - j^2)(p + j)!(p - j)!(n - p - 1)!} |a_j|.$$

The conjecture (2.1) has been verified for certain subclasses of p-valent functions. If $f(z)$ belongs to the class $K(p)$ of p-valent close-to-convex functions defined by Livingston [**9**], then (2.1) is known for $n = p + 1$ with no restriction on $a_1, \ldots a_p$ [**9**] and for $n \geqq p + 1$, provided $a_1 = \ldots = a_{p-2} = 0$ [**11**].

We recall that if $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in V_k$, then $|b_2| \leqq k/2$, $|b_3| \leqq (k^2 + 2)/6$ [**8**] and $|b_4| \leqq (k^3 + 8k)/4!$ [**15**], with equality if

$$g(z) = \frac{1}{k}\left[\left(\frac{1 + z}{1 - z}\right)^{k/2} - 1\right].$$

We will first consider functions $f(z) \in V_k(p)$ with all critical points at the origin.

THEOREM 2.1. *Let* $f(z) = a_p z^p + \ldots \in V_k(p)$. *Then*

$$(p+1)|a_{p+1}| \leqq p^2 k |a_p|$$
$$(p+2)|a_{p+2}| \leqq \left(\frac{p^2k^2}{2} + p\right) p|a_p|$$
$$(p+3)|a_{p+3}| \leqq \frac{p^2k}{6}(p^2k^2 + 6p + 2)|a_p|.$$

All of these results are sharp, with equality for $F'(z) = p\, a_p[g'(z)]^p$, *where*

$$g(z) = \frac{1}{k}\left[\left(\frac{1+z}{1-z}\right)^{k/2} - 1\right].$$

Proof. Let $g(z) = z + \sum_2^\infty b_n z^n$ be the function in V_k related to f by Theorem 1.5. We compute

$$\frac{f'(z)}{pa_p} = z^{p-1} + 2pb_2 z^p + [3pb_3 + 2p(p-1)b_2^{\,2}]z^{p+1}$$
$$+ \left[4pb_4 + p(p-1)6b_2b_3 + \frac{4p(p-1)(p-2)}{3} b_2^{\,3}\right] z^{p+2} + \ldots$$

Comparing coefficients we have

$$(p+1)\, a_{p+1} = 2p^2 b_2\, a_p$$
$$(p+2)\, a_{p+2} = [3pb_3 + 2p(p-1)b_2^2]\, a_p,$$
$$(p+3)\, a_{p+3} = [4pb_4 + 6p(p-1)b_2b_3 + (4/3)p(p-1)(p-2)b_2^3]a_p$$

and the result follows from the estimates for $|b_2|$, $|b_3|$, and $|b_4|$ after a short calculation.

We remark that if $k = 2$, we get the known results of Goodman [**4**] for p-valent convex functions; namely,

$$(p+1)|a_{p+1}| \leqq 2p^2|a_p|$$
$$(p+2)|a_{p+2}| \leqq (2p+1)p^2|a_p|$$
$$(p+3)|a_{p+3}| \leqq \tfrac{2}{3}(2p+1)(p+1)p|a_p|.$$

We omit the proof of the next lemma whose proof is similar to [**7**, Theorem 3.2] and [**8**, p. 7–10].

LEMMA 2.2. *Let* $g(z) = z + b_2 z^2 + \ldots \in V_k(p)$. *Then for any integer* $p \geqq 1$, $|3pb_3 - 2p(p-1)b_2^2| \leqq p^2k^2/2 - p$, *with equality for*

$$g(z) = \frac{1}{k}\left[\left(\frac{1+iz}{1-iz}\right)^{k/2} - 1\right].$$

THEOREM 2.3. *Let* $f(z) = a_{p-1}z^{p-1} + \ldots \in V_k(p)$. *Then*

$$(p+1)|a_{p+1}| \leqq p^2k|a_p| + (p-1)|a_{p-1}|(p^2k^2/2 - p + 1).$$

Proof. If $a_{p-1} = 0$, this reduces to Theorem 2.1 and our result is sharp in this case. We then assume that $a_{p-1} \neq 0$ and hence that $f(z)$ has a non-zero critical point z_0, since each function in $V_k(p)$ has precisely $p - 1$ critical points. Thus there is a function $g(z) = \sum_{n=1}^{\infty} b_n z^n \in V_k(1)$ such that

$$f'(z) = pa_p\, z^{p-2}\, (z - z_0)(1 - \bar{z}_0\, z)\, [g'(z)]^p.$$

We may assume without loss of generality that $b_1 = 1$. Let $[g'(z)]^p = \sum_{m=0}^{\infty} c_m z^m$. Then we have $c_0 = 1$, $c_1 = 2pb_2$, $c_2 = 3pb_3 + \binom{p}{2}4b_2^2$. Now

$$\sum_{n=p-1}^{\infty} na_n z^{n-1} = \sum_{m=0}^{\infty} [-\bar{z}_0 c_m + (1 + |z_0|^2)c_{m+1} - z_0 c_{m+2}]z^{m+p} + [(1 + |z_0|^2) - z_0 c_1]z^{p-1} - z_0 z^{p-2}$$

and thus comparing coefficients

$$\begin{aligned}(p - 1)a_{p-1} &= -z_0\\ pa_p &= (1 + |z_0|^2) - z_0 c_1\\ (p + 1)a_{p+1} &= -\bar{z}_0 + (1 + |z_0|^2)c_1 - z_0 c_2.\end{aligned}$$

Hence

$$\begin{aligned}(p + 1)a_{p+1} &= -\bar{z}_0 + c_1[(1 + |z_0|^2) - z_0 c_1] + z_0 c_1^2 - z_0 c_2\\ &= c_1 p\, a_p + (-z_0)[\bar{z}_0/z_0 + c_2 - c_1^2].\end{aligned}$$

Consequently

$$\begin{aligned}(p + 1)|a_{p+1}| &\leqq |c_1|p|a_p| + (p - 1)|a_{p-1}|[1 + |c_2 - c_1^2|]\\ &= 2p^2|b_2|a_p| + (p - 1)|a_{p-1}|[1 + |3pb_3 - 2p(p + 1)b_2^2|]\end{aligned}$$

and the result follows by Lemma 2.2, since $|b_2| \leqq k/2$.

We note that if $k = 2 + 2/p$, Theorem 2.3 yields the result

$$(p + 1)|a_{p+1}| \leqq 2p(p + 1)|a_p| + (p - 1)|a_{p-1}|[2(p + 1)^2 - p + 1]$$

which is certainly not sharp since (2.1) is known to be sharp for p-valently close-to-convex functions with $n = p + 1$. In order to obtain a sharp coefficient bound we restrict our attention to function $f(z) = a_{p-1}z^{p-1} + \ldots \in V_k(p)$ with real coefficients. The following lemma will be needed.

LEMMA 2.4. *Let $g(z) = z + b_2 z^2 + \ldots \in V_k$ have real coefficients. Then if $p \geqq 2$, $|1 + 3pb_3 - 2p(p + 1)b_2^2| \leqq p^2k^2/2 - p - 1$, and this result is sharp.*

Proof. By Lemma 2.2, $|3pb_3 - 2p(p + 1)b_2^2| \leqq p^2k^2/2 - p$, with equality for

$$g(z) = \frac{1}{k}\left[\left(\frac{1 + z}{1 - z}\right)^{k/2} - 1\right] = z + \frac{k}{2}z^2 + \frac{k^2 + 2}{6}z^3 + \ldots$$

for which $3pb_3 - 2p(p + 1)b_2^2 = p - p^2k^2/2$. Hence

$$1 + 3pb_3 - 2p(p + 1)b_2^2 \geqq 1 + p - p^2k^2/2.$$

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It remains to show that

$$1 + 3pb_3 - 2p(p+1)b_2^2 \leqq p^2k^2/2 - p - 1.$$

Suppose then that $3pb_3 - 2p(p+1)b_2^2 > p^2k^2/2 - p - 2$. Since $g \in V_k$, there is a function $\mu(t)$ of bounded variation on $[0, 2\pi]$ with

$$\int_0^{2\pi} d\mu(t) = 2 \text{ and } \int_0^{2\pi} |d\mu(t)| \leqq k$$

such that

$$g'(z) = \left[\int_0^{2\pi} \log\,(1 - ze^{-it})d\mu(t)\right].$$

A brief calculation shows that

$$3pb_3 - 2p(p+1)b_2^2 = \frac{p}{2}\left[\int_0^{2\pi} e^{-2it}d\mu(t) - p\left(\int_0^{2\pi} e^{-it}d\mu(t)\right)^2\right].$$

We have that

$$\int_0^{2\pi} e^{-it}d\mu(t) = 2b_2 \quad \text{is real,}$$

and hence

$$\frac{p^2k^2}{2} - p - 2 < \frac{p}{2}\int_0^{2\pi} e^{-2it}d\mu(t) \leqq \frac{pk}{2}$$

or,

(2.2) $\quad p^2k^2/2 - p(1 + k/2) - 2 < 0.$

The left hand side of (2.2) is an increasing function of p, ($p \geqq 2$) for any fixed value of $k \geqq 2$.

When $p = 2$, we have

$$2k^2 - 2(1 + k/2) - 2 \leqq 0,$$

which is impossible for any $k > 2$. Thus

$$1 + 3pb_3 - 2p(p+1)b_2^2 \leqq p^2k^2/2 - p - 1$$

and the result follows.

THEOREM 2.5. *Let* $f(z) = a_{p-1}z^{p-1} + \ldots \in V_k(p)\,(p > 2)$ *have real coefficients. Then*

$$(p+1)|a_{p+1}| \leqq p^2k|a_p| + (p-1)|a_{p-1}|(p^2k^2/2 - p - 1)$$

and there is a function in $V_k(p)$ *for which equality holds.*

Proof. If $f(z)$ has real coefficients, then $f(z)$ maps U onto a domain symmetric with respect to the real axis. Since $f(z)$ has precisely $p - 1$ critical points, f

has precisely one non-zero critical point z_0 which must be real since complex roots of the equation $f'(z) = 0$ occur in conjugate pairs. Using the notation of Theorem 2.3

$$(2.3)\quad (p+1)a_{p+1} = p\, a_p\, c_1 + (p-1)a_{p-1}[\bar{z}_0/z_0 + c_2 - c_1^2]$$
$$= 2pb_2\, a_p + (p-1)a_{p-1}[1 + 3pb_3 - 2p(p+1)b_2^2].$$

Since z_0 and the a_n are real, the c_n and hence the b_n are real. By Lemma 2.4, since the b_n are real,

$$|1 + 3pb_3 - 2p(p+1)b_2^2| \leqq p^2k^2/2 - p - 1.$$

Since $g(z) \in V_k$, $|b_2| \leqq k/2$ and the result follows.

To see that this result is sharp we consider

$$f'(z) = z^{p-2}(z - z_0)(1 - z_0 z)[g'(z)]^p,$$

where

$$g(z) = \frac{1}{k}\left[\left(\frac{1+z}{1-z}\right)^{k/2} - 1\right] \text{and } 0 < z_0 < \frac{pk - (p^2k^2 - 4)^{1/2}}{2}.$$

For this function it follows from (2.3) that

$$(p+1)a_{p+1} = p^2k\, a_p + (-z_0)[1 + p - p^2k^2/2]$$

and hence since $a_p > 0$ and $a_{p-1} < 0$,

$$(p+1)|a_{p+1}| = p^2k|a_p| + (p-1)|a_{p-1}|(p^2k^2/2 - p - 1).$$

We note that if $k = 2 + 2/p$, Theorem 2.5 reduces to a special case of the Goodman conjecture, which is known to be sharp, if correct.

3. Asymptotic coefficient estimates for $V_k(p)$.

THEOREM 3.1. *Let $f(z) \in V_k(p)$. Then*

$$\alpha = \lim_{r\to 1} (1 - r)^{\frac{1}{2}p(k+2)}\, M(r, f')$$

exists. If $\alpha > 0$, there is a unique θ_0 so that

$$\alpha = \lim_{r\to 1} (1 - r)^{\frac{1}{2}p(k+2)}\, |f'(re^{i\theta_0})|.$$

Proof. The result is known if $p = 1$ [**12**] and hence we may suppose $p \geqq 2$. If f has non-zero critical points $a_1, \ldots a_{p-q}$, then by Theorem 1.5 there are two univalent starlike functions $s_1(z)$ and $s_2(z)$ such that

$$f'(z) = z^{p-1} \prod_{j=1}^{p-q} \psi(z, a_j) \cdot \left[\frac{s_1(z)}{z}\right]^{\frac{1}{4}p(k+2)} \cdot \left[\frac{s_2(z)}{z}\right]^{-\frac{1}{4}p(k-2)}.$$

Since $z/s_2(z)$ and $\psi(z, a_j)$ are bounded near $|z| = 1$, it follows from [**14**] that

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$\alpha = 0$ unless $s_1(z) = z/(1 - e^{i\theta_0}z)^2$. Thus we may suppose that

$$\limsup_{r\to 1} (1 - r)^{\frac{1}{2}p(k+2)} M(r, f') > 0$$

and that $s_1(z)$ is of the form $z/(1 - e^{i\theta_0}z)^2$. We may assume without loss of generality that $\theta_0 = 0$.

Choose a sequence $r_n \to 1$ and a point z_n on $|z| = r_n$ with

$$\lim_{n\to\infty} (1 - r_n)^{\frac{1}{2}p(k+2)} |f'(z_n)| > 0.$$

We will show that the points z_n eventually lie interior to a fixed stoltz angle with vertex at $z = 1$. Suppose not. Then given $M > 0$, there is a subsequence z_j such that $|1 - z_j| > M(1 - r_j)$. If we set $L = \max |R(e^{i\theta})|$, where $R(z) = z^{p-1}\prod\psi(z, a_j)$, then for j sufficiently large we have

$$\begin{aligned} 2L\, 2^{\frac{1}{2}p(k-2)} &\geqq |R(z_j)| \left|\frac{z_j}{s_2(z_j)}\right|^{\frac{1}{4}p(k-2)} \\ &\geqq \left[(1 - r_j) \cdot \frac{M}{|1 - z_j|}\right]^{\frac{1}{2}p(k+2)} |R(z_j)| \left|\frac{z_j}{s_2(z_j)}\right|^{\frac{1}{4}p(k-2)} \\ &= M^{\frac{1}{2}p(k+2)} (1 - r_j)^{\frac{1}{2}p(k+2)} |f'(z_j)| \end{aligned}$$

which is impossible since $M > 0$ was arbitrary and

$$\lim_{n\to\infty} (1 - r_j)^{\frac{1}{2}p(k+2)} |f'(z_j)| > 0.$$

Since $s_2(z)$ is starlike in U, $\lim_{r\to 1} r/s_2(r)$ exists ([**14**] and the fact that

$$r\frac{\partial}{\partial r} (\log |f(re^{i\theta}|) = \operatorname{Re} \frac{zf'(z)}{f(z)} > 0)$$

and we have

$$\lim_{n\to\infty} \frac{z_n}{s_2(z_n)} = \lim_{n\to\infty} \frac{r_n}{s_2(r_n)}.$$

It follows that for any sequence z_n such that

$$\lim_{n\to\infty} (1 - r_n)^{\frac{1}{2}p(k+2)} |f'(z_n)| > 0,$$

$$\begin{aligned} \lim_{n\to\infty} |R(r_n)| \left|\frac{r_n}{s_2(r_n)}\right|^{\frac{1}{4}p(k-2)} &= \lim_{n\to\infty} |R(z_n)| \left|\frac{z_n}{s_2(r_n)}\right|^{\frac{1}{4}p(k-2)} \\ &\geqq \lim_{n\to\infty} \left(\frac{1 - r_n}{|1 - z_n|}\right)^{\frac{1}{2}p(k+2)} |R(z_n)| \left|\frac{z_n}{s_2(z_n)}\right|^{\frac{1}{4}p(k-2)} \\ &= \lim_{n\to\infty} (1 - r_n)^{\frac{1}{2}p(k+2)} |f'(z_n)|. \end{aligned}$$

Since $|f'(r)| \leqq M(r,f)$, we have

$$\lim_{n\to\infty} (1 - r_n)^{\frac{1}{2}p(k+2)} |f'(r_n)| = \lim_{n\to\infty} (1 - r_n)^{\frac{1}{2}p(k+2)} M(r_n, f')$$

and the result follows.

We note that

$$\alpha \leqq 2^{\frac{1}{2}p(k-2)} \max_{|z|=1} \left| \prod_{j=1}^{p-q} \psi(z, a_j) \right| .$$

THEOREM 3.2. *Let* $f(z) = \sum_1^\infty a_n z^n \in V_k(p)$. *Then*

$$\lim_{n\to\infty} \frac{|a_n|}{(n)^{\frac{1}{2}p(k+2)-2}} = \frac{\alpha}{\Gamma(\frac{1}{2}p(k+2))},$$

where α *is the constant of Theorem* 3.1.

Proof. The proof in the case $\alpha > 0$ follows by using the major-minor arc technique of Hayman [**5**, Theorem 5.7] as modified by Noonan [**12**].

Let us now consider the case $\alpha = 0$. Given $\epsilon > 0$, we may choose $r_0 < 1$ so that if $r_0 < r < 1$, f has no critical points and

$$\int_0^{2\pi} \left| \mathrm{Re}\left\{1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right\}\right| d\theta < (pk + \epsilon)\pi.$$

We may assume $p \geqq 2$ since the result is known for $p = 1$. An argument similar to that in [**2**] shows that

$$\text{(3.1)} \quad \int_0^{2\pi} r|f'(re^{i\theta})|d\theta < 2^{3/2} [2(pk + \epsilon) + 1]\pi M(r, f).$$

Since $M(r,f') = o(1 - r)^{-\frac{1}{2}p(k+2)}$,

$$\text{(3.2)} \quad M(r,f) = o(1 - r)^{-\frac{1}{2}p(k+2)+1}.$$

The result follows by using (3.1), (3.2) and the inequality

$$|a_n| < \frac{e}{n} \int_{r=1-1/n} |f'(re^{i\theta})|d\theta.$$

4. The class $V_k^*(p)$**.** We say that a function $f(z)$ meromorphic in U belongs to the class $V_k^*(p)$ if : $f'(z)$ has a finite number of zeros and poles in U, there is a $\rho < 1$ so that if $\rho < r < 1$,

$$\text{(4.1)} \quad \int_0^{2\pi} \mathrm{Re}\left\{1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right\} d\theta = -2p\pi$$

and

$$\text{(4.2)} \quad \limsup_{r\to 1^-} \int_0^{2\pi} \left|\mathrm{Re}\left\{1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right\}\right| d\theta \leqq pk\pi.$$

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Since f is meromorphic, each pole of f' must be of at least second order. We note that by the argument of Lemma 1.1,

$$\lim_{r\to 1^-} \int_0^{2\pi} \left| \operatorname{Re}\left\{1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right\}\right| d\theta \text{ exists,}$$

and that $V_2^*(p) = C(p)$, the class of p-valent meromorphic convex functions.

Pfaltzgraff and Pinchuk [**13**] defined the class Λ_k of meromorphic functions of bounded boundary rotation as the class of all functions

$$f'(z) = -\frac{1}{z^2}\exp\left[-\int_0^{2\pi} \log\,(1 - ze^{-it})d\mu(t)\right],$$

where

$$\int_0^{2\pi} d\mu(t) = 2, \int_0^{2\pi} |d\mu(t)| \leqq k \text{ and } \int_0^{2\pi} e^{-it}\,d\mu(t) = 0.$$

(The last condition ensures that f' does not have a simple pole at 0.) Since a function in Λ_k has no non-zero critical points, in general Λ_k is a proper subclass of $V_k^*(p)$.

We note that (4.1) implies that for $\rho < r < 1$, the argument of the vector tangent to $f(|z| = r)$ decreases by $2p\pi$ as θ increases from 0 to 2π ($z = re^{i\theta}$) and hence the curve $f(|z| = r)$ has at least p loops.

THEOREM 4.1. *Let* $f(z) \in V_k^*(p)$. *Then*:

$$p + 1 \leqq N(\infty, f') \leqq (pk + 2p + 4)/4$$

and

$$0 \leqq N(0, f') \leqq p(k - 2)/4.$$

Proof. We will show the inequalities hold in $|z| < r$, where r is chosen so that (4.1) holds. From (4.1) and the argument principle we obtain

$$N(0, f') - N(\infty, f) = -(p + 1)$$

and hence $f'(z)$ has at least $p + 1$ poles. Following Umezawa [**16**], we note that $N(w, f')$ is constant until w arrives at a value assumed by $f'(z)$ on $|z| = r$ and the jump of $N(w, f')$ at such a value must be an integer. Now

$$\int_0^{2\pi} \left|\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| d\theta \geqq \int_0^{2\pi} \left|\operatorname{Re}\frac{zf''(z)}{f'(z)}\right| d\theta - 2\pi$$

and hence if $\epsilon > 0$ is given we may choose $\rho < 1$ so that if $\rho < |z| < 1$,

$$(4.3)\quad 2\pi + (pk + \epsilon)\pi > \int_0^{2\pi} \left|\operatorname{Re}\frac{zf''(z)}{f'(z)}\right| d\theta \geqq [N(0, f') + N(\infty, f')]\,2\pi.$$

Since $N(0,f') - N(\infty,f') = -p + 1$, (4.3) yields

$$N(\infty, f') \leqq \frac{pk + 2p + 4}{4} + \frac{\epsilon}{4}$$

$$N(0, f') \leqq \frac{p(k-2)}{4} + \frac{\epsilon}{4}$$

and the result follows.

The next Lemma, due to Umezawa [**17**] will be used in estimating the valency of functions in $V_k^*(p)$.

LEMMA 4.2. *Let $f(z)$ be meromorphic for $|z| < R, f'(z) \neq 0$ on $|z| = R$. If*

$$\int_0^{2\pi} \left| \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| d\theta < 2\pi[M - N(\infty, f) + 1],$$

where [ ] denotes the greatest integer function, then f is at most M valent and at least max $[2N(\infty,f) - M, 1]$ *valent for* $|z| \leqq R$.

COROLLARY 4.3. *Let $f(z) \in V^*(p)$ have q poles in U. Then $f(z)$ is at least* max $[q + 1 - pk/2, 1]$ *valent and at most $pk/2 + q - 1$ valent in U.*

We note that if $k < 2 + 2/p$, then for r sufficiently near 1,

$$\frac{1}{2\pi}\int_0^{2\pi} \left| \operatorname{Re}\left\{1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right\}\right| d\theta < p + 1$$

and hence $f(z)$ belongs to the class $K^*(p)$ of meromorphic close-to-convex functions of order p defined by Livingston [**10**].

The following result is similar to Theorem 1.5 and its proof will be omitted.

THEOREM 4.4. *Let $f(z) \in V_k^*(p)$ and suppose $f'(z)$ has zeros at $\alpha_1 \ldots \alpha_n$ and poles at $\beta_1, \ldots \beta_{n+p+1}$, counting multiplicities. Then there are two univalent starlike functions $s_1(z)$ and $s_2(z)$ such that*

$$f'(z) = \frac{\dfrac{1}{z^{p+1}}\left[\displaystyle\prod_{j=1}^{n+p+1} \psi(z, B_j)\right]^{-1} \displaystyle\prod_{j=1}^{n} \psi(z, \alpha_j)\left[\frac{s_1(z)}{z}\right]^{\frac{1}{4}p(k-2)}}{\left[\dfrac{s_2(z)}{z}\right]^{\frac{1}{4}p(k+2)}}.$$

We note that Theorem 4.4 gives distortion theorems analogous to Theorems 1.6 and 1.7, but we do not state them here.

THEOREM 4.5. *Let $f(z) \in V_k^*(p)$. Then $\alpha = \lim_{r\to 1} (1 - r)^{\frac{1}{2}p(k-2)} M(r, f')$ exists. For $k > 2$, if $\alpha > 0$, there is a unique θ_0 such that*

$$\alpha = \lim_{r\to 1} (1 - r)^{\frac{1}{2}p(k-2)}|f'(re^{i\theta_0})|.$$

Proof. The proof is similar to Theorem 3.1, using Theorem 4.4 instead of Theorem 1.5.

We now turn to the problem of estimating the coefficients of a function $f(z) \in V_k^*(p)$.

THEOREM 4.6. *Let* $f(z) = \sum_{n=-q}^{\infty} a_n z^n$, *with* $k > 2 + 2/p$. *Then if* α *denotes the constant of* Theorem 4.5,

$$\lim_{n\to\infty} \frac{|a_n|}{n^{\frac{1}{2}p(k-2)-2}} = \frac{\alpha}{\Gamma[\frac{1}{2}p(k-2)]}.$$

Proof. The proof in the case $\alpha > 0$ follows by using the major-minor arc technique of Hayman [**5**, Theorem 5.7] as modified by Noonan [**12**]. Suppose $\alpha = 0$. Given $\epsilon > 0$ we may choose $r_0 < 1$ so that if $r_0 < r < 1$, (4.1) holds and

$$\int_0^{2\pi} \left|\operatorname{Re}\left\{1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right\}\right| d\theta < (pk + \epsilon)\pi.$$

Using an argument similar to that of Brannan and Kirwan [**2**], there is a constant $C = C(p, k)$ such that

$$\int_0^{2\pi} r|f'(re^{i\theta})|d\theta < C \cdot M(r, f).$$

Now

$$\begin{aligned} |f(re^{i\theta})| &\leqq \left|\int_{r_0}^{r} f'(\rho e^{i\theta})e^{i\theta}\, d\rho\right| + |f(r_0 e^{i\theta})| \\ &\leqq \int_{r_0}^{r} M(\rho, f')d\rho + |f(r_0 e^{i\theta})| \\ &= o(1-r)^{-\frac{1}{2}p(k-2)+1}, \end{aligned}$$

since $M(r, f') = o(1-r)^{-\frac{1}{2}p(k-2)}$. Therefore

$$\begin{aligned} r_1 I(r, f') &= \int_0^{2\pi} r|f'(re^{i\theta})|d\theta \\ &= o(1-r)^{-\frac{1}{2}p(k-2)+1} \end{aligned}$$

and the result follows from the standard inequality

$$|a_n| < \frac{e}{n} I_1\left(1 - \frac{1}{n}, f'\right).$$

We mention that Livingston [**10**] has shown that if $f(z) = \sum_{n=-p}^{\infty} a_n z^n$ belongs to the class $K^*(p)$ and has all its poles at $z = 0$, then $|a_n| = O(1/n)$. Consequently if $f(z) \in V_k^*(p)$ with $2 < k < 2 + 2/p$, we have $|a_n| = O(1/n)$. Since when $k = 2$, $V_k^*(p)$ is precisely the class of p-valent meromorphically convex functions and hence $|a_n| = O(1/n^2)$. To obtain an estimate on the growth of $|a_n|$ when $k = 2 + 2/p$ we note that $f(z) \in V_k^*(p)$ implies $f(z) \in V_{k'}^*(p)$ for all $k' > k$. Theorem 4.6 then yields that if $k = 2 + 2/p$, $|a_n| = O(n^{-1+\epsilon})$ for every $\epsilon > 0$.

References

1. D. A. Brannan, *On functions of bounded boundary rotation. I*, Proc. Edinburgh Math. Soc. *16* (1969), 339–347.
2. D. A. Brannan and W. E. Kirwan, *On some classes of bounded analytic functions*, J. London Math. Soc. *2* (1969), 431–443.
3. G. M. Goluzin, *Geometric theory of functions of a complex variable* (Amer. Math. Soc., Providence, R.I., 1969).
4. A. W. Goodman, *On the Schwarz-Christoffel transformation and p-valent functions*, Trans. Amer. Math. Soc. *68* (1950), 204–223.
5. W. K. Hayman, *Multivalent functions* (Cambridge University Press, Cambridge, 1958).
6. J. A. Hummel, *Extremal properties of weakly starlike p-valent starlike functions*, Trans. Amer. Math. Soc. *130* (1968), 544–551.
7. R. J. Leach, *Odd functions of bounded boundary rotation*, Can. J. Math. *26* (1974), 551–564.
8. O. Lehto, *On the distortion of conformal mappings with bounded boundary rotation*, Ann. Acad. Sci. Fenn. Ser. A, 1, *124* (1952), 14 pp.
9. A. E. Livingston, *p-valent close-to-convex functions*, Trans. Amer. Math. Soc. *115* (1965), 161–179.
10. ——— *Meromorphic multivalent close-to-convex functions*, Trans. Amer. Math. Soc. *119* (1965), 167–177.
11. ——— *The coefficients of multivalent close-to-convex functions*, Proc. Amer. Math. Soc. *21* (1969), 545–552.
12. J. Noonan, *Asymptotic behavior of functions with bounded boundary rotation*, Trans. Amer. Math. Soc. *164* (1972), 397–410.
13. J. A. Pfaltzgraff and B. Pinchuk, *Constrained extremal problems for classes of meromorphic functions*, Bull. Amer. Math. Soc. *75* (1969), 379–383.
14. C. Pommerenke, *On convex and starlike functions*, J. London Math. Soc. *37* (1962), 209–224.
15. M. Schiffer and O. Tammi, *On the fourth coefficient of univalent functions with bounded boundary rotation*, Ann. Acad. Sci. Fenn. Ser. A, 1, *396* (1967), 26 pp.
16. T. Umezawa, *Analytic functions convex in one direction*, J. Math. Soc. Japan *4* (1952), 194–202.
17. ——— *On the theory of univalent functions*, Tohoku Math. J. *7* (1955), 212–223.

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