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The CREAM conjecture for the subvarieties of certain abelian-by-nilpotent varieties

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It is proved that the subvarieties of the variety $\underline{\underline{A}}_m(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_n)$ are CREAM in the sense of Higman when m, n are coprime and n is an odd integer not divisible by q^4 for any prime q.

Brady, Bryce and Cossey [2] have made a claim (now withdrawn) that the subvarieties of $\underline{\underline{A}}_m(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_n)$, where m, n are coprime, are CREAM in the sense of Higman [4]. The work in [6] confirms this in the case when n is an odd integer not divisible by q^4 for any prime q. Our solution is essentially an application of the methods of Higman [4] to the case when $\underline{\underline{W}} = \underline{\underline{N}}_2 \wedge \underline{\underline{B}}_n$. As the calculations are both tedious and technical, we shall omit some of the details which can be found in [6]. We refer to [4] and [8] for the relevant terminology and concepts.

In view of results ([4], [6]) on the irreducible linear groups belonging to $\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_n$, we only need to show that for each closed class $\underline{\underline{X}}$ of irreducible linear groups belonging to $\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_{q^3}$, where q is an odd prime, the following function is CREAM:

$$n \mapsto c_n(\underline{\underline{X}}) , \quad n = 1, 2, \ldots .$$

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Here $c_n(\underline{\underline{X}}) = \sum_{X \in \underline{\underline{X}}} c_n(X)$, and $c_n(X) = (\deg X)^2 k_n(X)/|\text{lin aut}X|$, where $\deg X$ is the degree of the irreducible linear group X over a splitting field, $k_n(X)$ is the eulerian function of X and lin aut$X$ is the group of linear automorphisms of X .

1. Symplectic modules and linear automorphisms

The calculation of lin aut$X$ is reduced to a calculation of a certain subgroup $QSp(U)$ of the group of isometries $Sp(U)$ of a symplectic module U over Z_{q^α} , the ring of integers modulo q^α for a prime q . Although the structure of symplectic spaces over a field is well-known (see, for example, [1], [5]), that of symplectic modules over Z_{q^α} , a ring with zero divisors, is, to my knowledge, not explicitly mentioned anywhere in the literature. However, the latter is found (in [6]) to be very similar to the former, and we shall merely state the relevant results.

A finitely-generated Z_{q^α}-module U is a *symplectic module* (over Z_{q^α}) if there is an alternating Z_{q^α}-bilinear form f defined on U . U is said to be *non-degenerate* if f is non-degenerate. We write $V = V_1 \perp V_2$ to mean that $V = V_1 \oplus V_2$ and $f(v_1, v_2) = 0$ for all $v_i \in V_i$, $i = 1, 2$, and $\langle u_1, \ldots, u_r\rangle$ to denote the submodule generated by $u_1, \ldots, u_r$. Two non-degenerate symplectic modules U, V are isomorphic if there is a module isomorphism from U onto V which preserves their bilinear forms.

THEOREM 1.1. *Let U be a non-degenerate symplectic module over Z_{q^α} . Then*

$$U = U_1 \perp \ldots \perp U_\alpha ,$$

where for $1 \le i \le \alpha$, $U_i = 0$ *or*

$$U_i = \langle u_{i1}, v_{i1}\rangle \perp \ldots \perp \left\langle u_{in_i}, v_{in_i}\right\rangle , \quad n_i > 0 ,$$

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with $f(u_{ij}, v_{ij}) = q^{i-1}$, $j = 1, \ldots, n_i$.

U is determined up to isomorphism by the sequence of non-negative integers $(n_1, \ldots, n_\alpha)$.

An *isometry* of U is an automorphism of U which preserves the bilinear form f on U . We denote the group of isometries of U by $Sp(U)$ or $Sp(n_\alpha, \ldots, n_1)$ where $n_1, \ldots, n_\alpha$ are the invariants of U given by Theorem 1.1. Without loss of generality, we shall assume that $n_1 > 0$ so that there exist elements u and v in U such that $f(u, v) = 1$. An ordered pair (u, v) of elements of U is called a *hyperbolic pair* of U if $f(u, v) = 1$. We denote the set of hyperbolic pairs of U by $\Omega(U)$ and write $\omega(n_\alpha, \ldots, n_1) = |\Omega(U)|$. The order of $Sp(U)$ is calculated by an enumeration of hyperbolic pairs (*cf.* II.9.13 in [5]).

LEMMA 1.2. *$Sp(U)$ acts transitively on $\Omega(U)$.*

LEMMA 1.3. *$|Sp(U)| = \prod_{i=1}^{\alpha} \prod_{j=1}^{n_i} \omega(n_\alpha, \ldots, n_{i+1}, j)$, where*

$$\log_q \omega(m_\beta, \ldots, m_1) = (2m_1-1)\beta + 2m_1(\beta-1) + 4 \sum_{i=2}^{\beta} m_i(\beta-i+1) + \log_q\left(q^{2m_1}-1\right) .$$

LEMMA 1.4. *We have*

$$\log_q|Sp(n_\alpha, \ldots, n_1)| = \sum_{i=1}^{\alpha} \left\{(2\alpha-2i+1)n_i^2+(\alpha-i)n_i\right\} + 4 \sum_{i=1}^{\alpha-1} \sum_{j=i}^{\alpha-1} (\alpha-j)n_i n_{j+1} + \sum_{i=1}^{\alpha} \sum_{j=1}^{n_i} \log_q\left(q^{2j}-1\right) .$$

Kovács has shown (see Lemma 4.2.1 in [6]) that if X is an irreducible linear group in $\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_n$, then the group of linear automorphisms, lin aut$X$, of X is the group $\mathrm{aut}_Z X$ of automorphisms of X which act trivially on the centre $Z(X)$ of X . Now if $X \in \underline{\underline{N}}_2 \wedge \underline{\underline{B}}_{q^r}$, where q is an odd prime, then it is shown in [7] that X is uniquely determined by an abstract group of the form (see [2] for the definition of

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the $Q(n, r)$):

$$G = Q(n_1, r_1) \ldots Q(n_\alpha, r_\alpha) Q(l, l)^{\varepsilon_l} \ldots Q(1, 1)^{\varepsilon_1} ,$$

where $n_1 > \ldots . > n_\alpha > l$, $n_\alpha > r_1 > \ldots > r_\alpha \geq 0$,

$$0 < n_1 - r_1 < \ldots < n_\alpha - r_\alpha , \quad \alpha \geq 0 , \quad \varepsilon_i \geq 0 , \quad i = 1, 2, \ldots, l .$$

By factoring off by the centre $Z(G)$ of G , we can consider $U = G/Z(G)$ as a non-degenerate symplectic module over Z_{q^m} , where $q^m = |Z(G)|$, with the alternating bilinear form f given by:

$$\text{for all } x, y \in G , \quad f(\bar{x}, \bar{y}) = \lambda ,$$

where $\bar{x}, \bar{y}$ are the corresponding cosets in U , $[x, y] = z^\lambda$ and z is a fixed generator of $Z(G)$.

Let a_i, b_i, a_{jk}, b_{jk} , $i = 1, \ldots, \alpha$, $k = 1, \ldots, \varepsilon_j$, $1 \leq j \leq l$, be canonic generators (see [7]) of G . Define the following sets of elements of U :

$$A_i = \left\{\bar{x} \in U : x^{q^{r_i}} = a_i^{q^{r_i}}\right\} , \quad i = 1, \ldots, \alpha ,$$

$$B_i = \left\{\bar{x} \in U : x^{q^{r_i}} = 1\right\} , \quad i = 1, \ldots, \alpha ,$$

$$D_i = \left\{\bar{x} \in U : x^{q^i} = 1\right\} , \quad i = 1, \ldots, l .$$

Then the set of isometries φ of U such that

$$\bar{a}_i \varphi \in A_i , \quad \bar{b}_i \varphi \in B_i , \quad i = 1, \ldots, \alpha ,$$

$$\bar{a}_{jk}\varphi, \bar{b}_{jk}\varphi \in D , \quad k = 1, \ldots, \varepsilon_j , \quad 1 \leq j \leq l ,$$

forms a subgroup $QSp(U)$ of $Sp(U)$. If $\mathrm{inn}G$ denotes the group of inner automorphisms of G , it is not difficult to show that

THEOREM 1.5. $Qsp(U) \cong \mathrm{aut}_Z G/\mathrm{inn}G$.

A detailed analysis of $QSp(U)$ enables us to calculate its order.

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The calculations are found in [6] and we merely state

THEOREM 1.6. *Let* $n_0 = n_1$, $r_0 = l$, $n_{\alpha+1} = m = \max\{l, r_1, n_\alpha - r_\alpha\}$, $r_{\alpha+1} = 1$,

$$I_\beta = \{i : r_{\beta+1} \le i \le n_{\beta+1} - n_\beta + r_\beta\}, \quad 0 \le \beta \le \alpha,$$

$$J_\beta = \{i : n_{\beta+1} - n_\beta + r_\beta < i < r_\beta\}, \quad 1 \le \beta \le \alpha,$$

$$s_i = 2 \sum_{k=\beta+1}^{\alpha} r_k + \begin{cases} m - n_{\beta+1} & \text{if } i \in I_\beta, \quad 0 \le \beta \le \alpha, \\ m - n_\beta + r_\beta & \text{if } i \in J_\beta, \quad 1 \le \beta \le \alpha, \end{cases}$$

$$t_i = \begin{cases} 2\beta & \text{if } i \in I_\beta, \quad 0 \le \beta \le \alpha, \\ 2\beta - 1 & \text{if } i \in J_\beta, \quad 1 \le \beta \le \alpha. \end{cases}$$

Then

$$\log_q |QSp(U)| = 2\alpha m + 4 \sum_{i=1}^{\alpha} \sum_{j=i}^{\alpha} r_j - \sum_{i=1}^{\alpha} r_i - 2 \sum_{i=1}^{\alpha} n_i + 2 \sum_{i=1}^{l} (s_i + i t_i)\varepsilon_i + \log_q |Sp(U)|,$$

where

$$\log_q |Sp(U)| = \sum_{i=1}^{l} \left\{(2i-1)\varepsilon_i^2 + (i-1)\varepsilon_i\right\} + 4 \sum_{i=1}^{l-1} \sum_{j=1}^{i} j\varepsilon_j \varepsilon_{i+1} + \sum_{i=1}^{l} \sum_{j=1}^{\varepsilon_i} \log_q (q^{2j} - 1).$$

2. The closed classes of irreducible linear groups

We refer to [4] for the meaning of linear factors and closed classes. We summarize the following elementary observations (see [6]) on the relation $\prec$ of being a linear factor. In this section A, B, C, D will denote irreducible linear groups (over a splitting field) belonging to $\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_n$. Thus $A \prec B$ means that A is a linear factor of B.

LEMMA 2.1. *$C \prec B$ and $B \prec A$ implies that $C \prec A$.*

LEMMA 2.2. *$C \prec A$ and $D \prec B$ implies that $CD \prec AB$, where AB, CD*

denote central products with cyclic centres.

LEMMA 2.3. *Let* $A_0 \leq A$ *such that* A_0 *has cyclic centre. Then* A_0 *is an irreducible linear group and* $A_0 \prec A$.

LEMMA 2.4. *(i)* $Q(n, 0) \prec Q(n, 1) \prec \ldots \prec Q(n, n-1) \prec Q(n, n)$.

(ii) $Q(n-1, r-1) \prec Q(n, r)$, $0 < r < n$.

Henceforth q will denote an odd prime. In this section, we introduce a convenient way of partitioning the infinite closed classes of irreducible linear groups of a fixed exponent. This will be useful in Section 4. We denote the closed class of all irreducible linear groups in $\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_{q^i}$ by $\underline{\underline{Q}}_i$. We write $Q(k) = Q(k, k)$. Let $n > 1$ be a fixed integer. Suppose $\underline{\underline{X}} \subseteq \underline{\underline{Q}}_n$ is a closed class not contained in $\underline{\underline{Q}}_{n-1}$. Define the following (not necessarily closed) classes of linear groups:

$$\underline{\underline{S}}_i = \underline{\underline{X}} \cap Q(n, i)\underline{\underline{Q}}_{n-1} , \quad i = 0, 1, \ldots, n-1 ,$$

$$\underline{\underline{S}}^j = \underline{\underline{X}} \cap Q(n)^j\underline{\underline{Q}}_{n-1} , \quad j = 0, 1, \ldots ,$$

where $X\underline{\underline{Q}}_{n-1}$ denotes the set of central products with cyclic centres of X and Y for each $Y \in \underline{\underline{Q}}_{n-1}$ and $X = Q(n, i)$ or $Q(n)^j$. We call the $\underline{\underline{S}}_i$, $\underline{\underline{S}}^j$ the *derived classes* of $\underline{\underline{X}}$. Note that $\underline{\underline{S}}^0$ is closed.

We classify the derived classes as follows. We say that the *rank* of $\underline{\underline{S}}_i$ is 0 if $\underline{\underline{S}}_i$ is finite (or empty); that the *rank* of $\underline{\underline{S}}_i$ is 1 if $Q(n, i)Q(1)^{\varepsilon_1} \in \underline{\underline{S}}_i$ for all $\varepsilon_1 = 0, 1, \ldots$, but $Q(n, i)Q(2)^{\varepsilon_2} \notin \underline{\underline{S}}_i$ for some $\varepsilon_2 > 0$; and in general, that the *rank* of $\underline{\underline{S}}_i$ is k , where $1 \leq k < n$, if $Q(n, i)Q(k)^{\varepsilon_k} \ldots Q(1)^{\varepsilon_1} \in \underline{\underline{S}}_i$ for all $\varepsilon_1, \ldots, \varepsilon_k = 0, 1, \ldots$, but $Q(n, i)Q(k+1)^{\varepsilon_{k+1}} \notin \underline{\underline{S}}_i$ for some $\varepsilon_{k+1} > 0$. Likewise, for each $j > 0$, we say that the *rank* of $\underline{\underline{S}}^j$ is 0 if $\underline{\underline{S}}^j$ is finite (or empty); and that the *rank* of $\underline{\underline{S}}^j$ is k , where $1 \leq k < n$, if

$Q(n)^j Q(k)^{\varepsilon_k} \ldots Q(1)^{\varepsilon_1} \in \underline{\underline{S}}^j$ for all $\varepsilon_1, \ldots, \varepsilon_k = 0, 1, \ldots$, but $Q(n)^j Q(k+1)^{\varepsilon_{k+1}} \notin \underline{\underline{S}}^j$ for some $\varepsilon_{k+1} > 0$. Note that $\underline{\underline{S}}^0$ consists of linear groups in $\underline{\underline{Q}}_{n-1}$ and we do not define the rank of $\underline{\underline{S}}^0$.

We shall need the following observations.

LEMMA 2.5. *(i) If rank of* $\underline{\underline{S}}_0 = k > 0$, *then rank of* $\underline{\underline{S}}_i = k$ *for* $i = 0, 1, \ldots, k$.

(ii) The following chain of inequalities holds:

$$\text{rank of } \underline{\underline{S}}_0 \geq \text{rank of } \underline{\underline{S}}_1 \geq \ldots \geq \text{rank of } \underline{\underline{S}}_{n-1} \geq$$
$$\geq \text{rank of } \underline{\underline{S}}^1 \geq \text{rank of } \underline{\underline{S}}^2 \geq \ldots .$$

Proof. *(i)* Since $\underline{\underline{S}}_0$ has rank $k > 0$,

$$Q(n, 0)Q(k)^{\varepsilon_k} \ldots Q(1)^{\varepsilon_1} \in \underline{\underline{S}}_0$$

for all $\varepsilon_1, \ldots, \varepsilon_k = 0, 1, \ldots$. It is clear from Corollary 2.3 of [2] and Lemmas 2.1, 2.2, 2.3 that if $0 \leq i \leq k$, $Q(n, i)Q(k)^{\varepsilon_k} \ldots Q(1)^{\varepsilon_1} \in \underline{\underline{X}}$ for all $\varepsilon_1, \ldots, \varepsilon_k = 0, 1, \ldots$ and hence are in $\underline{\underline{S}}_i$. Moreover if $Q(n, i)Q(k+1)^{\varepsilon_{k+1}} \in \underline{\underline{S}}_i$ for all $\varepsilon_{k+1} = 0, 1, \ldots$, then $Q(n, 0)Q(k+1)^{\varepsilon_{k+1}} \in \underline{\underline{S}}_0$ for all $\varepsilon_{k+1} = 0, 1, \ldots$, which is not so. Hence $\underline{\underline{S}}_i$ has rank k.

(ii) Let $0 \leq i < n-1$. Suppose rank of $\underline{\underline{S}}_i = k_i$. Then rank of $\underline{\underline{S}}_{i+1}$ cannot exceed k_i. Otherwise $Q(n, i+1)Q(l)^{\varepsilon_l} \in \underline{\underline{S}}_{i+1}$ for all $\varepsilon_l = 0, 1, \ldots$, where $l = k_i + 1$, and since $Q(n, i) \prec Q(n, i+1)$ by Lemma 2.4 *(i)*, it follows from Lemma 2.2 that $Q(n, i)Q(l)^{\varepsilon_l} \in \underline{\underline{S}}_i$ for all $\varepsilon_l = 0, 1, \ldots$, which is a contradiction. Similarly, we can easily prove:

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rank of $\underline{\underline{S}}_{n-1} \geq$ rank of $\underline{\underline{S}}^1 \geq \ldots$. //

The next lemma is immediate.

LEMMA 2.6. $\underline{\underline{X}} = \left(\bigcup_{i=0}^{n-1} \underline{\underline{S}}_i\right) \cup \left(\bigcup_{j=0}^{\nu} \underline{\underline{S}}^j\right)$, *where the union is disjoint and* $\nu = \sup\{j : Q(n)^j \in \underline{\underline{X}}\}$.

We call ν in the above lemma the *index* of $\underline{\underline{X}}$. For our purposes, we need a detailed description of the derived classes in some simple cases.

LEMMA 2.7. *Suppose* $\underline{\underline{X}} \subseteq \underline{\underline{Q}}_2$ *is a closed class not contained in* $\underline{\underline{Q}}_1$. *The derived classes of* $\underline{\underline{X}}$ *of rank* 1 *are of the forms*

$$\underline{\underline{S}}_0 = \{Q(2, 0)Q(1)^r : r = 0, 1, \ldots\} ,$$

$$\underline{\underline{S}}_1 = \{Q(2, 1)Q(1)^r : r = 0, 1, \ldots\} ,$$

$$\underline{\underline{S}}^j = \{Q(2)^j Q(1)^r : r = 0, 1, \ldots\} , \quad j > 0 .$$

Proof. From [7], the irreducible linear groups of exponent q^2 are $Q(2, 0)Q(1)^r$, $Q(2, 1)Q(1)^r$, $Q(2)^j Q(1)^r$, $j > 0$, $r \geq 0$. //

LEMMA 2.8. *Suppose* $\underline{\underline{X}} \subseteq \underline{\underline{Q}}_3$ *is a closed class not contained in* $\underline{\underline{Q}}_2$. *The derived classes of* $\underline{\underline{X}}$ *of rank* 1 *are of the forms*

$$\underline{\underline{S}}_i = \bigcup_{s=0}^{\sigma_i} \underline{\underline{R}}_{i,s} , \quad 0 \leq \sigma_i < \infty , \quad i = 0, 1, 2 ,$$

$$\underline{\underline{S}}^j = \bigcup_{s=0}^{\mu_j} \underline{\underline{R}}^{(j,s)} , \quad 0 \leq \mu_j < \infty , \quad j > 0 ,$$

where

$$\underline{\underline{R}}_{i,0} = \{Q(3, i)Q(1)^r : r = 0, 1, \ldots\} , \qquad i = 0, 1, 2 ,$$

$$\underline{\underline{R}}_{i,s} = \{Q(3, i)Q(2)^s Q(1)^r : r = 0, 1, \ldots, \rho_{is} \leq \infty\} ,$$
$$0 < s \leq \sigma_i , \quad i = 0, 1, 2 ,$$

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$$\underline{\underline{R}}^{(j,0)} = \{Q(3)^j Q(1)^r : r = 0, 1, \ldots\} , \qquad j > 0 ,$$

$$\underline{\underline{R}}^{(j,s)} = \{Q(3)^j Q(2)^s Q(1)^r : r = 0, 1, \ldots, \lambda_{js} \le \infty\} ,$$

$$0 < s \le \mu_j , \quad j > 0 .$$

The derived classes of $\underline{\underline{X}}$ *of rank* 2 *are of the forms*

$$\underline{\underline{S}}_i = \{Q(3, i)Q(2)^s Q(1)^r : r, s = 0, 1, \ldots\} , \quad i = 0, 1, 2 ,$$

$$\underline{\underline{S}}^j = \{Q(3)^j Q(2)^s Q(1)^r : r, s = 0, 1, \ldots\} , \quad j > 0 .$$

Proof. From [7], the irreducible linear groups of exponent q^3 are $Q(3, i)Q(2)^s Q(1)^r$, $Q(3)^j Q(2)^s Q(1)^r$, $i = 0, 1, 2$, $j > 0$, $r, s \ge 0$. If the derived class $\underline{\underline{S}}_i$ has rank 1 , then by definition, there is a unique largest integer $\sigma_i \ge 0$ for which $Q(3, i)Q(2)^{\sigma_i} \in \underline{\underline{S}}_i$. If $\sigma_i > 0$, we define ρ_{is} for each $0 < s \le \sigma_i$, to be the largest integer for which $Q(3, i)Q(2)^s Q(1)^{\rho_{is}} \in \underline{\underline{S}}_i$. Note that ρ_{is} may be infinite. Similarly for $\underline{\underline{S}}^j$, $j > 0$, or rank 1 . The forms of the derived classes of rank 2 are obvious. //

LEMMA 2.9. *Suppose* $\underline{\underline{X}} \subseteq \underline{\underline{Q}}_2$ *is a closed class not contained in* $\underline{\underline{Q}}_1$. *If the derived class* $\underline{\underline{S}}_0$ *has rank* 1 , *then* $\underline{\underline{S}}^0 \cup \underline{\underline{S}}_0 \cup \underline{\underline{S}}_1$ *is the class of all irreducible linear groups in* $\underline{\underline{A}}_{q^2} \vee (\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_q)$.

Proof. $\underline{\underline{S}}^0 = \underline{\underline{Q}}_1$, $\underline{\underline{S}}_i = \{Q(2, i)Q(1)^r : r = 0, 1, \ldots\}$, $i = 0, 1$ .//

LEMMA 2.10. *Suppose* $\underline{\underline{X}} \subseteq \underline{\underline{Q}}_3$ *is a closed class not contained in* $\underline{\underline{Q}}_2$. *If the derived class* $\underline{\underline{S}}_0$ *has rank* 2 , *then* $\underline{\underline{S}}^0 \cup \left(\bigcup_{i=0}^{2} \underline{\underline{S}}_i\right)$ *is the class of all irreducible linear groups in* $\underline{\underline{A}}_{q^3} \vee \left(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_{q^2}\right)$.

Proof. $\underline{\underline{S}}^0 = \underline{\underline{Q}}_2$ and $\underline{\underline{S}}_i$, $i = 0, 1, 2$, are given by Lemma 2.8.

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Evidently the union of these classes gives all the irreducible linear groups in the stated variety. //

3. Some calculations of $c_n(X)$

In this section, we carry out some calculations of $c_n(X)$ for the relevant linear group X. We shall denote the q^i-cycle by C_i and write $C_iC_j = C_i \times C_j$, $C_i^k = C_iC_i^{k-1}$, $k \geq 2$.

LEMMA 3.1. *(i) For* $n \geq r \geq 0$, $\deg Q(n, r) = q^r$.

(ii) Suppose XY *is a central product with cyclic centre of* X, $Y \in \underline{\underline{Q}}_i$. *Then* $\deg(XY) = \deg X \cdot \deg Y$.

Proof. *(i)* Since we are working in a splitting field, the result is trivial if $r = 0$. So suppose $r > 0$. We may assume that the field contains a primitive q^r-th root of unity, ξ say. Then it is easily checked that

$$a = \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & & & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & & & \\ & \xi & & \\ & & \ddots & \\ & & & \xi^{q^r-1} \end{pmatrix}$$

gives a matrix representation of $Q(r, r)$ so that $\deg Q(r, r) = q^r$. Since $Q(n, 0)Q(n, r) \cong Q(n, 0)Q(r, r)$ by [2, Corollary 2.3], it follows that $\deg Q(n, r) = q^r$.

(ii) If $XY = X$, then Y is cyclic, and so the assertion follows from *(i)*. Now suppose $XY \neq X$ or Y. Let U and V be the vector spaces on which X and Y act respectively. Then the outer tensor product $U \# V$ is the space on which XY acts, and so $\dim(U\#V) = \dim U \cdot \dim V$. //

The next lemma is a special case of a theorem of Gaschütz (Satz 1 in [3]) and reduces the calculation of $k_n(X)$ to that of $k_n(C_1^m)$.

LEMMA 3.2. *Let* G *be a finite group and* $H \triangleleft G$, $H \leq \Phi(G)$. *Then*

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$k_n(G) = |H|^n k_n(G/H)$.

We can calculate $k_n\left(C_1^m\right)$ using another theorem of Gaschütz [3].

LEMMA 3.3. *For* $m \geq 1$, $k_n\left(C_1^m\right) = \left(q^n - q^{m-1}\right)k_n\left(C_1^{m-1}\right)$.

Proof. Consider C_1^m as a vector space V of dimension m over $\mathrm{GF}(q)$, generated by $v_1, \ldots, v_m$. Then by Satz 2 of [3],

$$k_n\left(C_1^m\right) = \left(q^n - c\right)k_n\left(C_1^{m-1}\right) ,$$

where c is the number of complements of $\langle v_1\rangle$ in V . Let

$$A = \{\alpha \in \operatorname{aut}V : v_1\alpha = \lambda v_1 \text{ for some } 0 \neq \lambda \in \mathrm{GF}(q)\} .$$

Then A is a subgroup consisting of linear transformations of V of the form

$$v_1\alpha = \lambda_1 v_1 , \quad \lambda_1 \neq 0 , \quad v_i\alpha = \lambda_i v_i + u_i , \quad i = 2, \ldots, m ,$$

where $\lambda_1, \ldots, \lambda_m \in \mathrm{GF}(q)$, $u_2, \ldots, u_m \in U = \langle v_2, \ldots, v_m\rangle$. It is easily checked that $u_2, \ldots, u_m$ are linearly independent. Hence $|A| = (q-1)q^{m-1}|\mathrm{GL}(m-1, q)|$. Let $\Omega = \{W \leq V : V = \langle v_1\rangle \oplus W\}$. For every $\alpha \in A$, $W \in \Omega$, we have $V = V\alpha = \langle v_1\rangle \oplus W$. For any two $W, W' \in \Omega$, there exists $\alpha \in A$ such that $W\alpha = W'$. We may define α by

$$v_1\alpha = v_1 , \quad w_i\alpha = w_i' , \quad i = 2, \ldots, m ,$$

where $\{w_2, \ldots, w_m\}$, $\{w_2', \ldots, w_m'\}$ are bases of W, W' respectively. Hence A acts transitively on Ω and so $|A| = |A_0| \cdot |\Omega|$, where $A_0 = \{\alpha \in A : W_0\alpha = W_0 \text{ and } W_0 = \langle v_2, \ldots, v_m\rangle\}$. Plainly A_0 consists of all those $\alpha \in A$ for which $\lambda_2, \ldots, \lambda_m$ are all zero. Hence $|A_0| = (q-1)|\mathrm{GL}(m-1, q)|$, and so $|\Omega| = q^{m-1}$. //

COROLLARY 3.4. *For* $m \geq 1$,

$$k_n\left(C_1^m\right) = q^{\frac{1}{2}m(m-1)} \prod_{i=1}^{m} \left(q^{n-i+1}-1\right) .$$

In the next four technical lemmas, we shall assume that $n_1 > l > 0$, $\varepsilon_l > 0$, and write $\lambda = 2\sum_{i=1}^{l} \varepsilon_i$, $\mu = 2\sum_{i=1}^{l} (i-1)\varepsilon_i$. In Lemmas 3.6 and 3.8, δ_{l,n_1-1} is the Kronecker delta.

LEMMA 3.5. *We have*

(i) $k_n\left(Q(l)^{\varepsilon_l} \ldots Q(1)^{\varepsilon_1}\right) = q^{(l+\mu)n} k_n\left(C_1^\lambda\right)$,

(ii) $k_n\left(Q\left(n_1, 0\right)Q(l)^{\varepsilon_l} \ldots Q(1)^{\varepsilon_1}\right) = q^{\left(n_1-1+\mu\right)n}\left(q^n-q^\lambda\right)k_n\left(C_1^\lambda\right)$,

(iii) $k_n\left(Q\left(n_1, 1\right)Q(l)^{\varepsilon_l} \ldots Q(1)^{\varepsilon_1}\right) =$
$$q^{\left(n_1-1+\mu\right)n}\left(q^n-q^{1+\lambda}\right)\left(q^n-q^\lambda\right)k_n\left(C_1^\lambda\right),$$

(iv) $k_n\left(Q\left(n_1, n_1-1\right)Q(l)^{\varepsilon_l} \ldots Q(1)^{\varepsilon_1}\right) =$
$$q^{\left(3n_1-5+\mu\right)n}\left(q^n-q^{1+\lambda}\right)\left(q^n-q^\lambda\right)k_n\left(C_1^\lambda\right).$$

Proof. As an example, we prove *(ii)*. The others are similarly proved. Put $X = Q\left(n_1, 0\right)Q(l)^{\varepsilon_l} \ldots Q(1)^{\varepsilon_1}$, so that $|X'| = q^l$ and $\overline{X} = X/X' \cong C_{n_1-l}C_l^{2\varepsilon_l} \ldots C_1^{2\varepsilon_1}$. By Lemma 3.2, $k_n(X) = q^{ln}k_n(\overline{X})$. Now put $\overline{N} = \langle \overline{x}^q : \overline{x} \in \overline{X}\rangle \leq \Phi(\overline{X})$, so that $|\overline{N}| = q^{n_1-l-1+\mu}$ and $\overline{X}/\overline{N} \cong C_1^{1+\lambda}$. Hence by Lemmas 3.2, 3.3, $k_n(\overline{X}) = |\overline{N}|^n\left(q^n-q^\lambda\right)k_n\left(C_1^\lambda\right)$. //

In the following lemma, $\mathrm{Sp}\left(\varepsilon_1, \ldots, \varepsilon_l\right)$ is the group in Section 1 and its order is given in Lemma 1.4. The next lemma is a direct application of the results in Section 1.

LEMMA 3.6. *We have*

(i) $$\left|\mathrm{lin\ aut}\left(Q(l)^{\varepsilon_l}\ \dots\ Q(1)^{\varepsilon_1}\right)\right| = \left|\mathrm{lin\ aut}\left(Q(n_1,\ 0)Q(l)^{\varepsilon_l}\ \dots\ Q(1)^{\varepsilon_1}\right)\right| = q^{\lambda}\left|\mathrm{Sp}(\varepsilon_1,\ \dots,\ \varepsilon_l)\right|\ ,$$

(ii) $$\left|\mathrm{lin\ aut}\left(Q(n_1,\ 1)Q(l)^{\varepsilon_l}\ \dots\ Q(1)^{\varepsilon_1}\right)\right| = q^{3+2\lambda}\left|\mathrm{Sp}(\varepsilon_1,\ \dots,\ \varepsilon_l)\right|\ ,$$

(iii) $$\left|\mathrm{lin\ aut}\left(Q(n_1,\ n_1-1)Q(l)^{\varepsilon_l}\ \dots\ Q(1)^{\varepsilon_1}\right)\right| = q^{5n_1+3\lambda+2\mu-7-2\varepsilon_l\delta_{l,n_1-1}}\left|\mathrm{Sp}(\varepsilon_1,\ \dots,\ \varepsilon_l)\right|\ .$$

The following two formulae are obtained by direct substitution into the formula for $c_n(X)$ using Lemmas 3.1, 3.5, 3.6.

LEMMA 3.7. *We have*

$$c_n\left(Q(n_1,\ 0)Q(l)^{\varepsilon_l}\ \dots\ Q(1)^{\varepsilon_1}\right) + c_n\left(Q(n_1,\ 1)Q(l)^{\varepsilon_l}\ \dots\ Q(1)^{\varepsilon_1}\right) = q^{(n_1-l)n-1}\left(q^{n-\lambda}-1\right)c_n\left(Q(l)^{\varepsilon_l}\ \dots\ Q(1)^{\varepsilon_1}\right)\ .$$

LEMMA 3.8. *Let* $\gamma = (3n_1-l-5)n - 3n_1 + 5 + 2\varepsilon_l\delta_{l,n_1-1}$. *Then*

$$c_n\left(Q(n_1,\ n_1-1)Q(l)^{\varepsilon_l}\ \dots\ Q(1)^{\varepsilon_1}\right) = q^{\gamma}\left\{q^{2(n-\lambda-\mu)}-(q+1)q^{n-\lambda-2\mu}+q^{1-2\mu}\right\}c_n\left(Q(l)^{\varepsilon_l}\ \dots\ Q(1)^{\varepsilon_1}\right)\ .$$

Henceforth we shall use the following notation:

$$c_n(r,\ s,\ t) = q^{2n(s+2t)+r(r-1)-(s+t)(s+3t)-2s-3t} \times \prod_{i=1}^{2(r+s+t)}\left(q^{n-i+1}-1\right)\prod_{i=1}^{r}\left(q^{2i}-1\right)^{-1}\prod_{i=1}^{s}\left(q^{2i}-1\right)^{-1}\prod_{i=1}^{t}\left(q^{2i}-1\right)^{-1}\ ,$$

where n, r, s, t are non-negative integers. For given r, s, t , $n \mapsto c_n(r,\ s,\ t)$ defines a function on the non-negative integers. In the

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next section, we reduce the study of the relevant CREAM problem to a study of some properties of $c_n(r, s, t)$. But first a straightforward observation.

LEMMA 3.9. *(i) For* $r \geq 1$, $c_n\left(Q(1)^r\right) = q^n c_n(r, 0, 0)$.

(ii) For $r \geq 0$, $s \geq 1$, $c_n\left(Q(1)^r Q(2)^s\right) = q^{2n} c_n(r, s, 0)$.

(iii) For $r, s \geq 0$, $t \geq 1$,

$$c_n\left(Q(1)^r Q(2)^s Q(3)^t\right) = q^{3n} c_n(r, s, t) .$$

4. The CREAM problem

In considering the CREAM problem for subvarieties of $\underline{\underline{A}}_r \underline{\underline{W}}$ where $\underline{\underline{W}} = \underline{\underline{N}}_2 \wedge \underline{\underline{B}}_s$ with r, s coprime, we only need to consider those subvarieties satisfying $\underline{\underline{W}} \leq \underline{\underline{V}} \leq \underline{\underline{A}}_p \underline{\underline{W}}$ where p is a prime and s is a prime power q^α with $p \neq q$. This follows from §§1.1, 1.3, 2.2, 2.7 of [4] and Lemma 4.4 below. Thus Theorem 4.1 follows from Theorem 4.2.

THEOREM 4.1. *Let r, s be coprime positive integers where s is odd and not divisible by q^4 for any prime q. Then the subvarieties of $\underline{\underline{A}}_r\left(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_s\right)$ are CREAM.*

THEOREM 4.2. *Let p, q be distinct primes with q odd. Then the subvarieties of $\underline{\underline{A}}_p\left(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_{q^3}\right)$ are CREAM.*

We shall use the notations and concepts of Section 2. We say that a non-empty (not necessarily closed) class $\underline{\underline{S}}$ of irreducible linear groups is CREAM if the following function is CREAM:

$$n \mapsto c_n(\underline{\underline{S}}) , \quad n = 1, 2, \ldots .$$

Theorem 4.2 then follows from the next theorem.

THEOREM 4.3. *Every non-empty closed class $\underline{\underline{X}} \subseteq \underline{\underline{Q}}_3$ is CREAM.*

Proof. (i) $\underline{\underline{X}} \subseteq \underline{\underline{Q}}_1$. If $\underline{\underline{X}} = \underline{\underline{Q}}_1$, then $\underline{\underline{X}}$ is CREAM by Lemma 4.5.

If $\underline{\underline{X}} \neq \underline{\underline{Q}}_1$, then $\underline{\underline{X}}$ is clearly finite and so is CREAM.

(ii) $\underline{\underline{X}} \subseteq \underline{\underline{Q}}_2$, $\underline{\underline{X}} \nsubseteq \underline{\underline{Q}}_1$. Suppose $\underline{\underline{S}}_0, \underline{\underline{S}}_1, \underline{\underline{S}}^j$, $j \geq 0$, are the derived classes of $\underline{\underline{X}}$. We may assume that the index of $\underline{\underline{X}} = \nu < \infty$. Otherwise $\underline{\underline{X}} = \underline{\underline{Q}}_2$ and so is CREAM by Lemma 4.5. Also $\underline{\underline{S}}^0 \subseteq \underline{\underline{Q}}_1$ is closed and hence CREAM by (i).

If $\underline{\underline{S}}_0$ has rank 0, then so do $\underline{\underline{S}}_1, \underline{\underline{S}}^j$, $j > 0$, by Lemma 2.5 *(ii)*. Hence $\underline{\underline{S}}_0 \cup \underline{\underline{S}}_1 \cup \underline{\underline{S}}^1 \cup \ldots \cup \underline{\underline{S}}^\nu$ is finite and CREAM. Thus, by Lemma 2.6, $\underline{\underline{X}}$ is CREAM.

If $\underline{\underline{S}}_0$ has rank 1, then $\underline{\underline{S}}^0 \cup \underline{\underline{S}}_0 \cup \underline{\underline{S}}_1$ is CREAM by §2.4 of [4] and Lemma 2.9. If $\underline{\underline{S}}^j$ has rank 0 for $j = 1, \ldots, \nu$, then $\bigcup_{j=1}^{\nu} \underline{\underline{S}}^j$ is finite and hence CREAM. So suppose $\underline{\underline{S}}^j$ has rank 1 for some $1 \leq j \leq \nu$. Let $1 \leq \lambda \leq \nu$ be the largest integer for which $\underline{\underline{S}}^\lambda$ has rank 1. Then $\underline{\underline{S}}^j = \{Q(2)^j Q(1)^r : r = 0, 1, \ldots\}$, $1 \leq j \leq \lambda$, and so is CREAM by Lemma 4.12. Since $\bigcup_{j=\lambda+1}^{\nu} \underline{\underline{S}}^j$ is finite (or empty), it follows that $\bigcup_{j=1}^{\nu} \underline{\underline{S}}^j$ is CREAM. Hence $\underline{\underline{X}}$ is CREAM.

(iii) $\underline{\underline{X}} \subseteq \underline{\underline{Q}}_3$, $\underline{\underline{X}} \nsubseteq \underline{\underline{Q}}_2$. Suppose $\underline{\underline{S}}_0, \underline{\underline{S}}_1, \underline{\underline{S}}_2, \underline{\underline{S}}^j$, $j \geq 0$, are the derived classes of $\underline{\underline{X}}$. Again we may assume that the index of $\underline{\underline{X}} = \nu < \infty$. Since $\underline{\underline{S}}^0 \subseteq \underline{\underline{Q}}_2$ is closed, $\underline{\underline{S}}^0$ is CREAM by (i) and (ii).

By Lemma 2.5 *(ii)*, we may assume rank of $\underline{\underline{S}}_0 > 0$. If $\underline{\underline{S}}_0$ has rank 1, then so has $\underline{\underline{S}}_1$ by Lemma 2.5 *(i)*. We have from Lemma 2.8,

$$\underline{\underline{S}}_0 = \bigcup_{s=0}^{\sigma_0} \{Q(3, 0)Q(2)^s Q(1)^r : r = 0, 1, \ldots, \rho_{0s}\},$$

$$\underline{\underline{S}}_1 = \bigcup_{s=0}^{\sigma_1} \{Q(3, 1)Q(2)^s Q(1)^r : r = 0, 1, \ldots, \rho_{1s}\},$$

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where $\sigma_0, \sigma_1 < \infty$. Clearly $\rho_{0s} = \infty$ if and only if $\rho_{1s} = \infty$. Let $0 \le \tau \le \max\{\sigma_0, \sigma_1\}$ be the largest integer for which $\rho_{0\tau} = \infty$. Then

$$\underline{\underline{S}}_0 = \bigcup_{s=0}^{\tau} \{Q(3, 0)Q(2)^s Q(1)^r : r = 0, 1, \ldots\} \cup \underline{\underline{S}}_0' ,$$

$$\underline{\underline{S}}_1 = \bigcup_{s=0}^{\tau} \{Q(3, 1)Q(2)^s Q(1)^r : r = 0, 1, \ldots\} \cup \underline{\underline{S}}_1' ,$$

where the unions are disjoint and $\underline{\underline{S}}_0'$, $\underline{\underline{S}}_1'$ are finite (or empty). By Lemma 4.14, $\{Q(3, 0)Q(2)^s Q(1)^r, Q(3, 1)Q(2)^s Q(1)^r : r = 0, 1, \ldots\}$ is CREAM for every $0 \le s \le \tau$. Hence $\underline{\underline{S}}_0 \cup \underline{\underline{S}}_1$ is CREAM.

Without loss of generality, suppose rank of $\underline{\underline{S}}_2 > 0$. If it is 1, then by Lemma 2.8,

$$\underline{\underline{S}}_2 = \bigcup_{s=0}^{\sigma_2} \{Q(3, 2)Q(2)^s Q(1)^r : r = 0, 1, \ldots, \rho_{2s}\} , \quad \sigma_2 < \infty .$$

Let $0 \le \tau' \le \sigma_2$ be the largest integer for which $\rho_{2\tau'} = \infty$. Then

$$\underline{\underline{S}}_2 = \bigcup_{s=0}^{\tau'} \{Q(3, 2)Q(2)^s Q(1)^r : r = 0, 1, \ldots\} \cup \underline{\underline{S}}_2'$$

where the union is disjoint and $\underline{\underline{S}}_2'$ is finite (or empty). By Lemma 4.15, each of the infinite classes in the above union is CREAM. Hence $\underline{\underline{S}}_2$ is CREAM, and so is $\bigcup_{i=0}^{2} \underline{\underline{S}}_i$.

If $\underline{\underline{S}}_0$ has rank 2, then by Lemma 2.10 and [4, §2.4], $\underline{\underline{S}}^0 \cup \left(\bigcup_{i=0}^{2} \underline{\underline{S}}_i\right)$ is CREAM. Thus it remains to prove that $\underline{\underline{S}}^j$, $1 \le j \le \nu$, is CREAM for $\underline{\underline{S}}^j$ of rank 1 or 2. Firstly if $\underline{\underline{S}}^j$ has rank 1, then from Lemma 2.8,

$$\underline{\underline{S}}^j = \bigcup_{s=0}^{\mu_j} \{Q(3)^j Q(2)^s Q(1)^r : r = 0, 1, \ldots, \lambda_{js}\} , \quad \mu_j < \infty .$$

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Let $0 \le \delta_j \le \mu_j$ be the largest integer for which $\lambda_{j\delta_j} = \infty$. Then

$$\underline{\underline{S}}^j = \bigcup_{s=0}^{\delta_j} \{Q(3)^j Q(2)^s Q(1)^r : r = 0, 1, \ldots\} \cup \underline{\underline{T}}^j$$

where the union is disjoint and $\underline{\underline{T}}^j$ is finite (or empty). Each of the infinite classes in this union is CREAM by Lemma 4.13. Hence $\underline{\underline{S}}^j$ is CREAM.

Finally if $\underline{\underline{S}}^j$ has rank 2, then by Lemma 2.8,

$$\underline{\underline{S}}^j = \{Q(3)^j Q(2)^s Q(1)^r : r, s = 0, 1, \ldots\}$$

and so is CREAM by Lemma 4.16. //

We now prove the lemmas used above.

LEMMA 4.4. $|F_n(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_s)| = s^{\frac{1}{2}n(n+1)}$.

Proof. We have

$$G_n = F_n(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_s) = \langle x_1, \ldots, x_n : x_i^s = [x_i, x_j]^s =$$
$$= [[x_i, x_j], x_k] = 1, i \ne j, i, j, k, = 1, \ldots, n\rangle .$$

Since G_n is of class 2, $[x_i x_j, x_k] = [x_i, x_k][x_j, x_k]$ and so $G_n' = \langle [x_i, x_j] : i, j = 1, \ldots, n\rangle$. Now G_n' is a direct product of $\frac{1}{2}n(n-1)$ s-cycles and G_n/G_n' is a direct product of n s-cycles. Hence $|G_n| = |G_n/G_n'| \cdot |G_n'| = s^{\frac{1}{2}n(n+1)}$. //

LEMMA 4.5. *$\underline{\underline{Q}}_i$ is CREAM for $i \ge 1$.*

Proof. From [4, §2.4], $c_n(\underline{\underline{Q}}_i) = \left|F_n\left(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_{q^i}\right)\right|$, and hence $\underline{\underline{Q}}_i$ is CREAM by Lemma 4.4. //

We say that $f(n)$ is CREAM if the function $n \mapsto f(n)$, $n = 0, 1, \ldots$, is CREAM. Note that there is no difficulty in adding $n = 0$ to the domain. This is done to facilitate the proofs of the following lemmas. Note also that in each of the following (apparently) infinite sums there are only finitely many terms for each fixed non-negative integer n so that these sums are finite.

LEMMA 4.6. *$\sum_{r=1}^{\infty} c_n(r, 0, 0)$ is CREAM.*

Proof. For $n > 0$, we have

$$c_n(\underline{\underline{Q}}_1) = c_n(1) + c_n(Q(1, 0)) + \sum_{r=1}^{\infty} c_n(Q(1)^r)$$

$$= 1 + (q^n-1) + q^n \sum_{r=1}^{\infty} c_n(r, 0, 0)$$

using Lemma 3.9 *(i)*. Hence from the proof of Lemma 4.5,

$$\sum_{r=1}^{\infty} c_n(r, 0, 0) = q^{\frac{1}{2}n(n-1)} - 1 , \quad n > 0 .$$

Clearly $c_0(r, 0, 0) = 0$ for $r > 0$. So the above relation holds for $n \geq 0$. Hence the result. //

If we use the explicit expression for $c_n(r, 0, 0)$ in the above proof, we obtain the following polynomial identity.

LEMMA 4.7. *For any real number x and positive integer n,*

$$\sum_{r=0}^{[\frac{1}{2}n]} x^{r(r-1)} \prod_{i=1}^{2r} (x^{n-i+1}-1) \prod_{i=1}^{r} (x^{2i}-1)^{-1} = x^{\frac{1}{2}n(n-1)} .$$

LEMMA 4.8. *$\sum_{r=0}^{\infty} \sum_{s=1}^{\infty} c_n(r, s, 0)$ is CREAM.*

Proof. From Lemma 2.9, we have

$$\underline{\underline{Q}}_2 = \underline{\underline{Y}} \cup \{Q(1)^r Q(2)^s : r = 0, 1, \ldots , s = 1, 2, \ldots\} ,$$

where $\underline{\underline{Y}}$ is the class of all irreducible linear groups in $\underline{\underline{W}}_0 = \underline{\underline{A}}_{q^2} \vee (\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_q)$. It is not difficult to deduce from §2.4 of [4] that $c_n(\underline{\underline{Y}}) = |F_n(\underline{\underline{W}}_0)|$. Since $\underline{\underline{A}}_{q^2} \wedge (\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_q) = \underline{\underline{A}}_q$, §2.2 of [4] gives

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$$|F_n(\underline{\underline{A}}_q)| \cdot |F_n(\underline{\underline{W}}_0)| = \left|F_n\left(\underline{\underline{A}}_{q^2}\right)\right| \cdot |F_n(\underline{\underline{N}}_2 \wedge \underline{\underline{B}}_q)| .$$

Hence by Lemmas 3.9 *(ii)*, 4.4 and the proof of 4.5,

$$\sum_{r=0}^{\infty} \sum_{s=1}^{\infty} c_n(r, s, 0) = q^{-2n}\{c_n(\underline{\underline{Q}}_2)-c_n(\underline{\underline{Y}})\} = q^{n(n-1)} - q^{\frac{1}{2}n(n-1)} , \quad n > 0 .$$

Clearly the above relation is also true for $n = 0$. Hence the result. //

LEMMA 4.9. $\sum_{r=1}^{\infty} q^{-2kr}c_n(r, s, 0)$ *is CREAM for* $s, k \geq 0$.

Proof. Write $b_{n,k}(r, s, 0) = q^{-2kr}c_n(r, s, 0)$, $s, k \geq 0$ and $f_k(n) = \sum_{r=1}^{\infty} b_{n,k}(r, 0, 0)$. We use induction on k to show that $f_k(n)$ is CREAM for $k \geq 0$. Clearly $f_0(n)$ is CREAM by Lemma 4.6. Suppose now $k > 0$ and $f_{k-1}(n)$ is CREAM. It is easily verified that for $n \geq 2$, $r \geq 1$,

$$\left(q^{2r}-1\right)b_{n,k}(r, 0, 0) = q^{-2k}\left(q^n-1\right)\left(q^{n-1}-1\right)b_{n-2,k-1}(r-1, 0, 0) ,$$

or

$$b_{n,k}(r, 0, 0) = b_{n,k-1}(r, 0, 0) - q^{-2k}\left(q^n-1\right)\left(q^{n-1}-1\right)b_{n-2,k-1}(r-1, 0, 0) .$$

Summing from $r = 1$ to $r = \infty$, we have for $n \geq 2$,

(1) $$f_k(n) = f_{k-1}(n) - f(n)\{1+f_{k-1}(n-2)\} ,$$

where $f(n) = q^{-2k}\left(q^n-1\right)\left(q^{n-1}-1\right)$ is obviously CREAM. By hypothesis, $f_{k-1}(n)$ is CREAM and hence so is $f_{k-1}(n-2)$. Since $f_k(0) = f_k(1) = 0$, $k \geq 0$, relation (1) also holds for $n = 0, 1$. Hence $f_k(n)$ is CREAM.

For a fixed $k \geq 0$, write $g_s(n) = \sum_{r=1}^{\infty} b_{n,k}(r, s, 0)$, $s \geq 0$. We use induction on s to show that $g_s(n)$ is CREAM for $s \geq 0$. Clearly

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$g_0(n) = f_k(n)$ is CREAM. So suppose $s > 0$ and $g_{s-1}(n)$ is CREAM. It can be checked that for $r, s \geq 1$, $n \geq 2$,

$$b_{n,k}(r, s, 0) = g(n)b_{n-2,k}(r, s-1, 0) ,$$

where $g(n) = q^{2n-2s-5}(q^n-1)(q^{n-1}-1)(q^{2s}-1)^{-1}$. Hence for $n \geq 2$,

$$g_s(n) = g(n)g_{s-1}(n-2) . \tag{2}$$

Now $g_s(0) = g_s(1) = 0$ for $s \geq 0$. Hence relation (2) also holds for $n = 0, 1$. Since $g(n)$ is obviously CREAM, $g_s(n)$ is CREAM. //

LEMMA 4.10. $\sum_{r=1}^{\infty} c_n(r, s, t)$ *is CREAM for* $s, t \geq 0$.

Proof. For a fixed $s \geq 0$, write $h_t(n) = \sum_{r=1}^{\infty} c_n(r, s, t)$, $t \geq 0$. Since the proof by induction on t is similar to the second part of the proof of Lemma 4.9, we omit the details. We merely note that $h_0(n)$ is CREAM by the preceding lemma and that for $t > 0$, $n \geq 0$,

$$h_t(n) = c(n)h_{t-1}(n-2)$$

where $c(n)$ is CREAM. //

LEMMA 4.11. $\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} c_n(r, s, t)$ *is CREAM for* $t \geq 0$.

Proof. Write $u_t(n) = \sum_{r=0}^{\infty} c_n(r, 0, t) + \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} c_n(r, s, t)$, $t \geq 0$. It is clear that $u_0(n)$ is CREAM by Lemma 4.6 and 4.8. Also $u_t(0) = u_t(1) = 0$, $t > 0$. As in the proof of Lemma 4.10, we have

$$u_t(n) = c(n)u_{t-1}(n-2) , \quad t > 0 , \quad n \geq 0 ,$$

where $c(n)$ is CREAM. We then induct on t. //

We shall now relate the above CREAM results to the classes of irreducible linear groups in $\underset{=}{Q}_3$.

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LEMMA 4.12. $\{Q(2)^s Q(1)^r : r = 0, 1, \ldots\}$ *is CREAM for* $s > 0$.

Proof. By Lemma 3.9 *(ii)*, we have for $n = 1, 2, \ldots$,

$$\sum_{r=0}^{\infty} c_n\left(Q(2)^s Q(1)^r\right) = q^{2n}\left\{c_n(0, s, 0) + \sum_{r=1}^{\infty} c_n(r, s, 0)\right\} ,$$

which is CREAM by Lemma 4.9 (with $k = 0$). //

LEMMA 4.13. $\{Q(3)^t Q(2)^s Q(1)^r : r = 0, 1, \ldots\}$ *is CREAM for* $s \geq 0$, $t > 0$.

Proof. From Lemma 3.9 *(iii)*, we have for $n = 1, 2, \ldots$,

$$\sum_{r=0}^{\infty} c_n\left(Q(3)^t Q(2)^s Q(1)^r\right) = q^{3n}\left\{c_n(0, s, t) + \sum_{r=1}^{\infty} c_n(r, s, t)\right\} .$$

The result then follows from Lemma 4.10. //

LEMMA 4.14. $\{Q(3, 0)Q(2)^s Q(1)^r, Q(3, 1)Q(2)^s Q(1)^r : r = 0, 1, \ldots\}$ *is CREAM for* $s \geq 0$.

Proof. It is sufficient to show that

$$\{Q(3, 0)Q(2)^s Q(1)^r, Q(3, 1)Q(2)^s Q(1)^r : r = 1, 2, \ldots\}$$

is CREAM. Considering the cases $s = 0$ and $s > 0$ separately, we have from Lemmas 3.7 and 3.9,

$$\sum_{r=1}^{\infty} \left\{c_n\left(Q(3, 0)Q(2)^s Q(1)^r\right) + c_n\left(Q(3, 1)Q(2)^s Q(1)^r\right)\right\}$$

$$= q^{4n-2s-1} \sum_{r=1}^{\infty} q^{-2r} c_n(r, s, 0) - q^{3n-1} \sum_{r=1}^{\infty} c_n(r, s, 0)$$

for $n = 1, 2, \ldots$. The result then follows from Lemma 4.9. //

LEMMA 4.15. $\{Q(3, 2)Q(2)^s Q(1)^r : r = 0, 1, \ldots\}$ *is CREAM for* $s \geq 0$.

Proof. It is sufficient to show that

$$\{Q(3, 2)Q(2)^s Q(1)^r : r = 1, 2, \ldots\}$$

is CREAM. Considering the cases $s = 0$ and $s > 0$ separately, we have from Lemmas 3.8 and 3.9,

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$$\sum_{r=1}^{\infty} c_n\left(Q(3,\ 2)Q(2)^s Q(1)^r\right) = q^{6n-6s-4} \sum_{r=1}^{\infty} q^{-4r} c_n(r,\ s,\ 0)$$

$$- (q+1)q^{5n-4s-4} \sum_{r=1}^{\infty} q^{-2r} c_n(r,\ s,\ 0) + q^{4n-2s-3} \sum_{r=1}^{\infty} c_n(r,\ s,\ 0)$$

for $n = 1, 2, \ldots$. The result then follows from Lemma 4.9. //

LEMMA 4.16. *$\{Q(3)^t Q(2)^s Q(1)^r : r,\ s = 0,\ 1,\ \ldots\}$ is CREAM for $t > 0$.*

Proof. By Lemma 3.9 *(iii)*,

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} c_n\left(Q(3)^t Q(2)^s Q(1)^r\right) = q^{3n} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} c_n(r,\ s,\ t)$$

for $n = 1, 2, \ldots$. Hence the result follows from Lemma 4.11. //

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