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TOTALLY NULL SETS FOR $A(X)$

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Abstract

For a compact subset K of the boundary of a compact Hausdorff space X, six properties that K may have in relation to the algebra $A(X)$ are considered. It is shown that in relation to the algebra $A(D^n)$, where D^n denotes the n-dimensional polydisc, the property of being totally null is weaker than the other five properties. A general condition is given on the algebra $A(X)$ which implies the existence of a totally null set that is not null, and several conditions are stated for $A(X)$, each of which is sufficient for a totally null set to be a null set.

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1. Introduction

For a compact Hausdorff space X, let $C(X)$ denote the space of continuous complex-valued functions on X with the supremum norm. A *function algebra* A on X is a closed subalgebra of $C(X)$ that contains the constant functions and separates the points of X. The set of all multiplicative linear functionals on A is called the *maximal ideal space* of A and is denoted by M_A. A *representing measure* for $\phi \in M_A$ is a positive measure μ on X such that $\phi(f) = \int_X f\, d\mu$ for all $f \in A$. We denote the set of representing measures for ϕ by M_ϕ. For any ϕ in M_A, the set M_ϕ is a nonempty convex weak*-compact set. A measure that represents the evaluation functional ϕ_x at a point x in X is simply called a representing measure for x. An *annihilating measure* for A is a regular complex Borel measure μ on X that satisfies $\int_X f\, d\mu = 0$ for all $f \in A$.

Throughout this paper, we shall be concerned with the algebra $A(X)$ of functions continuous on X and holomorphic on the interior of X. In [7], Rudin presents six properties that a compact subset of the unit sphere S_n in $\mathbb{C}^n$ can have in relation to the ball algebra $A(B_n)$. Below we define these properties for the algebra $A(X)$, where X is a compact subset of a Banach space, x_0 is a fixed element in M_A, and K is a compact subset of X.

Definition 1.1.

(1) K is a Z-set (*zero set*) if K is the zero set of a function f in the algebra $A(X)$.

(2) K is a P-set (*peak set*) if there is an f in the algebra $A(X)$ such that $f(x) = 1$ for every $x \in K$ and $|f(y)| < 1$ for every $y \in X \setminus K$.
(3) K is an I-set (*interpolation set*) if every complex-valued continuous function on K extends to a member of $A(X)$.
(4) K is a PI-set (*peak-interpolation set*) if K is a peak set and an interpolation set.
(5) K is an N-set (*null set*) if $|\nu|(K) = 0$ for every measure ν that annihilates $A(X)$. (Here $|\nu|$ is the total variation of ν.)
(6) K is a TN-set (*totally null set*) if $\rho(K) = 0$ for every representing measure ρ for x_0.

All of these properties are equivalent when X is

$$B_n = \Big\{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^2 \le 1\Big\},$$

the unit ball in $\mathbb{C}^n$, x_0 is zero, and K is a compact subset of the unit sphere S_n; for a proof of this see [7, Theorem 10.1.2]. In the case where X is

$$D^n = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |z_j| \le 1,\ j = 1, 2, \ldots, n\},$$

the unit polydisc in $\mathbb{C}^n$, x_0 is zero, and K is a compact subset of the distinguished boundary

$$T^n = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |z_j| = 1,\ j = 1, 2, \ldots, n\},$$

then the first five properties are known to be equivalent; see [6] or [9] for a proof.

In Section 3 we give a simple counterexample that shows the property of being a TN-set is weaker than the other five properties in the case of the n-dimensional and infinite-dimensional polydiscs. In Section 4 we give a general condition on an algebra such that the property of being a TN-set is weaker than the property of being an N-set. Finally, in Section 5 we present several conditions on an algebra that are sufficient for TN-sets to be N-sets.

2. Preliminaries

Definition 2.1. Let A be a function algebra on X, and let ϕ, ψ be elements of M_A. We say that ϕ and ψ are in the same *Gleason part* (or *part*) of M_A if $\|\phi - \psi\|_{A^*} < 2$.

Theorem 2.2 [1, Theorem 2.6.3]. *For $\phi, \psi \in M_A$, write $\phi \sim \psi$ if ϕ and ψ are in the same Gleason part. Then $\sim$ is an equivalence relation on M_A.*

An important property of the parts of M_A is that if ϕ and ψ lie in distinct parts, then every representing measure for ϕ is singular to every representing measure for ψ. It turns out that the converse is also true.

Theorem 2.3 [4, Theorem VI.1.1]. *Suppose ϕ and ψ belong to the same part of M_A. Then there are mutually absolutely continuous representing measures for ϕ and ψ.*

Consequently, two elements ϕ and ψ of M_A lie in the same part if and only if they have mutually absolutely continuous representing measures.

Definition 2.4. If a Gleason part consists of only one element of M_A, then it is called a *trivial part*. Otherwise, it is called a *nontrivial part*.

The following definition provides examples of trivial Gleason parts.

Definition 2.5. A peak set that consists of a single point is called a *peak point*.

Theorem 2.6 [4, Theorem II.11.3]. *Let A be a function algebra on a compact metric space X. Then $x \in X$ is a peak point for A if and only if the point mass at x is the only representing measure for x. This occurs if and only if $\mu(\{x\}) = 1$ for every representing measure for x.*

It follows from Theorems 2.3 and 2.6 that every peak point for the algebra A is a trivial part.

Example 2.7. The maximal ideal space $M_{A(D)}$ of the disc algebra $A(D)$ is the closed unit disc $D \subset \mathbb{C}$. If w is any point in the unit circle S, then the function $f : D \to \mathbb{C}$ defined by $f(z) = \frac{1}{2}(1 + z\bar{w})$ is in the disc algebra $A(D)$ and peaks at w. If z and w are points in the open unit disc $\mathrm{Int}(D)$, then Cauchy's formula gives representing measures for z and w that are mutually absolutely continuous. Consequently, the Gleason parts of $M_{A(D)}$ are the trivial parts consisting of points in the unit circle and the nontrivial part consisting of the open unit disc.

We will make use of the following lemma in Section 4.

Lemma 2.8 [7, Rainwater's lemma]. *Let X be a compact Hausdorff space and M a nonempty convex weak*-compact subset of the regular Borel measures on X. Suppose ν is a measure on X and $\nu \perp \rho$ for every $\rho \in M$. Then ν is concentrated on an F_σ set E of X, such that $\rho(E) = 0$ for every $\rho \in M$. (Recall that to say ν is concentrated on E means $|\nu|(X \setminus E) = 0$.)*

3. TN is a weaker property than N for $A(D^n)$

For a compact set $K \subset T^n$ and $x_0 = 0$ the five properties Z, P, I, PI and N are equivalent. For a proof of this see [6, Theorem 6.1.2]. We show that TN is a weaker property than the others.

Theorem 3.1. *For a compact set $K \subset T^n$ and $x_0 = 0$ each of the five properties Z, P, I, PI and N implies TN, but TN is not equivalent to these properties.*

Proof. We show that P implies TN (following the proof in [6] for $A(B_n)$), and construct a counterexample to show that TN does not imply Z.

To show P implies TN: if $f \in A(D^n)$ peaks on K, and ρ is a representing measure for zero, then for $m = 1, 2, 3, \ldots$, we have $f^m(0) = \int_{D^n} f^m \, d\rho$. The integral converges to $\rho(K)$ as $m \to \infty$ by Lebesgue's dominated convergence theorem, and $f^m(0) \to 0$ since $|f(0)| < 1$. Thus $\rho(K) = 0$.

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To show TN does not imply Z: let D be the closed unit disc and T be the unit circle. We show TN does not imply Z for the bidisc $D \times D$; it is clear how to generalise the proof to the n-dimensional polydisc D^n. Consider the set $(\{z_0\} \times T) \subset (D \times D)$ where $z_0 \in T$.

Because z_0 is in the unit circle T, z_0 is a peak point for $A(D)$. So there is a function $g \in A(D)$ with $g(z_0) = 1$ and $|g(z)| < 1$ for all $z \in D \setminus \{z_0\}$. Then any representing measure ρ for $x_0 = 0$ satisfies

$$\rho(\{z_0\} \times T) = \lim_{n\to\infty} \int_{D\times D} g(z)^n \, d\rho(z, w) = \lim_{n\to\infty} g^n(0) = 0.$$

So $\{z_0\} \times T$ is a TN-set.

Suppose $f \in A(D \times D)$ and $f(z_0 \times T) = 0$. Then $f(z_0, \cdot)$ is in $A(D)$, and $f(z_0, \cdot) = 0$ on T. But then, by the maximum modulus principle, $f(z_0, \cdot) \equiv 0$ on D. So $\{z_0\} \times T$ is not a Z-set because we cannot find a function $f \in A(D \times D)$ with $f = 0$ only on $\{z_0\} \times T$. □

The proof for the finite-dimensional polydisc carries through to the infinite-dimensional polydisc $D^\infty = \{(z_1, z_2, \ldots) \in \ell^\infty : |z_i| \leq 1 \text{ for } i = 1, 2, \ldots\}$ with only minor changes. (In this case, we let $A(D^\infty)$ be the algebra of complex functions weak*-continuous on D^∞ whose restriction to the interior of D^∞ is holomorphic.) So in the infinite-dimensional case the five properties Z, P, I, PI and N are also equivalent, and TN is a weaker property.

4. Condition for a TN-set that is not an N-set

We now present a condition on a function algebra A which implies the existence of a TN-set that is not an N-set.

Theorem 4.1. *Suppose A is a function algebra on X and x_o is an element of X. Let Q be the Gleason part of x_o in M_A. If M_A contains a nontrivial Gleason part other than Q, then there exists a compact subset K of X that is a TN-set but not an N-set with respect to A.*

Proof. Suppose there is an element $y \in X$ such that y belongs to a nontrivial Gleason part other than Q. Let μ_y be a representing measure for y that is not the point mass at y. Then since x_o and y are in different Gleason parts, the measure μ_y is singular to every representing measure for x_o. By Lemma 2.8, there is an F_σ-set E of X such that $\rho(E) = 0$ for all $\rho \in M_{x_o}$, while $|\mu_y|(X \setminus E) = 0$. Since μ_y is not the point mass at $\{y\}$, there exists some $\varepsilon > 0$ for which $\mu_y(E) > \mu_y(\{y\}) + \varepsilon$. Choose a compact set $K \subset E$ for which $\mu_y(K) \geq \mu_y(\{y\}) + \varepsilon$. Then it must be that $K \neq \{y\}$.

Choose an $f \in A$ with $f(y) = 0$ and $f \not\equiv 0$ on $K \cap \operatorname{supp}(\mu_y)$, where $\operatorname{supp}(\mu_y)$ denotes the support of μ_y. Then $f\mu_y$ annihilates A and $f\mu_y(K) \neq 0$, and, because K is a subset of E, $\rho(K) = 0$ for all $\rho \in M_{x_o}$. Thus K is a subset of X that is a TN-set but not an N-set. □

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It is an open question whether the converse of Proposition 4.1 is true. In other words, even if M_A has no nontrivial Gleason parts besides Q, we do not know that TN-sets are N-sets. However, there are several properties of function algebras which do imply that a TN-set is an N-set with respect to the algebra.

5. Sufficient conditions for TN-sets to be N-sets

We state several properties that an algebra can have, each of which is a sufficient condition for a TN-set to be an N-set.

For the remainder of the section we let X be a dual Banach space and U be an open subset of X with weak*-compact closure. Let $C(\bar{U})$ be the algebra of functions that are weak*-continuous on the closure $\bar{U}$ of U, and let $A = A(\bar{U})$ be the algebra of functions that are weak*-continuous on $\bar{U}$ and holomorphic in U. Let $M(\partial U)$ be the space of finite regular complex Borel measures on the boundary ∂U of U. Fix a point x_0 in U. We say a subset K of $\bar{U}$ is a TN-set if K is totally null for any representing measure for x_0.

Definition 5.1. A bounded sequence f_n in A is a *Montel sequence* if $f_n(z) \to 0$ for every $z \in U$. A measure $\mu \in M(\partial U)$ is called a *Henkin measure* if $\int f_n \, d\mu \to 0$ for every Montel sequence f_n in A.

Example 5.2. If z is a point in U, then every representing measure for z is a Henkin measure. Also, every annihilating measure is a Henkin measure.

In 1968 Henkin proved the following for the unit ball B_n in $\mathbb{C}^n$.

Theorem 5.3 [5, Theorem 4]. *If λ is a Henkin measure on the sphere S_n, and μ is absolutely continuous with respect to λ, then μ is a Henkin measure.*

Definition 5.4. We say that *Henkin's theorem holds for A* if every measure absolutely continuous with respect to a Henkin measure is also a Henkin measure.

In 1972 Cole and Range [3] considered the case where U is a relatively compact domain in a complex manifold, and U has a smooth strictly pseudoconvex boundary. They showed that when Henkin's theorem holds for the algebra A, then every Henkin measure is absolutely continuous with respect to some representing measure in $M(\partial U)$ for a point z in U.

Definition 5.5. We say the *Cole–Range theorem holds for A* if every Henkin measure is absolutely continuous with respect to a representing measure for x_0.

If the Cole–Range theorem holds for the algebra A, then clearly TN-sets for A are N-sets for A, because every annihilating measure is a Henkin measure.

Definition 5.6. A $(0, 1)$-form f on U is *$\bar{\partial}$-closed* if $\bar{\partial} f = 0$, and f is *$\bar{\partial}$-exact* if there exists a $g \in C(\bar{U})$ with $\bar{\partial} g = f$. (Recall that $\bar{\partial} f = \partial f / \partial \bar{z}$.) The *$\bar{\partial}$-equation is solvable on U* if the closed forms are exact. More precisely, the $\bar{\partial}$-equation is solvable on U if, for any $(0, 1)$-form f with $\bar{\partial} f = 0$, there is a function $g \in C(\bar{U})$ that satisfies $\bar{\partial} g = f$.

Let $L_{(0,1)}(U)$ denote the $\bar{\partial}$-closed (0, 1)-forms on U. We say the *$\bar{\partial}$-equation is solvable on U with a gain in smoothness* if there is a linear operator $S : L_{(0,1)}(U) \to C(\bar{U})$ so that:

(i) if $g = S(f)$, then $\bar{\partial} g = f$ on U; and
(ii) if $\{f_n\}$ converges weak* to 0 in $L_{(0,1)}(U)$, then $\{S(f_n)\}$ converges uniformly to 0 in $C(\bar{U})$.

Cole and Range [3] showed that if U is a set for which the $\bar{\partial}$-equation is solvable with a gain in smoothness, then the Cole–Range theorem holds for $A(\bar{U})$. An example of a set for which the $\bar{\partial}$-equation is solvable with a gain in smoothness is the open unit ball B_n in $\mathbb{C}^n$. In 1982, Cole and Gamelin [2] introduced a generalisation of this property, which they called tightness.

Definition 5.7 [2]. The algebra A is a *tight algebra on* $\bar{U}$ if the *Hankel-type operator* $S_g : A \to C(\bar{U})/A$ defined by

$$S_g(f) = fg + A$$

is weakly compact for every $g \in C(\bar{U})$.

As observed by Cole and Gamelin in [2], the following are all tight algebras:

(i) $A(\bar{U})$ where U is any bounded open subset of the complex plane;
(ii) $R(K)$ where K is any compact subset of the complex plane, and, more generally, T-invariant algebras on K (see [2, Section 17] for the definition of T-invariance);
(iii) $A(\bar{U})$ where U is any domain in $\mathbb{C}^n$ on which the $\bar{\partial}$-equation is solvable with a gain in smoothness; and
(iv) $A(\bar{U})$ where U is a strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary.

Definition 5.8 [2]. The algebra A is a *strongly tight algebra on* $\bar{U}$ if the Hankel-type operator S_g is norm compact for every $g \in C(\bar{U})$.

If A is a strongly tight algebra, then clearly A is a tight algebra. In 1995 Saccone [8] showed that if A is a tight algebra, then Henkin's theorem holds for A. Thus each of the following conditions implies the next.

(1) A is a strongly tight algebra.
(2) A is a tight algebra.
(3) Henkin's theorem holds for A.
(4) The Cole–Range theorem holds for A.
(5) TN-sets for A are N-sets for A.

One consequence of these implications is that any algebra that satisfies the hypothesis of Theorem 4.1 fails to have each of these properties. In particular, the finite-dimensional and infinite-dimensional polydisc algebras fail to have these properties. We can also conclude that the $\bar{\partial}$-equation is not solvable with a gain in smoothness (as described in Definition 5.6) on the finite-dimensional or the infinite-dimensional open polydisc.

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