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INTERACTION OF INTERNAL WAVES IN A CONTINUOUS THERMOCLINE MODEL

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Abstract

Weak nonlinear interactions are studied for systems of internal waves when the Brunt-Väisälä frequency is proportional to $\operatorname{sech}^2 z$, where $z = 0$ is the centre of the thermocline. Explicit results expressed in terms of gamma functions have been obtained for the interaction coefficients appearing in the amplitude evolution equations. The cases considered include resonant triads as well as second and third harmonic resonance. In the non-resonant situation, the Stokes frequency correction due to finite-amplitude effects has been computed and the extension to wave packets is outlined. Finally, the effect of a mean shear on resonant interactions is discussed.

1. Introduction

Thermocline regions of the ocean and other large bodies of water are known to be sites of considerable internal wave activity. Such waves are believed to play an important role in the vertical transport of momentum and energy. Consequently, there is at present much interest in semi-empirical models describing the spectral distribution of internal wave energy in the ocean. Weak nonlinear interactions can significantly influence the distribution of energy within the spectrum and it is such interactions that occupy the principal part of this paper.

Resonant interactions, in particular, appear to be significant in the foregoing context. Although there have been a number of analytical studies of resonantly interacting internal waves, the vast majority have dealt with the case where N^2, the Brunt-Väisälä frequency, is constant (*e.g.* Phillips [13]). This quantity is

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defined by

$$N^2 = -\frac{g}{\bar{\rho}} \frac{d\bar{\rho}}{dz}, \tag{1.1}$$

$\bar{\rho}$ being the mean density.

The constant N^2 representation is particularly inaccurate in thermocline regions where N^2 is often sharply peaked. A profile more appropriate to this region which is employed in the present study, is $N^2 = g\beta \operatorname{sech}^2 z$, corresponding to the dimensionless density profile $\bar{\rho} = \exp(-\beta \tanh z)$. In the unbounded case, a closed-form solution to the linear problem is known for this profile. There are an infinite number of modes, the first four of which are tabulated in Section 2.

Given the existence of a closed-form solution to the linear problem it is natural to employ it in a finite-amplitude study, where the linear solution becomes the first term in a perturbation expansion in powers of ε, a dimensionless amplitude parameter. This is the approach due originally to Stokes and it was employed in the context of internal waves by Thorpe [16] who determined the first effects of nonlinearity on the wave profile. By continuing to the next term in the expansion the first correction to the frequency due to nonlinearity can be computed and this we have done in Section 5 of the present article.

Returning to the subject of resonant interactions, the past investigation most related to our own was the analytical and experimental study of Davis and Acrivos [3] on interacting waves in a pycnocline region. These authors also employed the profile $N^2 = g\beta \operatorname{sech}^2 z$, but in order to conform with their experimental configuration they treated the finite-depth case. Closed-form solutions to the linear eigenvalue problem are then no longer available, so they used approximate solutions valid for long wavelengths. It will be seen below that some interesting special cases are excluded as a result.

Although from a mathematical viewpoint resonant interaction theory describes an energy sharing process, it was found in the experiments reported in [3] that localized instabilities often resulted. Similar observations for standing waves in a roughly constant N^2 fluid were reported by McEwan [8]. Hence, the theory is relevant to the breaking and subsequent dissipation of internal waves. For two-dimensional, horizontally propagating waves it is well-known that the necessary conditions for triad resonance among three waves with wavenumbers k_j and frequencies ω_j, $j = 1, 2, 3$, are

$$k_1 \pm k_2 \pm k_3 = 0 \quad \text{and} \quad \omega_1 \pm \omega_2 \pm \omega_3 = 0. \tag{1.2}$$

Denoting the corresponding wave amplitudes by A_j, it is found that the evolution of the system is governed by

$$\frac{dA_1}{d\tau} = i\gamma_1 A_2^* A_3, \qquad \frac{dA_2}{d\tau} = i\gamma_2 A_1^* A_3$$

and

$$\frac{dA_3}{d\tau} = i\gamma_3 A_1 A_2, \tag{1.3}$$

where $\tau = \varepsilon t$ is a slow time scale and * denotes the complex conjugate. The coupled equations describe a sharing and exchange of the total energy as discussed, for example, in the survey article by Phillips [14]. To interpret these equations in a stability context (in keeping with the experimental observation of [3] and [8]), one takes A_2 and $A_3 \ll A_1$ and linearizes equations (1.3) on that basis. If γ_2 and γ_3 have the appropriate signs, then A_2 and A_3 can exhibit exponential growth for intervals of time such that they remain small in comparison to A_1. For the purposes of this investigation, however, all the A_j will be regarded as $O(1)$ in magnitude.

A particularly interesting interaction involving only two waves can occur if a pair of modes have the same phase speed, but their wavenumbers differ by a factor of two. This phenomenon, known as "second harmonic resonance", has been analyzed previously in connection with capillary-gravity waves by Simmons [15] and McGoldrick [11]. In Section 4, its occurrence in the internal wave context is illustrated with the present thermocline model. The reader familiar with this topic will recognize that second harmonic resonance can be viewed as a special case of triad resonance in which two of the waves coincide; we will, in fact, take advantage of that viewpoint in order to condense the mathematical presentation. At the same time, however, it should be pointed out that this case offers significant advantages in making comparisons with experiment due to the gain in simplicity of having only to deal with two wave components. This was noted by McGoldrick [10] in his surface wave experiments and it is equally true in the numerical work reported in Section 4, where the finite-amplitude results are compared with solutions of the full nonlinear governing equations.

Before proceeding to the analysis, we cite briefly the somewhat related work on vertical propagation of internal waves through a thermocline region. The latter problem has been treated by Mied and Dugan [12] using a modification of the density profile employed here, namely, $N^2 = N_0^2(1 + B\,\mathrm{sech}^2\,\sigma z)$. By having a constant, but nonzero, stratification outside of the pycnocline region a solution oscillatory in the z-direction can be obtained and interpreted as a vertically propagating wave. These authors were able to choose values for the constants B and σ that enabled a good fit to be achieved with observational data. It is also possible to treat this sort of problem in the framework of resonant interaction theory. Cases in which the interaction coefficients, γ_j in (1.3), vanish are equivalent to what Phillips [13] has termed a "window". It will be seen below that such cases arise frequently with the density profile employed here, which is effectively a horizontal wave-guide model.

2. Basic equations

We consider the propagation of two-dimensional waves in a stratified fluid with a length scale l characterizing the thickness of the stratified region. Outside of this wave-guide region the fluid is taken to have a constant density. Making a Boussinesq-like approximation the dimensionless equations of motion governing the density and vorticity are

$$\rho_t + \mathcal{J}(\rho, \psi) = 0 \tag{2.1}$$

and

$$\Delta\psi_t + \{\mathcal{J}(\Delta\psi, \psi)\} - g\rho_x = 0, \tag{2.2}$$

where $\mathcal{J}$ is the Jacobian with respect to x and z and Δ is the Laplacian operator. The stream function ψ is related to the velocity components by $u = \psi_z$ and $v = -\psi_x$. Time has been scaled with N_0, a characteristic value of the Brunt-Väisälä frequency, whereas the velocity components are nondimensionalized with respect to $N_0 l$.

A perturbation approach will be employed in which ψ and ρ are expanded in powers of ε, a dimensionless amplitude parameter. The wave amplitudes are assumed to vary on a slow time scale $\tau = \varepsilon t$, this scaling being appropriate to resonantly interacting monochromatic waves. The modifications required to deal with wave packets will be outlined subsequently.

We introduce scaled perturbation quantities by writing $\psi = \varepsilon\hat{\psi}(x, z, t, \tau)$, $\rho = \varepsilon\hat{\rho}(x, z, t, \tau)$ and transform time derivatives according to

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial \tau}.$$

It is then found that $\hat{\psi}$ and $\hat{\rho}$ satisfy

$$\hat{\rho}_t - \bar{\rho}'\hat{\psi}_x + \varepsilon(\hat{\rho}_x\hat{\psi}_z - \hat{\psi}_x\hat{\rho}_z) = 0, \tag{2.3}$$

and

$$\Delta\hat{\psi}_{tt} + N^2\hat{\psi}_{xx} + \varepsilon\left[2\Delta\hat{\psi}_{t\tau} + \{\mathcal{J}(\hat{\rho}, \hat{\psi})\}_x + \{\mathcal{J}(\Delta\hat{\psi}, \hat{\psi})\}_t\right] = 0. \tag{2.4}$$

To recover the already known results of linear theory, we set $\varepsilon = 0$ and write

$$\hat{\psi} = \Phi_1(z)E \quad \text{and } \hat{\rho} = P_1(z)E, \tag{2.5}$$

where

$$E = \exp\{i(kx - \omega t)\}. \tag{2.6}$$

After substituting into (2.3), (2.4) and separating variables, it is found that P_1 can be eliminated from the system and Φ_1 satisfies the ODE

$$\mathcal{L}\Phi_1 \equiv \Phi_1'' - k^2\Phi_1 + (g\beta/c^2)\operatorname{sech}^2 z\,\Phi_1 = 0, \tag{2.7}$$

where the phase speed $c = \omega/k$. For the unbounded flow considered here, $\Phi_1 \to 0$ as $z \to \pm\infty$.

This eigenvalue problem was first solved in 1948 by P. Groen who transformed (2.7) into the hypergeometric equation (alternatively, the change of variable $T = \tanh z$ leads to the associated Legendre equation); a more accessible reference, however, is the paper of Thorpe [16]. A complete set of modes exists and for future reference the first four of these are given in Table 1.

TABLE 1. The first four internal wave modes.

Mode Number	Φ_1	Dispersion Relation	Symmetry of Φ_1
1	S^k	$c^2 = \dfrac{g\beta}{k(1+k)}$	Even
2	S^kT	$c^2 = \dfrac{g\beta}{(1+k)(2+k)}$	Odd
3	$S^k(1 - \dfrac{k+3/2}{k+1}S^2)$	$c^2 = \dfrac{g\beta}{(2+k)(3+k)}$	Even
4	$S^kT(1 - \dfrac{k+5/2}{k+1}S^2)$	$c^2 = \dfrac{g\beta}{(3+k)(4+k)}$	Odd

The notation $S = \operatorname{sech} z$ and $T = \tanh z$ has been used in expressing the results in Table 1 and will be employed in the analysis to follow.

3. Triad interactions

To derive the evolution equations (1.3), we expand $\hat{\psi}$ and $\hat{\rho}$ in the following perturbation series

$$\varepsilon\hat{\psi} \sim \varepsilon\psi^{(1)} + \varepsilon^2\psi^{(2)} + \cdots, \qquad \varepsilon\hat{\rho} \sim \varepsilon\rho^{(1)} + \varepsilon^2\rho^{(2)} + \cdots,$$

and substitute these series into (2.3) and (2.4). The lowest-order terms satisfy the equations of linear theory and are expressed as a superposition of normal modes in the form

$$\psi^{(1)} = \sum_{j=1}^{3} \{A_j(\tau)\Phi_{1j}(z)E_j + *\}$$

and

$$\rho^{(1)} = \sum_{j=1}^{3} \{A_j(\tau)P_{1j}(z)E_j + *\},$$

where $E_j = \exp\{i(k_jx - \omega_jt)\}$ and * denotes complex conjugate. Without loss of generality, we consider the case where the k_j and ω_j are all positive so that the

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waves propagate to the right and the resonance conditions satisfied are

$$k_1 + k_2 = k_3 \quad \text{and} \quad \omega_1 + \omega_2 = \omega_3. \tag{3.1}$$

Substituting now into (2.3) and (2.4) leads at $O(\varepsilon^2)$ to equations for $\rho^{(2)}$ and $\psi^{(2)}$ and (2.4) yields

$$\Delta\psi_{tt}^{(2)} + g\beta\,\mathrm{sech}^2 z\,\psi_{xx}^{(2)} = -2\Delta\psi_{t\tau}^{(1)} + \left\{\mathcal{J}\left(\psi^{(1)}, \rho^{(1)}\right)\right\}_x - \left\{\mathcal{J}\left(\Delta\psi^{(1)}, \psi^{(1)}\right)\right\}_t. \tag{3.2}$$

After examining the right hand side of (3.2), it becomes clear that in order to separate variables, $\psi^{(2)}$ must have the form

$$\psi^{(2)} = A_2^* A_3 \Phi_{21} E_1 + A_1^* A_3 \Phi_{22} E_2 + A_1 A_2 \Phi_{23} E_3 + * + \text{nonresonant terms}, \tag{3.3}$$

and the amplitudes A_j satisfy (1.3).

By employing the resonance conditions (3.1) and the separation of variables (3.3), it is found after a moderate amount of algebra that (3.2) yields the following equations for the quantities Φ_{2j}:

$$\begin{aligned}\mathcal{L}\Phi_{2j} &= 2\gamma_j\omega_j\left(g\beta S^2/c_j^2\right)\Phi_{1j} \\ &\quad + \left(g\beta S^2/c_i^2c_3^2\right)\left\{2\omega_j^2(c_i + c_3)T\Phi_{1i}\Phi_{13}\right. \\ &\quad\quad \left. + (c_i - c_3)\left[k_j c_i c_3 + \omega_j(c_i + c_3)\right]\left(k_i\Phi_{1i}\Phi_{13}' + k_3\Phi_{13}\Phi_{1i}'\right)\right\},\end{aligned} \tag{3.4}$$

where $j = 1$ or 2 and $i = j - (-1)^j$; Φ_{23} satisfies

$$\begin{aligned}\mathcal{L}\Phi_{23} &= 2\gamma_3\omega_3\left(g\beta S^2/c_3^2\right)\Phi_{13} \\ &\quad + \left(g\beta S^2/c_1^2c_2^2\right)\left\{2\omega_3^2(c_1 + c_2)T\Phi_{11}\Phi_{12}\right. \\ &\quad\quad \left. + (c_2 - c_1)\left[c_1c_2k_3 + \omega_3(c_1 + c_2)\right]\left(k_1\Phi_{11}\Phi_{12}' - k_2\Phi_{12}\Phi_{11}'\right)\right\}.\end{aligned} \tag{3.5}$$

The γ_j are determined by multiplying both sides of equations (3.4) and (3.5) by their appropriate adjoint functions and integrating from $z = -\infty$ to $z = \infty$. This procedure is facilitated in the present case by the fact that the operator $\mathcal{L}$ is self-adjoint; hence, Φ_{13}, for example, is the required adjoint in imposing the orthogonality condition upon equation (3.5). Because of the symmetry of the Φ_{ij}, we can determine by inspection whether or not a particular $\gamma_j = 0$. There are two cases in which resonance ($\gamma_j \neq 0$) takes place; these are when (i) all three eigenfunctions are odd, or (ii) two eigenfunctions are even, while the third is odd. In all other cases, the γ_j are zero.

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When there is a resonance, the interaction coefficient γ_1, for example, is given by

$$\gamma_1 = -\frac{c_1^2 \int_0^1 \{\ \} \Phi_{11} dT}{2\omega_1 c_2^2 c_3^2 \int_0^1 \Phi_{11}^2 dt}, \tag{3.6}$$

where the quantity in curly brackets is given in equation (3.4). Also, $T = \tanh z$, as before, and the range of integration has been halved by taking advantage of the evenness of the integrands. The definite integrals, such as those appearing in equation (3.6), can all be expressed in terms of beta functions.

To illustrate, we consider the interaction of two mode one waves with a mode two wave; *i.e.*, the relevant eigenfunctions are the following:

$$\Phi_{11} = S^{k_1}, \quad \Phi_{12} = S^{k_2}T \quad \text{and} \quad \Phi_{13} = S^{k_3}.$$

An integral typical of the sort that must be evaluated is

$$I = \int_0^1 T\Phi_{11}\Phi_{12}\Phi_{13} dT = \int_0^1 T^2 S^{k_1+k_2+k_3} dT.$$

Noting that $S = (1 - T^2)^{1/2}$ and letting $\theta = T^2$, we obtain

$$I = \frac{1}{2}\int_0^1 \theta^{1/2}(1-\theta)^{(k_1+k_2+k_3)/2} d\theta = \frac{1}{2}\beta\left(\frac{3}{2}, 1 + \frac{(k_1+k_2+k_3)}{2}\right).$$

The β functions can readily be evaluated using the identity $\beta(m, n) = \Gamma(m)\Gamma(n)/\Gamma(m+n)$. A further simplification is gained if $k_1 + k_2 + k_3$ is an even integer. Most of the integrals can then be evaluated explicitly. In the case for which results will now be presented, $k_1 + k_2 + k_3 = 2$ and $I = 2/15$.

The numerical values of the interaction coefficients have been computed for the following triad:

$$\begin{array}{lll} k_1 = 0.1217 & k_2 = 0.8783 & k_3 = 1.0 \\ \omega_1 = 0.3294 & \omega_2 = 0.3777 & \omega_3 = 0.7071. \end{array}$$

Corresponding to these modes it was found that $\gamma_1 = 0.1561$, $\gamma_2 = -1.5685$ and $\gamma_3 = 0.0319$.

The foregoing results can readily be generalized to include interacting wave packets by introducing a slow spatial variable $X = \varepsilon x$ which the amplitudes depend upon, in addition to τ. It is well-known that equations (1.3) then take the form

$$\frac{\partial A_j}{\partial \tau} + \omega_j' \frac{\partial A_j}{\partial X} = i\gamma_j A_l A_m, \qquad j = 1, 2, 3, \tag{3.7}$$

where $\omega'(k)$ is the group velocity. Equations (3.7) arise in plasma physics, as well as in a number of other applications, and the system can be solved, in principle, by the inverse scattering method. The reader is referred to the paper of Case and Chiu [2], where special soliton solutions are presented and previous results are reviewed.

4. Harmonic resonances

The phenomenon of harmonic resonance can occur when there are two waves having the same phase speed, but their wavenumbers differ by some integer n. A substantial simplification of the results above is gained in the case of second harmonic resonance, *i.e.* $n = 2$. Equations (3.4) and (3.5) were, in fact, written in a form that clearly exhibits this simplification—when $c_1 = c_2 = c_3$ most of the nonhomogeneous terms vanish. The somewhat more difficult case $n = 3$ will be examined subsequent to that of $n = 2$ which we now consider.

The stream function for this case is expressed as a sum of two modes by writing

$$\psi^{(1)} = A_1(\tau)\Phi_{11}(z)E + A_2(\tau)\Phi_{12}(z)E^2 + *,$$

where $\tau = \varepsilon t$, as before. The amplitude evolution equations are given by

$$\frac{dA_1}{d\tau} = i\Gamma_1 A_2 A_1^* \quad \text{and} \quad \frac{dA_2}{d\tau} = i\Gamma_2 A_1^2. \tag{4.1}$$

An energy integral can be derived from (4.1) which takes the form

$$|A_1|^2 + (\Gamma_1/\Gamma_2)|A_2|^2 = E, \tag{4.2}$$

where E is a constant. It is to be expected that Γ_1 and Γ_2 have the same sign so that the total energy is conserved and the interaction is in the nature of an energy exchange.

Relatively simple expressions for Γ_1 and Γ_2 will now be obtained directly from the triad results. First, we note from symmetry considerations that Γ_1 and Γ_2 are nonzero only when Φ_{11} and Φ_{12} are both odd functions. Therefore, the reduction from the triad case consists of evaluating initially a triad interaction in which $k_1 = k_2$, $\omega_1 = \omega_2$ and all three eigenfunctions are odd.

We identify A_1 in the triad case with A_1 in (4.1) and make the subscript transformations $2 \to 1$ and $3 \to 2$ in (3.4) and (3.5). By carefully considering the derivation of the triad results it is concluded that $\gamma_1 \to \Gamma_1$ and $\gamma_3 \to 2\Gamma_2$, *i.e.* the interactions contributing to Γ_2 have been counted twice in (3.5). Hence, we arrive at the following expressions for the interaction coefficients:

$$\Gamma_1 = -2k_1 \frac{\int_0^1 T\Phi_{11}^2\Phi_{12}\,dT}{\int_0^1 \Phi_{11}^2\,dT} \quad \text{and} \quad \Gamma_2 = -k_2 \frac{\int_0^1 T\Phi_{11}^2\Phi_{12}\,dT}{\int_0^1 \Phi_{12}^2\,dT}. \tag{4.3}$$

To illustrate the application of these results, consider an interaction involving modes 2 and 4. The resonance conditions are satisfied when $c^2 = g\beta/30$ in which case

$$\Phi_{11} = S^2T(1 - 3T^2), \qquad k_1 = 2 \qquad \text{(mode 4)}$$

and

$$\Phi_{12} = S^4T, \qquad k_2 = 4 \qquad \text{(mode 2)}.$$

Substituting into (4.3) leads to the following results: $\Gamma_1 = -0.2896$ and $\Gamma_2 = -0.2168$.

It was noted earlier that the relative simplicity of the two-wave interaction makes it an attractive example to employ in connection with numerical studies. For small amplitudes and moderate time intervals the theory provides a check on the numerical code. Conversely, once the accuracy of the computer program has

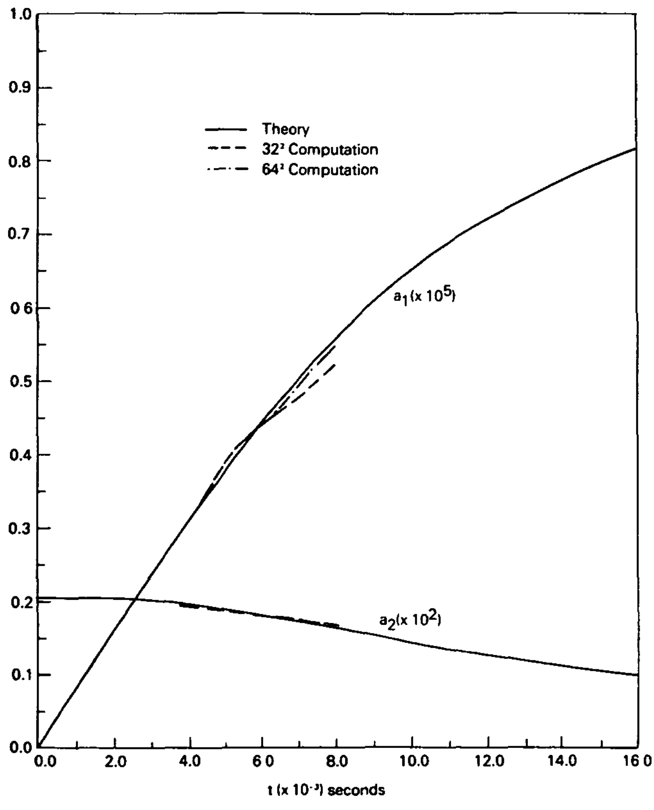

Figure 1. Amplitude variation as a function of time for a resonant pair of modes with the initial maximum isopycnal slope of the short wave equal to 0.21.

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been established, it can be used to assess the limitations of the theory. An investigation of this nature has been made by Lucas, Metcalfe and Maslowe [7] who employed a pseudospectral code with 64 Fourier modes in each direction to solve eqs. (2.3) and (2.4). The initial conditions were chosen to permit an exact solution of the amplitude (4.1) in terms of Jacobian elliptic functions.

From the results in Figure 1 for a case in which the amplitudes are relatively small, it is seen that impressive agreement is possible for long periods of time when using 64 modes. With 32 modes, the agreement is satisfactory, but after 5000 seconds there is some departure of the numerical solution from the correct values due to the buildup of phase errors. In Figure 2, the initial amplitude of the shorter wave has been doubled and we see that the theory begins to break down as the real amplitude a_1 becomes substantial after about 1600 seconds. A further doubling of $a_2(0)$ produced a continuation of this trend with the numerical solution departing from the theoretical curve after 400 seconds. However, in the latter case (which is not shown here) both the a_1 and the a_2 curves diverge from their theoretical trajectories.

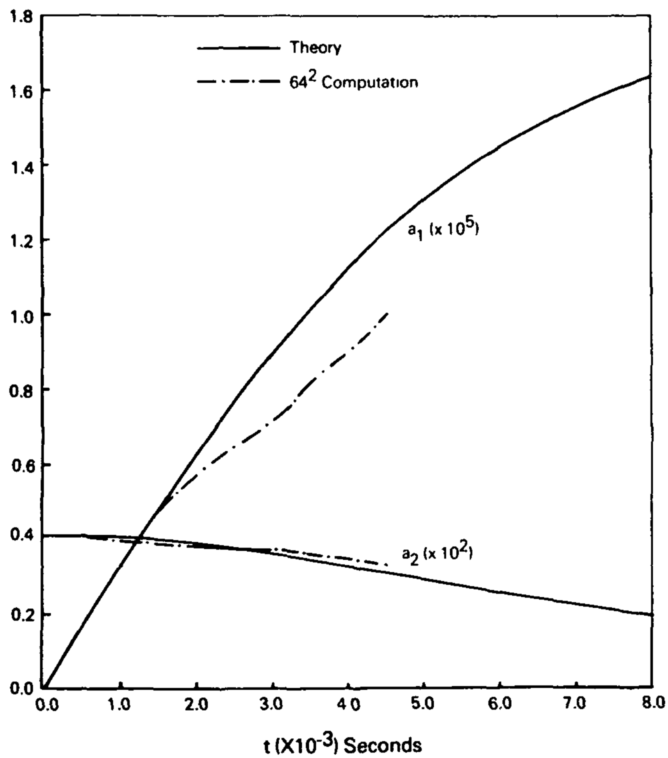

Figure 2. Amplitude variation as a function of time for a resonant pair of modes with the initial maximum isopycnal slope of the short wave equal to 0.42.

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Let us now briefly consider the theory for $n = 3$ corresponding to third harmonic resonance. This interaction, which has been discussed by Bretherton [1], is somewhat less important because it occurs on a slower time scale, namely, $\varepsilon^2 t$. The necessary condition for its occurrence is that $\omega(3k) = 3\omega(k)$ and to obtain nonzero interaction coefficients both eigenfunctions must have the same parity, *i.e.* be even or odd. Referring to the mode tabulation in Section 2, it can be seen that modes 2 and 4 satisfy these conditions when $k = 3$ and $k = 1$, respectively, and $c^2 = g\beta/20$. We omit the details for this case, but note that the nonlinearity is cubic and the amplitude equation is given in Section 7 of [1].

An interesting possibility that has not been discussed previously in the literature is that of simultaneous second and third harmonic resonance. The basic disturbance for each triad takes the form

$$\psi^{(1)} = A_1\Phi_{11}E + A_{12}\Phi_{12}E^2 + A_3\Phi_{13}E^3 + *,$$

where the amplitudes again evolve on an εt time scale. Although the triad conditions (3.1) are satisfied, the evolution equations that result from the analysis are not identical with those obtained in Section 3 due to the presence of second harmonic resonance. Instead, we are led to the following set of amplitude equations:

$$\frac{dA_1}{d\tau} = i(\gamma_{21}A_2A_1^* + \gamma_{32}A_3A_2^*), \tag{4.4a}$$

$$\frac{dA_2}{d\tau} = i(\gamma_{11}A_1^2 + \gamma_{31}A_3A_1^*), \tag{4.4b}$$

and

$$\frac{dA_3}{d\tau} = i\gamma_{12}A_1A_2. \tag{4.4c}$$

A conservation equation can be derived from (4.4), namely,

$$\frac{|A_1|^2}{\gamma_{21}} + \frac{|A_2|^2}{\gamma_{11}} + \frac{|A_3|^2}{\gamma_{12}}\left(\frac{\gamma_{32}}{\gamma_{21}} + \frac{\gamma_{31}}{\gamma_{11}}\right) = E.$$

To demonstrate that the requisite conditions for such an interaction can be satisfied with the present thermocline model, we have worked out the following example:

$$\Phi_{11} = S\left(1 - \tfrac{5}{4}S^2\right) \quad (\text{mode } 3, k_1 = 1),$$
$$\Phi_{12} = S^2T \quad (\text{mode } 2, k_2 = 2),$$
$$\Phi_{13} = S^3 \quad (\text{mode } 1, k_3 = 3).$$

The values obtained for the interaction coefficients in (4.4) are $\gamma_{11} = \gamma_{12} = -2/11$, $\gamma_{21} = -64/1155$, $\gamma_{31} = -8/33$ and $\gamma_{32} = -3.5 \times 10^{-3}$. Although we have not studied in detail the properties of (4.4), it seems likely that they are similar to those of (4.1) due to the existence of similar conservation laws.

5. Nonlinear self-interaction and wave packets

In this section, we depart from the analysis of resonant interactions and consider the more traditional weakly nonlinear expansion appropriate to a monochromatic wave train. This is followed by an extension of the analysis to include wave packets.

For a monochromatic wave, the expansion for $\hat{\psi}$ is written

$$\varepsilon\hat{\psi} \sim \varepsilon\left[A(\tau_1)\Phi_1 E + A^*\Phi_1 E^*\right] + \varepsilon^2\left(A^2\Phi_2 E^2 + *\right) + \varepsilon^3\left[A^2A^*\Phi_{31}E + A^3\Phi_{33}E^3 + *\right] + \cdots, \tag{5.1}$$

where $\tau_1 = \varepsilon^2 t$ and a similar expansion is employed for $\hat{\rho}$. At $O(\varepsilon^3)$ secular terms arise due to the interaction of E^2 and E^* terms so that Φ_{31} satisfies a nonhomogeneous equation of the form

$$A^2A^*\mathfrak{L}\Phi_{31} = f(z)\frac{dA}{d\tau_1} + g(z)A^2A^*. \tag{5.2}$$

Separation of variables is achieved by writing the amplitude equation

$$\frac{dA}{d\tau_1} = i\gamma_{NL}A^2A^*, \tag{5.3}$$

where γ_{NL} is fixed by imposing a solvability condition on (5.2). Anticipating the extension to wave packets, we will write down a more general result from which γ_{NL} is obtained by neglecting the contribution due to packet effects. This result, found independently by Koop and Redekopp [5] and Liu and Benney [6], can be written

$$\hat{\gamma} = \gamma_{NL} + \gamma_0 = \frac{c}{2k}\frac{\int_{-\infty}^{\infty}\Phi_1(Q_0 + Q_2)dz}{\int_{-\infty}^{\infty}N^2\Phi_1^2 dz}, \tag{5.4}$$

where Q_0 represents the contribution of the packet-induced distortion, so it will be neglected initially. Equivalent results were derived previously by Grimshaw [4] using an averaging method.

We have computed γ_{NL} for the first four modes in the case $\bar{\rho} = \exp(-\beta \tanh z)$. However, due to the length of these expressions, only the results for modes 1 and 2 are presented here; they are, respectively,

$$\gamma_{NL} = \left(\frac{4k^2}{\omega}\right)\frac{(2k+1)(61k^2+67k+18)}{(3k+2)(4k+1)(4k+3)}\frac{\Gamma(2k)\Gamma(k+1/2)}{\Gamma(k)\Gamma(2k+1/2)} \tag{5.5a}$$

and

$$\gamma_{NL} = \left(\frac{12k^2}{\omega}\right)\frac{(2k+1)(2k+3)(9k^3-13k^2-56k-28)}{(3k+5)(4k+1)(4k+3)(4k+5)(4k+7)}\frac{\Gamma(2k)\Gamma(k+1/2)}{\Gamma(k)\Gamma(2k+1/2)}. \tag{5.5b}$$

The foregoing theory can be generalized to treat wave packets by introducing the slow spatial variable $X = \varepsilon x$ and then transforming to a coordinate system moving at the group velocity through the change of variables $\xi = X - \omega'\tau$ and $\tau_1 = \varepsilon\tau$. The amplitude $A(\xi, \tau_1)$ now satisfies the nonlinear Schrödinger equation

$$\frac{\partial A}{\partial \tau_1} = i\left\{\frac{\omega''}{2}\frac{\partial^2 A}{\partial \xi^2} + \hat{\gamma}A^2A^*\right\}, \tag{5.6}$$

whose general solution characteristics have been established in recent years; a summary of these characteristics in the context of water waves is given by Yuen and Lake [17]. As noted below, the properties of the solution depend crucially on the signs of ω'' and $\hat{\gamma}$.

Before presenting the results for these quantities, we consider the interesting phenomenon of mean flow modification due to packet effects. The ξ-dependence of A leads to additional terms at $O(\varepsilon^2)$ and (5.1) must be modified accordingly. Specifically, we include terms of the form

$$i\Phi_{21}(A_\xi E + A_\xi^* E^*) + \Phi_0(z)|A|^2; \tag{5.7}$$

the reader is referred to (2.11) of [5] for the equation satisfied by Φ_0, whereas $\Phi_{21} = -\partial\Phi_1/\partial k$.

To illustrate certain complexities that arise, we consider the case of a mode 1 packet. After transforming to $T = \tanh z$ as the independent variable, it is found that Φ_0 satisfies the ODE

$$\frac{d}{dT}\left\{(1-T^2)\frac{d\Phi_0}{dT}\right\} + 4k(1+k)^3\Phi_0 = 4(k/c)(1+k)^2(1-T^2)^kT. \tag{5.8}$$

The solution to (5.8) can be expressed as an infinite series of Legendre polynomials which are bounded at $T = \pm 1$, but Φ_0 does not vanish there. When $k = 1$, for example, the solution of (5.8) is simply

$$\Phi_0 = (4/5c)T(1-T^2) - (8/75c)T \tag{5.9}$$

and the second term leads to an $O(\varepsilon^3)$ vertical velocity as $z \to \pm\infty$. This indicates that the expansion scheme should be modified slightly to deal with unbounded flows. Fortunately, there is no difficulty in computing γ_0, the packet contribution to γ, because Q_0 contains $d\Phi_0/dz$ rather than Φ_0 which does vanish as $|z| \to \infty$ so that Q_0 is well-behaved. Similarly, the mean density term, denoted $\Sigma(z)$ in [5], causes no difficulty and is found to be

$$\Sigma = -8S^2T(15S^2 + 8)/75 \tag{5.10}$$

when $k = 1$ and after correction of a sign error in (2.12) of [5] (*cf.* equation (3.14) of [6]). Finally, after a lengthy calculation we obtain

$$\gamma_0 = -0.1097/\omega. \tag{5.11}$$

Thus, γ_0 turns out to make a very small contribution to $\hat{\gamma}$, at least for the lowest mode with $k = 1$. For purposes of comparison, when $k = 1$, (5.5a) yields

$$\gamma_{NL} = 6.674/\omega \tag{5.12}$$

and, consequently, $\hat{\gamma} = 6.564/\omega$.

With the coefficients of (5.6) determined, the stability of the wave train to sideband modes can be established. If $\omega''\hat{\gamma} > 0$, instability occurs, where this instability means physically that envelope solitons will develop in the primary wave over long periods of space and time. For the mode 1 wave

$$\omega'' = -\frac{(1 + 4k)}{4k^2(1 + k)^2}\,\omega, \tag{5.13}$$

and, comparing with the results above, we see that $\omega''\hat{\gamma} < 0$ when $k = 1$. This means that dispersion and nonlinearity act in concert to cause the wave packet to disperse and decay.

The above conclusion was reached for $k = 1$, but comparing with the results of Liu and Benney [6; Section 4.1] for a constant N^2 case it seems likely to be true for short waves in general. These authors found, however, that $\hat{\gamma}$ changes sign for long waves which consequently tend to form envelope solitons. This probably occurs with our thermocline model too, because γ_{NL} decreases in magnitude with k; it is reduced by 66%, for example, when $k = \frac{1}{2}$. Should γ_0, which is negative, increase in magnitude as $k \to 0$ it is clear that $\hat{\gamma}$ will become negative for small values of k.

6. Concluding remarks

Resonant interactions have been considered in the case of an unbounded thermocline model and closed-form expressions obtained for the interaction coefficients. The advantages of having explicit results for these constants are evident; one can readily identify circumstances in which strong interactions will take place and, conversely, it is clearly demonstrated that there exist cases where no interaction occurs to this order despite satisfaction of the resonance conditions.

These results should be relevant to the real ocean even though it cannot happen often that three waves exactly satisfying the triad conditions arrive at the same place at the same time. McEwan and Plumb [9] in an experimental and theoretical study of a constant N^2 fluid have considered a more realistic detuned situation involving two waves initially; the third component is generated by sums and differences of the wavelengths and frequencies of the interacting pair. Even though their configuration and point of view (based on parametric instability) differs somewhat from our own, many of the ideas discussed in [9] concerning the nature of resonant interactions in the ocean would appear to be equally applicable to the model employed in this paper.

In the foregoing context, the case of second harmonic resonance is of particular interest because it really requires only one wave to get started; the second harmonic is generated automatically by weak nonlinear effects. The ease with which this phenomenon can be observed experimentally was noted by McGoldrick [10] in his investigation of capillary-gravity waves and it is somewhat surprising that similar observations have not yet been reported for internal waves.

There are several obvious directions in which the present analytical study can and should be extended. For example, the long-wave/short-wave resonance occuring for a three-layer model, as reported in [5], can also take place with the present thermocline model and a corresponding analysis is possible. The effects of a mean shear are also of interest in many regions of the ocean. It is clear that the interaction coefficients computed in Sections 3 and 4 will be greatly modified by the presence of a mean current because the eigenfunctions of the linear problem are then no longer symmetric. Certainly, the "window" cases in which the interaction coefficients vanish, will occur much less frequently. The modification due to finite Richardson number effects, even in the cases where Γ_1 and Γ_2 are nonzero, is substantial according to the numerical results of Lucas *et al.* [7]. For example, with a velocity profile $\bar{u} = \tanh z$ the interaction time scale is doubled when the Richardson number is unity in comparison with the infinite Richardson number situation where shear is absent. Numerical difficulties are encountered sooner when shear is present according to [7], so more Fourier modes are required.

The propagation of wave packets may be of particular interest when there is shear, because the possibility arises of a group-velocity critical layer, where $\omega' = \bar{u}(y)$. In such a case, the analysis of Liu and Benney [6] of wave packets in stratified shear flows would require some modification, as noted in their paper. Finally, a brief comment is in order about the nonvanishing of Φ_0, the mean flow distortion, as $|z| \to \infty$. In retrospect, this is not surprising because the mean flow induced by the packet is analogous to a long wave and the modifications leading to the Benjamin-Ono equation in that case are well known. We have not extended our analysis to deal with that difficulty because, as noted in Section 5, it seems to have little effect on the value of $\hat{\gamma}$, the coefficient of the nonlinear term in the amplitude equation.

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