Canad. Math. Bull. Vol. **52** (1), 2009 pp. 28–38

Right and Left Weak Approximation Properties in Banach Spaces

Changsun Choi, Ju Myung Kim, and Keun Young Lee

Abstract. New necessary and sufficient conditions are established for Banach spaces to have the approximation property; these conditions are easier to check than the known ones. A shorter proof of a result of Grothendieck is presented, and some properties of a weak version of the approximation property are addressed.

1 Introduction

A Banach space X is said to have the *approximation property* (AP) if for every compact subset K of X and $\epsilon > 0$, there is a finite rank operator T on X such that $\|Tx - x\| < \epsilon$ for all $x \in K$. Grothendieck [6] initiated the investigation of the AP, and one important tool he used was the following topology which is strictly weaker than the topology given by the operator norm :

For compact $K \subset X$, $\epsilon > 0$, and $T \in \mathcal{B}(X, Y)$, the space of bounded linear operators from a Banach space X into another Banach space Y, we let

$$\mathcal{N}(T; K, \epsilon) = \{S \in \mathcal{B}(X, Y) : \sup_{x \in K} \|Sx - Tx\| < \epsilon\}.$$

We denote by τ the topology on $\mathcal{B}(X, Y)$ generated by the collection of all $\mathcal{N}(T; K, \epsilon)$'s.

Observe that the τ topology is a locally convex and completely regular vector topology on $\mathcal{B}(X, Y)$. For a net (T_α) and T in $\mathcal{B}(X, Y)$,

$$T_\alpha \xrightarrow{\tau} T \text{ if and only if } \sup_{x \in K} \|T_\alpha x - Tx\| \longrightarrow 0$$

for each compact $K \subset X$; and that for $\mathcal{A} \subset \mathcal{B}(X, Y)$ and $T \in \mathcal{B}(X, Y)$: $T \in \overline{\mathcal{A}}^{\tau}$ if and only if for each compact $K \subset X$ and $\epsilon > 0$, there is a $S \in \mathcal{A}$ such that $\sup_{x \in K} \|Sx - Tx\| < \epsilon$.

Therefore, a Banach space X has the AP if the following property holds :

$$I_X \in \overline{\mathcal{F}(X, X)}^{\tau}.$$

Here I_X is the identity and $\mathcal{F}(X, X)$ is the space of finite rank operators on X.

We now have the following simple characterizations of the AP through straightforward verifications.

Received by the editors June 30, 2006; revised November 27, 2006.
The first and third authors were supported by BK 21 project.
AMS subject classification: Primary: 46B28; secondary:46B10.
Keywords: approximation property, quasi approximation property, weak approximation property.

https://doi.org/10.4153/CMB-2009-004-5 Published online by Cambridge University Press

Fact 1 Let X be a Banach space. Then the following are equivalent.

(a) X has the AP.
(b) For every Banach space Y, $\mathcal{B}(Y,X) = \overline{\mathcal{F}(Y,X)}^{\tau}$.
(c) For every Banach space Y, $\mathcal{B}(X,Y) = \overline{\mathcal{F}(X,Y)}^{\tau}$.

In this paper we are concerned with Banach spaces having the following properties:

$$\text{for every Banach space } Y, \mathcal{K}(Y,X) \subset \overline{\mathcal{F}(Y,X)}^{\tau} \tag{1.1}$$

and

$$\text{for every Banach space } Y, \mathcal{K}(X,Y) \subset \overline{\mathcal{F}(X,Y)}^{\tau}. \tag{1.2}$$

Here $\mathcal{K}(X,Y)$ is the space of compact operators from a Banach space X into another Banach space Y.

A Banach space X is said to have the *weak approximation property* (WAP) if $\mathcal{K}(X,X) \subset \overline{\mathcal{F}(X,X)}^{\tau}$, and X is said to have the *quasi approximation property* (QAP) if $\mathcal{K}(X,X) = \overline{\mathcal{F}(X,X)}$ (the operator norm closure of $\mathcal{F}(X,X)$). Choi and Kim [2] first introduced the WAP and QAP and corresponding results were presented in Kim [7–9]. In this paper we say that a Banach space X is said to have the *left weak approximation property* (lWAP) (resp. *right weak approximation property* (rWAP)) if X has the property (1.1) (resp. (1.2)).

We now have the following diagram by Fact 1, (1.1), (1.2), and the definitions.

$$\begin{tikzcd}
& \mathrm{AP} \arrow[dl] \arrow[d] \arrow[dr] & \\
l\mathrm{WAP} \arrow[dr] & \mathrm{QAP} \arrow[d] & r\mathrm{WAP} \arrow[dl] \\
& \mathrm{WAP} &
\end{tikzcd}$$

In Section 2, we will obtain the following diagram (1.3), and for reflexive Banach spaces we will also obtain the diagram (1.4).

$$\begin{tikzcd}
l\mathrm{WAP} \arrow[r, Leftrightarrow] & \mathrm{AP} \arrow[d] \arrow[dr] & \\
& \mathrm{QAP} \arrow[d] & r\mathrm{WAP} \arrow[dl] \\
& \mathrm{WAP} &
\end{tikzcd}$$

(1.3)

$$\begin{tikzcd}
l\mathrm{WAP} \arrow[r, Leftrightarrow] & \mathrm{AP} \arrow[r, Leftrightarrow] \arrow[d] & r\mathrm{WAP} \\
& \mathrm{QAP} \arrow[r, Leftrightarrow] & \mathrm{WAP}
\end{tikzcd}$$

(1.4 : Reflexive Banach spaces)

https://doi.org/10.4153/CMB-2009-004-5 Published online by Cambridge University Press

We introduce a famous open problem in the area of the AP (cf. [11, Problem 1.e.9] and [1, Problem 2.7]).

Problem If a Banach space X has the QAP, then does X have the AP ?

For a long time this problem remained unsolved even for the reflexive case. By diagram (1.4), if the above question had an affirmative answer for a reflexive Banach space X, then all our approximation properties would be equivalent for X.

In Section 2, characterizations of the AP and relations between the approximation properties are established. A new short proof of a result of Grothendieck is provided. In Section 3, some properties for rWAP are established.

2 Relations Between the Approximation Properties

Notation We start by listing some notations.

X, Y: Banach spaces.
X^*: The dual space of X.
T^* : The adjoint of an operator T.
$\mathcal{B}(X, Y)$: The space of bounded linear operators from X into Y.
$\mathcal{F}(X, Y)$: The space of bounded and finite rank linear operators from X into Y.
$\mathcal{K}(X, Y)$: The space of compact operators from X into Y.
$\mathcal{F}^*(X, Y)$: The space of finite rank adjoint operators from Y^* into X^*.
$\mathcal{K}^*(X, Y)$: The space of compact adjoint operators from Y^* into X^*.

For convenience we denote $\mathcal{B}(X, X), \ldots$ by $\mathcal{B}(X), \ldots$.

We review the following well-known results of Grothendieck [6].

Fact 2 (a) $(\mathcal{B}(X, Y), \tau)^*$ consists of all functionals f of the form $f(T) = \sum_n y_n^*(Tx_n)$, where $(x_n) \subset X, (y_n^*) \subset Y^*$, and $\sum_n \|x_n\|\|y_n^*\| < \infty$.
(b) X has the AP if and only if for every Banach space Y, $\mathcal{K}(Y, X) = \overline{\mathcal{F}(Y, X)}$.
(c) X^* has the AP if and only if for every Banach space Y, $\mathcal{K}(X, Y) = \overline{\mathcal{F}(X, Y)}$.

The following is due to Feder and Saphar [4, Theorem 1].

Lemma 2.1 *If X^{**} or Y^* has the Radon–Nikodym property, then $\mathcal{K}(X, Y)^*$ consists of all functionals g of the form $g(T) = \sum_n x_n^{**}(T^* y_n^*)$, where $(x_n^{**}) \subset X^{**}, (y_n^*) \subset Y^*$, and $\sum_n \|x_n^{**}\|\|y_n^*\| < \infty$.*

We now have the following.

Theorem 2.2 *Suppose that X is reflexive. Then $\mathcal{K}(X, Y) \subset \overline{\mathcal{F}(X, Y)}^{\tau}$ if and only if $\mathcal{K}(X, Y) = \overline{\mathcal{F}(X, Y)}$.*

Proof We only need to show the "only if" part. Let $T \in \mathcal{K}(X, Y)$. Then there is a net $(T_\alpha) \subset \mathcal{F}(X, Y)$ such that $T_\alpha \xrightarrow{\tau} T$. From Fact 2(a) it follows that

$$\sum_n y_n^*(T_\alpha x_n) \longrightarrow \sum_n y_n^*(Tx_n)$$

https://doi.org/10.4153/CMB-2009-004-5 Published online by Cambridge University Press

for each $(x_n) \subset X$ and $(y_n^*) \subset Y^*$ with $\sum_n \|x_n\|\|y_n^*\| < \infty$. Since X is reflexive, by Lemma 2.1 we have

$$g(T_\alpha) \longrightarrow g(T)$$

for each $g \in \mathcal{K}(X,Y)^*$. Hence $T \in \overline{co}^{weak}(\{T_\alpha\}) = \overline{co}(\{T_\alpha\}) \subset \overline{\mathcal{F}(X,Y)}$. ∎

The following is a result of Lindenstrauss [10, Proposition 1].

Lemma 2.3 *Let X be a reflexive Banach space. If X_0 is a separable subspace of X, then there is a separable space Z satisfying $X_0 \subset Z \subset X$ such that there is a projection of norm one from X onto Z.*

We now have the following characterizations of the AP, which are easier to check than the ones in Fact 2(b) and (b3) in [13, Theorem 2].

Theorem 2.4 *The following are equivalent.*

(a) *X has the AP.*
(b) *X has the lWAP*
(c) *For every separable and reflexive Banach space Y, $\mathcal{K}(Y,X) \subset \overline{\mathcal{F}(Y,X)}^{\tau}$.*
(d) *For every reflexive Banach space Y, $\mathcal{K}(Y,X) \subset \overline{\mathcal{F}(Y,X)}^{\tau}$.*

Proof We show (a) ⇒(b)⇒(c)⇒(d)⇒(a). But (a)⇒(b) and (b)⇒(c) are clear.

(c)⇒(d) Let Y be a reflexive Banach space. Let $T \in \mathcal{K}(Y,X)$, compact $K \subset Y$, and $\epsilon > 0$. Since the closed linear span $[K]$ of K is a separable subspace of Y, by Lemma 2.3 there is a separable subspace Z of Y such that $[K] \subset Z \subset Y$ and there is a projection P of norm one from Y onto Z. By the assumption (c) there is a $T_0 \in \mathcal{F}(Z,X)$ such that

$$\sup_{x \in K} \|T_0 x - TI_Z x\| < \epsilon,$$

where I_Z is the inclusion from Z into Y. Now consider $T_0 P \in \mathcal{F}(Y,X)$. Then we have

$$\sup_{x \in K} \|T_0 P x - Tx\| = \sup_{x \in K} \|T_0 x - TI_Z x\| < \epsilon.$$

Hence $T \in \overline{\mathcal{F}(Y,X)}^{\tau}$, proving our implication.

(d)⇒(a) Let Y be a Banach space and $T \in \mathcal{K}(Y,X)$. Then there exist a reflexive Banach space Z, $R \in \mathcal{K}(Y,Z)$, and $S \in \mathcal{K}(Z,X)$ such that $T = SR$ (cf. Figiel [5, Corollary 3.3]). By the assumption there is a net $(S_\alpha) \subset \mathcal{F}(Z,X)$, $S_\alpha \xrightarrow{\tau} S$. Consider $(S_\alpha R) \subset \mathcal{F}(Y,X)$. Then we have

$$\|S_\alpha R - T\| = \|S_\alpha R - SR\| = \sup_{z \in R(B_Y)} \|S_\alpha z - Sz\| \longrightarrow 0,$$

which shows $T \in \overline{\mathcal{F}(Y,X)}$. Hence X has the AP by Fact 2(b). ∎

Corollary 2.5 *Suppose that X is reflexive. Then the following are equivalent.*

(a) *X has the AP.*
(b) *X has the lWAP.*
(c) *X has the rWAP.*

https://doi.org/10.4153/CMB-2009-004-5 Published online by Cambridge University Press

Proof In view of Theorem 2.4 we only need to show (c)⇒(a). Assume (c). Then by Theorem 2.2 and Fact 2(c) X^* has the AP. It is well known that if X^* has the AP, then X has the AP (cf. [1, 11]). Hence X has the AP. ■

In [7, Corollary 1.5], it was shown that for every separable and reflexive Banach space X, X having QAP and WAP are equivalent. But by Theorem 2.2 the assumption of separability can be removed.

Corollary 2.6 *Suppose that X is reflexive. Then X has the QAP if and only if X has the WAP.*

Lemma 2.7 *For all Banach spaces X and Y, $\mathcal{F}(Y^*, X^*) \subset \overline{\mathcal{F}^*(X,Y)}^{\tau}$.*

Proof It suffices to show that every rank one $T_0 \in \mathcal{F}(Y^*, X^*)$ belongs to $\overline{\mathcal{F}^*(X,Y)}^{\tau}$. So let $x_0^* \in X^*$, $y_0^{**} \in Y^{**}$ and write $T_0 y^* = y_0^{**}(y^*)x_0^*$ for $y^* \in Y^*$. Let $K \subset Y^*$ be compact and $\epsilon > 0$. First choose a small $\delta > 0$ so that $\delta\|x_0^*\|(1 + 2\|y_0^{**}\|) < \epsilon$. Since K is compact, there are $y_1^*, \dots, y_m^*$ in K such that for each $y^* \in K$, $\|y^* - y_i^*\| < \delta$ for some $1 \le i \le m$. By Goldstine's theorem there is $y_0 \in Y$ such that $\|y_0\| \le \|y_0^{**}\|$ and $|y_i^* y_0 - y_0^{**} y_i^*| < \delta$ for all $1 \le i \le m$. Now consider $T \in \mathcal{F}^*(X,Y)$ given by $Ty^* = (y^* y_0)x_0^*$ for $y^* \in Y^*$. Using the triangle inequality one checks that

$$\sup_{y^* \in K} \|T_0 y^* - Ty^*\| < \epsilon,$$

which proves the lemma. ■

In the following theorem, (a)⇔(b) is Fact 2(c) and (a)⇔(d) is known [13, Theorem 5]. However, we would like to present its full proof, which may be noted for its brevity and elegance.

Theorem 2.8 *The following are equivalent.*

(a) *X^* has the AP.*
(b) *For every Banach space Y $\mathcal{K}(X,Y) = \overline{\mathcal{F}(X,Y)}$.*
(c) *For every separable and reflexive Banach space Y $\mathcal{K}(X,Y) = \overline{\mathcal{F}(X,Y)}$.*
(d) *For every reflexive Banach space Y $\mathcal{K}(X,Y) = \overline{\mathcal{F}(X,Y)}$.*

Proof We show (a)⇒(b)⇒(c)⇒(d)⇒(a).

(a)⇒(b) Let Y be a Banach space, $T \in \mathcal{K}(X,Y)$, and $\epsilon > 0$. Since $T^* \in \mathcal{K}(Y^*, X^*)$, $T^*(B_{Y^*})$ is a relatively compact set in X^*, where B_{Y^*} is the unit ball in Y^*. By the assumption there is a $T_0 \in \mathcal{F}(X^*)$ such that

$$\sup_{y^* \in B_{Y^*}} \|T_0 T^* y^* - T^* y^*\| < \frac{\epsilon}{2}.$$

Also, by Lemma 2.7 there is a $S_0^* \in \mathcal{F}^*(X)$

$$\sup_{y^* \in B_{Y^*}} \|S_0^* T^* y^* - T_0 T^* y^*\| < \frac{\epsilon}{2}.$$

https://doi.org/10.4153/CMB-2009-004-5 Published online by Cambridge University Press

Now consider $TS_0 \in \mathcal{F}(X, Y)$. Then we have

$$\|T - TS_0\| = \|S_0^* T^* - T^*\| \leq \|S_0^* T^* - T_0 T^*\| + \|T_0 T^* - T^*\| < \epsilon.$$

Hence $T \in \overline{\mathcal{F}(X, Y)}$.

(b)$\Rightarrow$(c) Clear.

(c)$\Rightarrow$(d) Let Y be a reflexive Banach space. Let $T \in \mathcal{K}(X, Y)$ and $\epsilon > 0$. Since $T(X)$ is a separable subspace of Y, by Lemma 2.3 there is a separable subspace Z of Y such that $T(X) \subset Z \subset Y$ and there is a projection P of norm one from Y onto Z. By the assumption there is a $T_0 \in \mathcal{F}(X, Z)$ such that $\|T_0 - PT\| < \epsilon$. Now consider $I_Z T_0 \in \mathcal{F}(X, Y)$, where I_Z is the inclusion from Z into Y. Then we have

$$\|I_Z T_0 - T\| = \|T_0 - PT\| < \epsilon.$$

Hence $T \in \overline{\mathcal{F}(X, Y)}$.

(d)$\Rightarrow$(a) By Theorem 2.4(d) it is enough to show that for every reflexive Banach space Y, $\mathcal{K}(Y, X^*) = \overline{\mathcal{F}(Y, X^*)}$. The proof is a review of the proof of [11, Theorem 1.e.5]. Let Y be a reflexive Banach space. Let $T \in \mathcal{K}(Y, X^*)$ and $\epsilon > 0$. Then $T^* Q_X \in \mathcal{K}(X, Y^*)$, where Q_X is the natural map from X into X^{**}. By the assumption there is a $\sum_{k=1}^{n} x_k^*(\cdot) y_k^* \in \mathcal{F}(X, Y^*)$ such that

$$\Big\| T^* Q_X - \sum_{k=1}^{n} x_k^*(\cdot) y_k^* \Big\| < \epsilon.$$

Now consider $\sum_{k=1}^{n} y_k^*(\cdot) x_k^* \in \mathcal{F}(Y, X^*)$. Then we have

$$\begin{aligned}
\Big\| T - \sum_{k=1}^{n} y_k^*(\cdot) x_k^* \Big\| &= \sup_{\|y\| \leq 1} \Big\| Ty - \sum_{k=1}^{n} y_k^*(y) x_k^* \Big\| \\
&= \sup_{\|y\|, \|x\| \leq 1} \Big\| (Ty)x - \sum_{k=1}^{n} y_k^*(y) x_k^*(x) \Big\| \\
&= \sup_{\|y\|, \|x\| \leq 1} \Big\| Q_X x(Ty) - \sum_{k=1}^{n} y_k^*(y) x_k^*(x) \Big\| \\
&= \sup_{\|y\|, \|x\| \leq 1} \Big\| T^* Q_X x(y) - \sum_{k=1}^{n} x_k^*(x) y_k^*(y) \Big\| \\
&= \sup_{\|x\| \leq 1} \Big\| T^* Q_X x - \sum_{k=1}^{n} x_k^*(x) y_k^* \Big\| \\
&= \Big\| T^* Q_X - \sum_{k=1}^{n} x_k^*(\cdot) y_k^* \Big\| < \epsilon.
\end{aligned}$$

Hence $T \in \overline{\mathcal{F}(Y, X^*)}$. ■

We introduce another topology on $\mathcal{B}(X,Y)$, which is induced by a subspace of $\mathcal{B}(X,Y)^\sharp$, the space of linear functionals on $\mathcal{B}(X,Y)$.

Let $\mathcal{Z}$ be the space of linear functionals φ on $\mathcal{B}(X,Y)$ of the form

$$\varphi(T) = \sum_n y_n^*(Tx_n)$$

where $(x_n) \subset X$ and $(y_n^*) \subset Y^*$ with $\sum_n \|x_n\|\|y_n^*\| < \infty$. Then the *$\nu$ topology* (ν) on $\mathcal{B}(X,Y)$ is the topology induced by $\mathcal{Z}$.

From elementary facts about topologies induced by spaces of linear functionals on vector spaces, ν is a locally convex topology. Also $(\mathcal{B}(X,Y),\nu)^* = \mathcal{Z}$, and for a net (T_α) and T in $\mathcal{B}(X,Y)$,

$$T_\alpha \xrightarrow{\nu} T \text{ if and only if } \sum_n y_n^*(T_\alpha x_n) \longrightarrow \sum_n y_n^*(Tx_n)$$

for each $(x_n) \subset X$ and $(y_n^*) \subset Y^*$ with $\sum_n \|x_n\|\|y_n^*\| < \infty$.

Recall Fact 2(a). Then $(\mathcal{B}(X,Y),\nu)^* = (\mathcal{B}(X,Y),\tau)^*$. Hence by [12, Corollary 2.2.29], $\overline{\mathcal{C}}^\nu = \overline{\mathcal{C}}^\tau$ for every convex set $\mathcal{C}$ in $\mathcal{B}(X,Y)$.

We now have the following characterization of the AP for dual spaces, which is easier to check than the ones in Theorem 2.8.

Corollary 2.9 *X^* has the AP if and only if for every separable and reflexive Banach space Y, $\mathcal{K}^*(X,Y) \subset \overline{\mathcal{F}^*(X,Y)}^\nu$ in $\mathcal{B}(Y^*,X^*)$.*

Proof We only need to show the "if" part. If, for every separable and reflexive Banach space Y, $\mathcal{K}^*(X,Y) \subset \overline{\mathcal{F}^*(X,Y)}^\nu$, then, as in the proof of Theorem 2.2, we obtain $\mathcal{K}(X,Y) = \overline{\mathcal{F}(X,Y)}$. Hence X^* has the AP by Theorem 2.8(c). ∎

Remark 2.10 (a) In [9, Corollary 2.7], it was shown that for every Banach space X with separable dual, X has the QAP if and only if $\mathcal{K}^*(X) \subset \overline{\mathcal{F}^*(X)}^\nu$ in $\mathcal{B}(X^*)$. But as in the proof of Theorem 2.2, even when the dual space has the Radon–Nikodym property, we have the same result.

(b) In [7, Theorem 1.3], it was shown that for every Banach space X with the separable dual, if X^* has the WAP, then X has the QAP. But by (a) and Lemma 2.7 even when the dual space has the Radon–Nikodym property, we have the same result.

3 Some properties for rWAP

It is known that the AP, QAP, and WAP are inherited from X^* to X [1, 2, 9]. For the rWAP we have the following.

Theorem 3.1 *If X^* has the rWAP, then for every Banach space Y, $\mathcal{K}^*(Y,X) \subset \overline{\mathcal{F}^*(Y,X)}^\nu$ in $\mathcal{B}(X^*,Y^*)$; in particular, X has the lWAP and rWAP.*

Proof If X^* has the rWAP, then for every Banach space Y, $\mathcal{K}(X^*,Y^*) \subset \overline{\mathcal{F}(X^*,Y^*)}^\tau$. Recall the ν topology. Then by Lemma 2.7, for every Banach space Y, we have

$$\mathcal{K}^*(Y,X) \subset \mathcal{K}(X^*,Y^*) \subset \overline{\mathcal{F}(X^*,Y^*)}^\tau = \overline{\mathcal{F}^*(Y,X)}^\tau = \overline{\mathcal{F}^*(Y,X)}^\nu.$$

https://doi.org/10.4153/CMB-2009-004-5 Published online by Cambridge University Press

In particular, for every Banach space Y, $\mathcal{K}(Y,X) \subset \overline{\mathcal{F}(Y,X)}^{\tau}$ in $\mathcal{B}(Y,X)$. By Theorem 2.4, X has the AP; consequently, X has the lWAP and the rWAP. ∎

The following example shows that AP, rWAP, QAP, WAP are not inherited from X by X^* in general.

Example 3.2 There is a dual space Z with a boundedly complete basis such that Z^* is separable and does not have the WAP [7, Theorem 1.9].

By simple calculations one may see that the AP, QAP, and WAP are inherited by complemented subspaces. For the rWAP we also have the following.

Proposition 3.3 *Let Z be a complemented subspace of X. If X has the rWAP, then Z has the rWAP.*

Proof Suppose that X has the rWAP. Let P be a projection from X onto Z. Let Y be a Banach space, $T \in \mathcal{K}(Z,Y)$, compact $K \subset Z$, and $\epsilon > 0$. Since $TP \in \mathcal{K}(X,Y)$, there is a $T_0 \in \mathcal{F}(X,Y)$ such that $\sup_{x\in K} \|TPx - T_0x\| < \epsilon$. Now consider $T_0I_Z \in \mathcal{F}(Z,Y)$, where I_Z is the inclusion of Z into X. Then we have

$$\sup_{x\in K} \|Tx - T_0I_Zx\| = \sup_{x\in K} \|TPx - T_0x\| < \epsilon.$$

Hence T_0I_Z is a desired finite rank operator and so Z has the rWAP. ∎

The following is well known (cf. Diestel [3, Exercises I.6 and II.6(1)]).

Fact 3 Let (X_n) be a sequence of Banach spaces. If $1 \le p < \infty$ and K is a relatively compact subset of $(\sum_n \oplus X_n)_{l_p}$, then for every $\epsilon > 0$ there is a positive integer N_ϵ such that $\sum_{n>N_\epsilon}^{\infty} \|k_n\|_{X_n}^p < \epsilon$ for all $(k_n) \in K$. Also, if a subset K of $(\sum_n \oplus X_n)_{c_0}$ is relatively compact, then for every $\epsilon > 0$ there is a positive integer N_ϵ such that $\sup_{n>N_\epsilon} \|k_n\|_{X_n} < \epsilon$ for all $(k_n) \in K$.

The AP passes through l_p-sums and c_0-sums [1, Proposition 2.14]. For the rWAP we also have

Theorem 3.4 *If (X_n) is a sequence of Banach spaces with the rWAP, then for every $1 \le p < \infty$, $(\sum_n \oplus X_n)_{l_p}$ and $(\sum_n \oplus X_n)_{c_0}$ have the rWAP.*

Proof First we show that $X = (X_1 \oplus X_2)_{l_p}$ has the rWAP. Let Y be a Banach space and let $T \in \mathcal{K}(X,Y)$. Let $K \subset X$ be compact and $\epsilon > 0$. Now $Ti_n \in \mathcal{K}(X_n,Y)$, where i_n is the map from X_n into X defined by $i_1x_1 = (x_1,0), i_2x_2 = (0,x_2)$. Let $P_n \colon X \longrightarrow X_n$ be the projection given by $P_n((x_1,x_2)) = x_n$ for $n = 1,2$. Since $P_n(K)$ is compact in X_n and X_n has the rWAP for $n = 1,2$, there is a $T_n \in \mathcal{F}(X_n,Y)$ such that

$$\|T_nP_n(k_1,k_2) - Ti_nP_n(k_1,k_2)\|_Y < \frac{\epsilon}{2}$$

https://doi.org/10.4153/CMB-2009-004-5 Published online by Cambridge University Press

for all $(k_1, k_2) \in K$. Put $T_0 = T_1P_1 + T_2P_2 \in \mathcal{F}(X, Y)$. Then for all $(k_1, k_2) \in K$ we have

$$\|T_0(k_1, k_2) - T(k_1, k_2)\| \le \|T_1P_1(k_1, k_2) - Ti_1P_1(k_1, k_2)\| \\ + \|T_2P_2(k_1, k_2) - Ti_2P_2(k_1, k_2)\| < \epsilon.$$

Hence X has the rWAP. Similarly it follows that for each m $(\sum_{n=1}^{m} \oplus X_n)_{l_p}$ has the rWAP. Now put $X = (\sum_n \oplus X_n)_{l_p}$. Let Y be a Banach space and let $T \in \mathcal{K}(X, Y)$. Let $K \subset X$ be compact and $\epsilon > 0$. Using Fact 3 find a positive integer N such that

$$\|T\|^p \sum_{n>N} \|k_n\|_{X_n}^p < \left(\frac{\epsilon}{2}\right)^p$$

for all $(k_n) \in K$. Now $Ti \in \mathcal{K}((\sum_{n=1}^{N} \oplus X_n)_{l_p}, Y)$, where i is the map from $(\sum_{n=1}^{N} \oplus X_n)_{l_p}$ into X defined by $i(x_1, \dots, x_N) = (x_1, \dots, x_N, 0, \dots)$. Let $P\colon X \longrightarrow (\sum_{n=1}^{N} \oplus X_n)_{l_p}$ be the projection given by $P((x_n)) = (x_1, \dots, x_N)$. Since $P(K)$ is compact in $(\sum_{n=1}^{N} \oplus X_n)_{l_p}$ and $(\sum_{n=1}^{N} \oplus X_n)_{l_p}$ has the rWAP, there is a

$$T_0 \in \mathcal{F}\Big(\big(\sum_{n=1}^{N} \oplus X_n\big)_{l_p}, Y\Big) \text{ such that } \|T_0P(k_n) - TiP(k_n)\| < \frac{\epsilon}{2}$$

for all $(k_n) \in K$. Now $T_0P \in \mathcal{F}(X, Y)$ and for all $(k_n) \in K$

$$\|T_0P(k_n) - T(k_n)\| \le \|T_0P(k_n) - TiP(k_n)\| + \|T\|\Big(\sum_{n>N} \|k_n\|_{X_n}^p\Big)^{\frac{1}{p}} < \epsilon.$$

Hence X has the rWAP. Also, similarly, we can show that $(\sum_n \oplus X_n)_{c_0}$ has the rWAP. ■

The authors are naturally led to the following question related to the problem in the introduction.

Question If X and Y have the WAP (resp. QAP), then does $X \oplus Y$ have the WAP (resp. QAP) ?

But we have the following.

Theorem 3.5 *If X has the WAP (resp. QAP) and Y has the AP (resp. Y^* has the AP), then $X \oplus Y$ has the WAP (resp. QAP).*

Proof Suppose that X has the WAP and Y has the AP. Let $T \in \mathcal{K}(X \oplus Y)$, compact $K \subset X \oplus Y$, and $\epsilon > 0$. Now $Ti_x \in \mathcal{K}(X, X \oplus Y)$ and $Ti_y \in \mathcal{K}(Y, X \oplus Y)$ where i_x (resp. i_y) is the map from X (resp. Y) into $X \oplus Y$ defined by $i_x x = (x, 0)$ (resp. $i_y y = (0, y)$). Let P_x (resp. P_y) : $X \oplus Y \longrightarrow X$ (resp. Y) be the projection given by $P_x(x, y) = x$ (resp. $P_y(x, y) = y$). Consider $P_xTi_x \in \mathcal{K}(X)$, $P_yTi_x \in \mathcal{K}(X, Y)$, $P_xTi_y \in \mathcal{K}(Y, X)$, and

https://doi.org/10.4153/CMB-2009-004-5 Published online by Cambridge University Press

$P_yTi_y \in \mathcal{K}(Y)$. Since X and Y have the WAP, there are $T_{xx} \in \mathcal{F}(X)$ and $T_{yy} \in \mathcal{F}(Y)$ such that

$$\|T_{xx}x - P_xTi_xx\| < \frac{\epsilon}{4} \text{ and } \|T_{yy}y - P_yTi_yy\| < \frac{\epsilon}{4}$$

for all $(x, y) \in K$. Since Y has the AP, by Fact 1 there are $T_{xy} \in \mathcal{F}(X, Y)$ and $T_{yx} \in \mathcal{F}(Y, X)$ such that

$$\|T_{xy}x - P_yTi_xx\| < \frac{\epsilon}{4} \text{ and } \|T_{yx}y - P_xTi_yy\| < \frac{\epsilon}{4}$$

for all $(x, y) \in K$. Let the map $T_0\colon X \oplus Y \longrightarrow X \oplus Y$ defined by

$$T_0(x, y) = (T_{xx}x + T_{yx}y, T_{xy}x + T_{yy}y).$$

Observe $T_0 \in \mathcal{F}(X \oplus Y)$ and for all $(x, y) \in K$

$$\begin{aligned}
&\|T_0(x, y) - T(x, y)\|_{X\oplus Y}\\
&\le \|(T_{xx}x + T_{yx}y, T_{xy}x + T_{yy}y) - (P_xTi_xx + P_xTi_yy, P_yTi_xx + P_yTi_yy)\|_{X\oplus Y}\\
&\le \|(T_{xx}x - P_xTi_xx, T_{xy}x - P_yTi_xx)\|_{X\oplus Y} + \|(T_{yx}y - P_xTi_yy, T_{yy}y - P_yTi_yy)\|_{X\oplus Y}\\
&= \|T_{xx}x - P_xTi_xx\|_X + \|T_{xy}x - P_yTi_xx\|_Y + \|T_{yx}y - P_xTi_yy\|_X\\
&\qquad\qquad + \|T_{yy}y - P_yTi_yy\|_Y < \epsilon,
\end{aligned}$$

where we used l_1-sum for $X \oplus Y$, hence $X \oplus Y$ has the WAP. Similarly we can show the other part using the fact that Y^* having the AP implies

$$\mathcal{K}(X, Y) = \overline{\mathcal{F}(X, Y)} \text{ and } \mathcal{K}(Y, X) = \overline{\mathcal{F}(Y, X)}$$

for every Banach space X. ■

Acknowledgment The authors would like to express a sincere gratitude to the anonymous referee for valuable comments which resulted in stylistic improvement of the paper.

References

[1] P. G. Casazza, *Approximation Properties.* In: Handbook of the geometry of Banach spaces, Vol. 1, North-Holland, Amsterdam, 2001, pp. 271–316.

[2] C. Choi and J. M. Kim, *Weak and quasi approximation properties in Banach spaces.* J. Math. Anal. Appl. **316**(2006), no. 2, 722–735.

[3] J. Diestel, *Sequences and series in Banach spaces.* Graduate Texts in Mathematics 92, Springer-Verlag, New York, 1984.

[4] M. Feder and P. Saphar, *Spaces of compact operators and their dual spaces.* Israel J. Math. **21**(1975), no. 1, 38–49.

[5] T. Figiel, *Factorization of compact operators and applications to the approximation problem.* Studia Math. **45**(1973), 191–210.

[6] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16**(1955), no. 16.

[7] J. M. Kim, *Dual problems for weak and quasi approximation properties.* J. Math. Anal. Appl. **321**(2006), 569–575.

[8] ———, *On relations between weak approximation properties and their inheritances to subspaces.* J. Math. Anal. Appl.**324**(2006), no. 1, 721–727.
[9] ———, *Compact adjoint operators and approximation properties.* J. Math. Anal. Appl. **327**(2007), 257–268.
[10] J. Lindenstrauss, *On nonseparable reflexive Banach spaces.* Bull. Amer. Math. Soc. **72**(1966), 967–970.
[11] J. Lindenstrauss and L. Tzafriri,*Classical Banach Spaces. I. Sequence spaces.* Ergebnisse der Mathematik und ihrer Grenzgebiete 92, Springer-Verlag, Berlin, 1977.
[12] R. E. Megginson, *An Introduction to Banach space theory.* Graduate Texts in Mathematics 183, Springer-Verlag, New York, 1998.
[13] E. Oja and A. Pelander, *The approximation property in terms of the approximability of weak*-weak continuous operators.* J. Math. Anal. Appl. **286**(2003), no. 2, 713–723.

Division of Applied Mathematics, KAIST, Daejeon 305-701, Korea
e-mail: cschoi@kaist.ac.kr
northstar21@hanmail.net

National Institute for Mathematical Sciences, Yuseong-gu, Daejeon 305-340, Korea
e-mail: kjm21@nims.re.kr