Canad. Math. Bull. Vol. **14** (3), 1971

ADMISSIBLE ESTIMATORS OF θ^r IN SOME EXTREME VALUE DENSITIES

BY

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1. **Introduction.** Let the random variables X, Y, Z have respectively the extreme value densities as

(1) $$f_X(x;\theta) = \begin{cases} n\theta^{-n}x^{n-1}, & 0 \le x \le \theta \\ 0, & \text{otherwise} \end{cases}$$

(2) $$f_Y(y;\lambda) = \begin{cases} n\lambda^n y^{-n-1}, & y \ge \lambda \\ 0, & \text{otherwise} \end{cases}$$

(3) $$f_Z(z;\mu) = \begin{cases} e^{-(z-\mu)}, & z \ge \mu \\ 0, & \text{otherwise} \end{cases}$$

where $\theta > 0$, $\lambda > 0$ and μ are real numbers.

If δ is any estimator of the parametric function $g(\theta)$, the loss function L_0 is assumed to be

$$L_0(\delta, \theta) = [(\delta - g(\theta))/g(\theta)]^2.$$

We prove the following:

THEOREM. *Let the loss function be* L_0.

(a) *If X has density* (1) *then*

$$\delta_0(X) = [(n+2r)/(n+r)]X^r$$

is an admissible minimax estimator of θ^r, $r > -n/2$.

(b) *If Y has density* (2) *then*

$$\delta_0(Y) = [(n-2s)/(n-s)]\,Y^s$$

is an admissible minimax estimator of λ^s, $s < n/2$.

(c) *If Z has density* (3) *then*

$$\delta_0(Z) = [(1+2\alpha)/(1+\alpha)]\,e^{-\alpha Z}$$

is an admissible minimax estimator of $e^{-\alpha\mu}$, $\alpha > -\frac{1}{2}$.

As a particular case of (a) we have

Received by the editors April 27, 1970 and, in revised form, November 10, 1970.

https://doi.org/10.4153/CMB-1971-071-9 Published online by Cambridge University Press

COROLLARY 1. *If $X_1, \ldots, X_n$ represent n independent observations from a rectangular density on* $(0, \theta)$ *then*

$$(n+2r)/(n+r)\,[\max, (X_1, \ldots, X_n)]^r$$

is an admissible minimax estimator of θ^r, $r>-n/2$, *with respect to the loss* L_0.

Before proving the theorem in §2 we make the following

REMARKS. (i) Karlin [1] proved two theorems (his Theorems 2 and 3 on pp. 418–419) which when applied to densities (1), (2), and (3) state, respectively,

(a′) the estimator $\delta_0(X)$ is an admissible estimator of θ^r, $r>0$;

(b′) $\delta_0(Y)$ is an admissible estimator of λ^r, $r<0$;

(c′) $\delta_0(Z)$ is an admissible estimator of $e^{-\alpha\mu}$, $\alpha>0$.

Thus our theorem extends the admissibility of δ_0 in (a) to negative powers r of θ, $r>-n/2$; in (b) to positive powers s of λ, $s<n/2$; in (c) to negative values of $\alpha>-\frac{1}{2}$.

Although we have not been able to extend Theorems 2 and 3 of Karlin (in their general form) to values of $\alpha>-\frac{1}{2}$, our theorem follows as we have been able to use, in our proof, the exact form of the densities involved.

(ii) The restriction $r>-n/2$ in (a), $r<n/2$ in (b) and $\alpha>-\frac{1}{2}$ in (c) is necessary, for otherwise the risk of the estimator δ_0 is not defined.

(iii) Under the loss function L_0, an estimator $\delta(X)=CX^r$, C constant, has risk

$$R(CX, \theta) = C^2(n/n+2r)-2C(n/n+r)+1$$

which is minimized with respect to C for $C=(n+2r)/(n+r)$. Thus the estimator $\delta_0(X)=[(n+2r)/(n+r)]X^r$ is the uniformly minimum risk estimator in the class of all estimators of the type CX^r. Its risk is given by

$$R(\delta_0, \theta) = [r/n+r]^2. \tag{4}$$

It may be noted that θ^r has a UMVUE given by $\delta_1(X)=[(n+r)/r]X^r$. This follows from the fact that $\delta_1(X)$ is a function of the sufficient and complete statistic X, and has expected value $E(\delta_1(X))=\theta^r$. But, being of the type CX^r, it is clearly uniformly inferior to $\delta_0(X)$.

Similar remarks apply to $\delta_0(Y)$ and $\delta_0(Z)$.

2. **Discussion of the proof.** Karlin's approach to establish admissibility can be briefly stated as follows. To prove the admissibility of δ_0 for $g(\theta)$, assume that there is an estimator δ better than δ_0. This leads to the inequality

$$\begin{aligned} T(\theta) &= \int (\delta_0-\delta)^2 f(x;\theta)\,dx \\ &\leq 2\int (\delta_0-\delta)(\delta_0-g(\theta))f(x;\theta)\,dx \end{aligned} \tag{5}$$

holding for all θ. In order to demonstrate that δ_0 is admissible it is enough to show

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that (5) is possible provided $\delta(x)=\delta_0(x)$ a.e. This is done as follows. Suppose there exists an increasing function $F(\theta)$ with the property that r.h.s. of (5), integrated with respect to $dF(\theta)$, reduces to zero. This implies that $\int T(\theta)\,dF(\theta)\le 0$, proving thereby that $T(\theta)=0$ a.e., which in turn implies the desired result $\delta(x)=\delta_0(x)$ a.e. One way to look for $F(\theta)$ is to choose it so that

$$\int g(\theta)f(x;\theta)\,dF(\theta) = \delta_0(x)\int f(x;\theta)\,dF(\theta).$$

Proof. (a) Let δ be an estimator better than δ_0. Then the inequality (5) in our case becomes

$$\text{(6)} \qquad \begin{aligned} T(\theta) &= \int_0^\theta (\delta_0-\delta)^2\theta^{-n}x^{n-1}\,dx \\ &\le 2\int_0^\theta (\delta_0-\delta)(\delta_0-\theta^r)\theta^{-n}x^{n-1}\,dx \end{aligned}$$

holding for all θ. On applying Schwarz inequality to the r.h.s. of (6) and then using (4) we get

$$\text{(7)} \qquad \int_0^\theta (\delta_0-\delta)(\delta_0-\theta^r)\theta^{-n}x^{n-1}\,dx \le k_1\theta^{2r}$$

$$\text{(8)} \qquad T(\theta)\le k_2\theta^{2r}$$

where k_1, k_2 depend on n and r but not on θ. Let $dF(\theta)=\theta^{-2r-1}\,d\theta$ and let $0<a<b$ be real numbers. Integrating (6) with respect to $dF(\theta)$ we have

$$\int_a^b T(\theta)\theta^{-2r-1}\,d\theta \le 2\int_a^b \left[\int_0^\theta (\delta_0-\delta)(\delta_0-\theta^r)\theta^{-n}x^{n-1}\,dx\right]\theta^{-2r-1}\,d\theta.$$

The r.h.s., after an interchange of order of integration (which is justified on account of (7) and (8)) is equal to I_b-I_a, where I_a is obtained on replacing b by a in

$$\begin{aligned} I_b &= 2\int_0^b (\delta_0-\delta)x^{n-1}\left[\int_x^b (\delta_0-\theta^r)\theta^{-2r-n-1}\,d\theta\right]dx \\ &= 2\int_0^b (\delta_0-\delta)x^{n-1}[b^{-n-r}-x^rb^{-n-2r}](n+r)^{-1}\,dx. \end{aligned}$$

Applying Schwarz inequality to the r.h.s. after taking absolute values we find

$$|I_b| \le k_3b^{-r}(T(b))^{1/2}$$

where $k_3=(2/n+r)[n^{-1/2}+(n+2r)^{-1/2}]$ is independent of θ. It follows therefore that

$$\text{(9)} \qquad \int_0^b T(\theta)\theta^{-2r-1}\,d\theta \le k_3[b^{-r}(T(b))^{1/2}+a^{-r}(T(a))^{1/2}].$$

Below we show that there exist two sequences of real numbers $\{a_i\}$ and $\{b_j\}$ such

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that $a_i \to 0 \Rightarrow T(a_i)a_i^{-2r} \to 0$ and $b_j \to \infty \Rightarrow T(b_j)b_j^{-2r} \to 0$. If we let $a \to 0$ along $\{a_i\}$ and $b \to \infty$ along $\{b_j\}$ in (9) we get

$$\int_0^\infty T(\theta)\theta^{-2r-1}\,d\theta = 0$$

implying the desired result $\delta(x) = \delta_0(x)$ a.e.

To prove the existence of the two sequences asserted above we first note that (8) and (9) imply

$$\int_0^\infty T(\theta)\theta^{-2r-1}\,d\theta < \infty. \tag{10}$$

Suppose $T(\theta)\theta^{-2r} > \epsilon > 0$ for $\theta > \theta_0$. Then

$$\int_0^\infty T(\theta)\theta^{-2r-1}\,d\theta > \int_{\theta_0}^\infty (\epsilon/\theta)\,d\theta = +\infty,$$

contradicting (10). Thus there exists a sequence $\{b_j\}$ with the desired property. The existence of the sequence $\{a_i\}$ can be proved in a similar way. This completes the proof of the admissibility part. The minimaxity of δ_0 follows from the fact that $R(\delta_0, \theta)$ is constant with respect to the loss L_0. The proof of part (b) and (c) can be either given in a similar way or deduced from that of (a) as follows.

(b) Let $Y = 1/X$ and $\lambda = 1/\theta$ in the density (2). We get the density of X as in (1). Therefore, $[(n+2r)/(n+r)]X^r$ is admissible for θ^r. That is, $[(n+2r)/(n+r)]Y^{-r}$ is admissible for λ^{-r} for $r > n/2$. If we set $s = -r$, we get the assertion in (b).

(c) Let $Y = e^Z$ and $\lambda = e^\mu$ in (3). Then Z has the same density as in (2) with $n = 1$. Therefore, $[(1-2s)/(1-s)]Y^s$ is admissible for λ^s. That is, $[(1-2s)/(1-s)]e^{sZ}$ is admissible for $e^{s\mu}$ for $s < n/2$. If we set $s = -\alpha$ we get the assertion in (c).

This completes the proof of our theorem.

Reference

1. S. Karlin, *Admissibility for estimation with quadratic loss*, Ann. Math. Stat. **29** (1958), 406–435.

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