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A NOTE ON BRØNDSTED'S FIXED POINT THEOREM

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Abstract

We show that for the case of uniformly convex Banach spaces, the conditions of Brøndsted's fixed point theorem can be relaxed.

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1. Introduction and main theorem

The object of this short note is a fixed point theorem by Arne Brøndsted. Let us formulate this theorem.

Let $(X, \|\cdot\|)$ be a Banach space and let $M \subset X$ be a closed set. We denote the closed unit ball by $B = \{x \in X \mid \|x\| \leq 1\}$. Assume that

$$M \cap B = \emptyset. \tag{1.1}$$

Consider a mapping $T : M \to M$ that maps each $x \in M$ in the direction of the ball: if $Tx \neq x$, then there exists $t > 1$ such that

$$x + t(Tx - x) \in B. \tag{1.2}$$

THEOREM 1.1 (Brøndsted [2]). *In addition to the assumptions above, suppose that*

$$\inf\{\|x\| \mid x \in M\} > 1. \tag{1.3}$$

Then the mapping T has a fixed point.

Observe that condition (1.3) is stronger than condition (1.1) only if $\dim X = \infty$.

To prove Theorem 1.1, Brøndsted endows the set M with a partial order in the following way.

DEFINITION 1.2. If $x, y \in M$, we write $x \leq y$ provided either $x = y$ or there exists $t > 1$ such that

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$$x + t(y - x) \in B.$$

The second possibility can equivalently be formulated as follows: there exist $\tilde{t} > 1$ and $a \in X$, $\|a\| = 1$, such that $x + \tilde{t}(y - x) = a$ and $x + t(y - x) \notin B$ for all $t < \tilde{t}$.

Equation (1.2) takes the form

$$x \preceq Tx \quad \text{for all } x \in M. \tag{1.4}$$

Then Brøndsted observes that this partial order is finer than that of the Caristi type [3] and, by some of his other results [1], the fixed point exists.

Our aim is to show that for the class of uniformly convex Banach spaces X, Theorem 1.1 remains valid even in the critical case when condition (1.3) is replaced by (1.1). This does not follow from Brøndsted's original method. We recall a definition.

DEFINITION 1.3. A Banach space $(X, \|\cdot\|)$ is said to be uniformly convex if for any $\sigma > 0$, there exists $\gamma > 0$ such that if $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \sigma$, then $\|x + y\| \leq 2 - \gamma$.

For example, the space L^p, $p \in (1, \infty)$, is uniformly convex. Similarly, ℓ_p is uniformly convex. Each uniformly convex Banach space is reflexive and a Hilbert space is uniformly convex (see [4] and references therein).

We now state our main result.

THEOREM 1.4. *Assume that X is a uniformly convex Banach space. If the mapping T satisfies condition (1.4) and condition (1.1) is fulfilled, then T has a fixed point.*

EXAMPLE 1.5. For the space X, take ℓ_p, $1 < p < \infty$. For each $n \in \mathbb{N}$, define

$$M_n = \{\mathbf{x} = \{x_k\} \in \ell_p \mid x_n \geq 1 + 1/n\}, \quad M = \bigcup_{n \in \mathbb{N}} M_n.$$

It is not hard to show that the set M is closed and $M \cap B = \emptyset$. A sequence

$$\mathbf{x}_j = (0, \dots, 0, 1 + 1/j, 0, \dots) \quad (\text{where } 1 + 1/j \text{ stands at the } j\text{th place}),$$

belongs to M and $\|\mathbf{x}_j\| \to 1$ as $j \to \infty$. Thus, the set M satisfies the hypotheses of Theorem 1.4, but not those of Theorem 1.1.

Now take any nonempty closed set $M \subset X$ with $M \cap B = \emptyset$ in a uniformly convex Banach space X and let $f : B \to B$ be a mapping. Construct T as follows. Take

$$\mathbf{x} \in M, \quad \mathbf{y} = f\Big(\frac{\mathbf{x}}{\|\mathbf{x}\|}\Big)$$

and let

$$\lambda_0 = \min\{\lambda \in [0, 1] \mid \lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in M\}.$$

It is clear that $\lambda_0 > 0$. Define $T\mathbf{x} = \lambda_0\mathbf{x} + (1 - \lambda_0)\mathbf{y}$. We obviously obtain $\mathbf{x} \preceq T\mathbf{x}$ and T has a fixed point. Since we assume only $M \cap B = \emptyset$, this fact follows from Theorem 1.4 and it does not follow from Theorem 1.1.

2. Proof of Theorem 1.4

The scheme of the proof is quite standard by itself. It is clear that a maximal element of the set M provides a fixed point. To prove that the maximal element exists, we check the conditions of Zorn's lemma. This argument and the technique developed below make it possible to give a direct proof of Theorem 1.1 as well.

PROPOSITION 2.1. *Suppose that vectors $a, x \in X$ have the properties*

$$\|(1-t)a + tx\| > 1 \quad \textit{for all } t \in (0,1),\ \|x\| > 1 \textit{ and } \|a\| = 1.$$

Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that inequality $\|x\| \le 1 + \delta$ implies $\|x - a\| \le \varepsilon$.

This proposition has a 'physical' interpretation. Let x be a light source placed away from the ball B, where $\|x\| > 1$. According to the proposition, the diameter of the light spot on the ball tends to zero as x approaches the ball, that is, $\|x\| \to 1$.

Here the uniform convexity of the norm is essential: such a feature fails for the norm $\|(p,q)\| = \max\{|p|, |q|\}$ in $\mathbb{R}^2$.

PROOF. Assume the opposite: there exist $\varepsilon > 0$ and sequences a_n, x_n, with

$$\|x_n\| > 1, \quad \|a_n\| = 1, \quad \|x_n\| \to 1 \quad \text{and} \quad \|(1-t)a_n + tx_n\| > 1, \tag{2.1}$$

such that

$$\|x_n - a_n\| > \varepsilon.$$

Consequently, for all sufficiently large n, the estimate

$$\|x_n - a_n\| = \left\| x_n - \frac{x_n}{\|x_n\|} + \frac{x_n}{\|x_n\|} - a_n \right\| \le \alpha_n + \left\| a_n - \frac{x_n}{\|x_n\|} \right\|,$$

where

$$\alpha_n = \|x_n\|\left(1 - \frac{1}{\|x_n\|}\right) \to 0,$$

implies

$$\left\| a_n - \frac{x_n}{\|x_n\|} \right\| \ge \varepsilon/2.$$

Substituting $t = 1/2$ in (2.1),

$$\|a_n + x_n\| > 2. \tag{2.2}$$

The inequality

$$\left\| a_n + \frac{x_n}{\|x_n\|} \right\| > 2 - \alpha_n$$

follows from (2.2) in the same way as above. This contradicts the hypothesis of uniform convexity of the space X. The proposition is proved. □

Let $C \subset M$ be a chain and put $\rho = \inf\{\|u\| \mid u \in C\}$ where $\rho \geq 1$. The inclusion $x \in C$ implies that $\|x\| > 1$ provided $\rho = 1$ and $\|x\| \geq \rho$ provided $\rho > 1$.

For any $x \in C$, define a set

$$K_x(\rho) = \{y \in M \mid \|y\| \geq \rho, x \preceq y\}.$$

The sets $K_x(\rho)$ are nonvoid: $x \in K_x(\rho)$ and

$$x_1 \preceq x_2 \Longrightarrow K_{x_2}(\rho) \subset K_{x_1}(\rho). \tag{2.3}$$

LEMMA 2.2. *The sets $K_x(\rho)$ are closed.*

PROOF. Indeed, let a convergent sequence $\{y_k\}$ belong to $K_x(\rho)$ and $y_k \to y \in M$. This means that there are sequences $\{\beta_k\} \subset (0,1)$ and $\{a_k\} \subset X$ with $\|a_k\| = 1$, such that

$$y_k = \beta_k a_k + (1-\beta_k)x.$$

The sequence $\{\beta_k\}$ contains a convergent subsequence; we keep the same notation for this subsequence, say $\beta_k \to \beta$. If $\beta = 0$, then $\|\beta_k a_k\| \to 0$ and $y = x \in K_\rho(x)$. If $\beta \neq 0$, put

$$a = \frac{1}{\beta}y + \Big(1 - \frac{1}{\beta}\Big)x$$

so that

$$a_k = \frac{1}{\beta_k}y_k + \Big(1 - \frac{1}{\beta_k}\Big)x \to a.$$

Since $\|a_k\| = 1$ and $a_k \to a$, we have $\|a\| = 1$. It follows that

$$y = \beta a + (1-\beta)x.$$

Since $y \in M$, the parameter β cannot be equal to 1. The lemma is proved. □

LEMMA 2.3. *Suppose that $z \in K_x(\rho)$ with $x \in C$. If $\rho > 1$, then*

$$\|z - x\| \leq (\|x\| - \rho)\frac{\|x\| + 1}{\rho - 1}.$$

If $\rho = 1$, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x\| \leq 1 + \delta \Longrightarrow \|z - x\| \leq \varepsilon.$$

PROOF. *The case $\rho > 1$.* The formula

$$x + t(z - x) = a, \quad \text{where } \|a\| = 1, t > 1 \text{ and } \|x\|, \|z\| \geq \rho > 1, \tag{2.4}$$

implies $z = (a + (t-1)x)/t$ and

$$\rho \leq \|z\| \leq \frac{1}{t} + \frac{t-1}{t}\|x\|, \quad \frac{1}{t} \leq \frac{\|x\| - \rho}{\|x\| - 1}.$$

Using (2.4) again,

$$\|z - x\| = \frac{1}{t}\|a - x\| \leq \frac{1}{t}(1 + \|x\|).$$

The case $\rho = 1$. The condition of the lemma that $z \in K_x(1)$ means

$$z = \tau a + (1-\tau)x, \quad \text{where } \|x\|, \|z\| > 1, \|a\| = 1 \text{ and } \tau \in (0,1).$$

Therefore, the assertion of the lemma follows from Proposition 2.1 and the formulae

$$z - x = \tau(a - x), \quad \|z - x\| \le \|x - a\|. \qquad \square$$

LEMMA 2.4. *For any* $\varepsilon > 0$, *there exists* $\tilde{x} \in C$ *such that*

$$C \ni x \ge \tilde{x} \Longrightarrow \operatorname{diam} K_x(\rho) \le \varepsilon.$$

PROOF. *The case* $\rho > 1$. By definition of the number ρ, for any $\varepsilon > 0$, there is an element $\tilde{x} \in C$ such that

$$\|\tilde{x}\| \le \varepsilon + \rho.$$

Take any elements $z_1, z_2 \in K_{\tilde{x}}$ and apply Lemma 2.3 for each summand on the right side of the inequality

$$\|z_1 - z_2\| \le \|z_1 - \tilde{x}\| + \|z_2 - \tilde{x}\|. \tag{2.5}$$

Observe also that (2.3) implies

$$\tilde{x} \le x \in C \Longrightarrow \operatorname{diam} K_x(\rho) \le \operatorname{diam} K_{\tilde{x}}(\rho). \tag{2.6}$$

The case $\rho = 1$. Fix $\varepsilon > 0$. By Lemma 2.3, there exists $\delta > 0$ such that if $\tilde{x} \in C$ and $\|\tilde{x}\| \le 1 + \delta$, then for any $z \in K_{\tilde{x}}(1)$, one has $\|\tilde{x} - z\| \le \varepsilon$. By definition of the number ρ, such an element $\tilde{x} \in C$ exists. Thus, (2.5), (2.6) remain valid.

The lemma is proved. $\square$

PROOF OF THEOREM 1.4. Therefore, we have a nested family of closed sets $K_x(\rho)$ whose diameters tend to zero. By a well-known theorem, their intersection is not empty and consists of a single point:

$$\bigcap_{x \in C} K_x(\rho) = \{m\}.$$

The point $m \in M$ is an upper bound for C. Indeed, for any $x \in C$, we have $m \in K_x(\rho)$ and thus $x \le m$. Theorem 1.4 is proved. $\square$

References

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