J. Austral. Math. Soc. **24** (Series A) (1977), 465–495.

PROPER COLOURINGS OF K_{15}

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(Received October 12, 1976; revised February 10, 1977)

Communicated by W. D. Wallis

Abstract

We denote the complete graph on n vertices by K_n. A proper k-colouring of K_n is a way of assigning colours from a set of k colours to the edges of K_n in such a way that no monochromatic triangles are formed. It is known that there are precisely two proper 3-colourings of K_{16} each of which has exactly one proper 3-colouring of K_{15} embedded in it. We show that these two are the only proper 3-colourings of K_{15}.

Subject classification (Amer. Math. Soc. 1970): *05C15, 05C99

Throughout this paper the usual definitions of graph theory are used. The complete graph on n vertices is denoted by K_n. A *proper k-colouring* of K_n is a way of assigning colours from a set of k colours to the edges of K_n in such a way that no monochromatic triangles are formed. Alternatively, a proper k-colouring is a factorization of K_n into k factors, none of which contains a triangle. These factors are called *monochromatic subgraphs* and the coloured K_n is obtained by assigning one colour to all edges of each monochromatic subgraph.

We shall customarily use the colours R, B and G; in figures we denote these by solid, broken and dotted lines respectively. Two vertices joined by an edge coloured R will be called R-adjacent, and so on. Two proper colourings are *isomorphic* if one can be obtained from the other by a relabelling of vertices or an exchange of colours. (If we say a proper colouring is "unique" we mean "unique up to isomorphism".) Suppose that C and D are k-colourings of K_n and K_{n+1} respectively and that there exists a vertex v in D such that $D - v$ is isomorphic to C. Then we say that C is *embedded* in D.

When $k = 2$ the maximum value of n such that K_n can be properly 2-coloured is 5 and the colouring is unique. This colouring and the two distinct proper 2-colourings of K_4 are shown in Figure 1. Only one of the K_4 colourings can be embedded in the K_5 colouring. When $k = 3$ the maximum

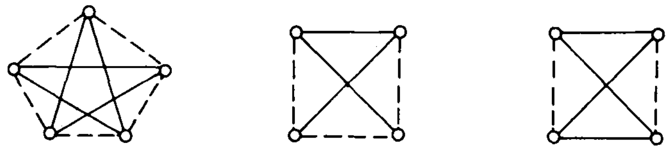


Figure 1.

value of n is 16 and (Kalbfleisch and Stanton) there are precisely two proper 3-colourings of K_{16}. Each colouring of K_{16} has only one colouring of K_{15} embedded in it as the automorphism groups of both K_{16} colourings are transitive (see Street and Wallis (1976)). These two colourings of K_{15} are not isomorphic. This leads to the question as to whether there are any other proper 3-colourings of K_{15}. Following the procedure used by Kalbfleisch and Stanton we show that, apart from the two colourings of K_{15} which can be embedded in the colourings of K_{16}, there are no others.

Let C be a proper 3-colouring of K_{15} and let C_R, C_B and C_G be the monochromatic subgraphs of C. In a monochromatic subgraph of C the maximum degree is five. For, suppose C_R contains a vetex v of degree six. Then the subgraph of C induced by the six vertices adjacent to v, a coloured K_6, must have each of its edges coloured either B or G. But we know that the maximum size of a complete graph that can be properly 2-coloured is K_5, a contradiction. If any vertex has degree three in one monochromatic subgraph then it must have degree at least six in one of the other two, a contradiction. Thus in any monochromatic subgraph of C all vertices have degree four or five. If all the veritces in any monochromatic subgraph of C have degree five the subgraph will have $\frac{15 \times 5}{2}$ edges which is obviously impossible. Therefore, at least one vertex has degree four and the maximum number of edges in a subgraph is $\frac{(14 \times 5) + 4}{2} = 37$. By a similar argument we see that in any monochromatic subgraph there is always an odd number of vertices with degree four.

We shall associate with C an *edge-vector* (x, y, z) which signifies that C_R has x edges, C_B has y edges and C_G has z edges. As K_{15} has $\binom{15}{2} = 105$ edges, $x + y + z = 105$. Obviously, we can assume that $x \geqq y \geqq z$. Thus there are seven possible edge-vectors for C. These are

$$(37, 37, 31),\ (37, 36, 32),\ (37, 35, 33),\ (37, 34, 34),$$

$$(36, 36, 33),\ (36, 35, 34),$$

$$(35, 35, 35).$$

First, we show that C satisfies a lemma which was proven by Kalbfleisch and Stanton to be valid for any proper 3-colouring of K_{16}; we use the lemma extensively in the remainder of the paper. In Theorem 1 we prove that the only possible edge-vector is (35, 35, 35) and in Theorem 2 we show that the only proper 3-colourings of K_{15} having this edge-vector are those two embedded in proper 3-colourings of K_{16}.

LEMMA 1. *Consider any two vertices of C. If the edge joining them is coloured R, then at most two vertices are adjacent to both of the given ones in C_B and in C_G.*

PROOF. Let the two vertices be labelled 1 and 2. Assume the three vertices 3, 4 and 5 are adjacent to both 1 and 2 in C_B. Label the remaining vertices $6, \cdots, 15$.

There are three cases to consider.

CASE 1. Vertices 1 and 2 both have degree four in C_B.

CASE 2. Vertex 1 has degree four and vertex 2 degree five in C_B.

CASE 3. Both vertices 1 and 2 have degree five in C_B.

For each case a figure will be drawn showing all possible ways of colouring the edges incident with vertices 1 and 2. Any edge for which there is no choice of colour or which can be coloured without loss of generality is then coloured.

Notice that:

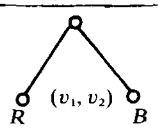
(*) In either proper 2-colouring of K_4 no vertex has degree three in either colour and in the unique proper 2-colouring of K_5 all vertices have degree two in each colour; see Figure 1. Thus, if we have a 2-coloured K_5, say the colours are R and B, on the vertices v_1, v_2, v_3, v_4 and v_5 and the edges (v_1, v_2) and (v_1, v_3) are R then the edges (v_1, v_4) and (v_1, v_5) must be B. A 2-coloured K_3, K_4 or K_5 arises as a subgraph of K_{15} when the three, four of five points involved are the endpoints of three, four or five edges of the same colour drawn from some other point.

An adjacency matrix $A = (a_{ij})$ showing the edges coloured so far is constructed. The symbol "B/G" means that either B or G can be used, but not R; for example, if a_{ij} and a_{jk} are both R, a_{ik} will be labelled B/G. Whenever the edge (i, j) can be coloured either R, B or G, a_{ij} is left empty.

We then attempt to colour the remaining edges. This is done in each case by using the given binary decision tree. At each point of the tree an edge for which there were only two possible colours, say R and B, is on one branch coloured R and on the other coloured B. This procedure is continued until a monochromatic triangle cannot be avoided.

Points of the tree are drawn as, $\circ$, ∇, $\square$, $\bullet$ or $\blacktriangledown$ and the meanings of these symbols are given in Table 1.

TABLE 1. Notation Used in Binary Decision Trees

Symbol	Meaning
	The edge (v_1, v_2) is chosen to be one of two possible colours, in this case R or B, and this is assumed for all other point symbols.
$\rho\sqsubset$	The edge (v_1, v_2) is coloured in the above manner. If we then apply the permutation ρ to the vertices of the graph we see that the colouring with (v_1, v_2) R is isomorphic to that with (v_1, v_2) B. This is illustrated in Diagram 1b.
$\nabla\{v_1, v_2, v_3, v_4, v_5\}$	The edge and its colour are chosen so that we have a 2-coloured K_5 on the vertices $\{v_1, v_2, v_3, v_4, v_5\}$ which either has a vertex v of degree two in one of the colours and so all edges incident with v in this K_5 can be coloured, or four of these vertices yield a properly 2-coloured K_4 which is then uniquely extended to a properly 2-coloured K_5 on these vertices.
$\nabla\{v_1, v_2, v_3, v_4\}$	In the 2-coloured K_4 on $\{v_1, v_2, v_3, v_4\}$ we have a vertex v of degree two in one of the colours and so all edges of the K_4 incident with v can be coloured.
$\bullet$ or $\blacktriangledown$	The meanings are as for $\circ$ and ∇ respectively but in both cases a monochromatic triangle can no longer be avoided.

CASE 1. Vertex 1 must have degree five in C_R so colour the edges $(1,7)$, $(1,8)$, $(1,9)$ and $(1,10)$ R. At least three of the edges $(2,7)$, $(2,8)$, $(2,9)$ and $(2,10)$ must be coloured G as at most one of them can be B. Colouring C as in Figure 2, we see that we cannot colour the edges $(7,8)$, $(8,9)$ and $(7,9)$ in such a way as to avoid a monochromatic triangle in C.

CASE 2. Here there are two subcases to consider.

2.1. The four vertices 3, 4, 5 and 6 are all adjacent to vertices 1 and 2 in C_B. This case is now eliminated using a similar argument to that of *CASE* 1.

2.2. Only the three vertices 3, 4 and 5 are adjacent to both 1 and 2 in C_B. Vertex 2 must have degree four in C_R otherwise an argument similar to that of *CASE* 1 can be applied. The edges incident with vertices 1 and 2 must now

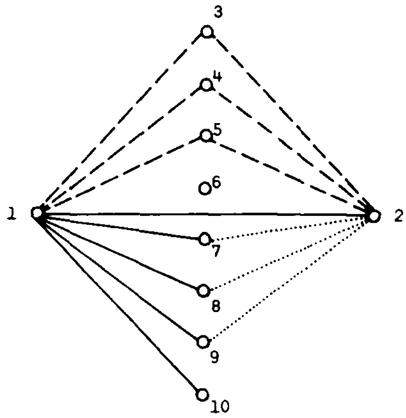


Figure 2.

be coloured as in Figure 3. If the edge $(2, 6)$ is G, then three of the vertices 11, 12, 13, 14 and 15 must be G-adjacent to 1 and R-adjacent to 2. This would lead us to a triangle on these three vertices with three B edges. Hence $(2, 6)$ is R.

Edges $(7, 8)$, $(9, 10)$ and $(14, 15)$ must be coloured as shown.

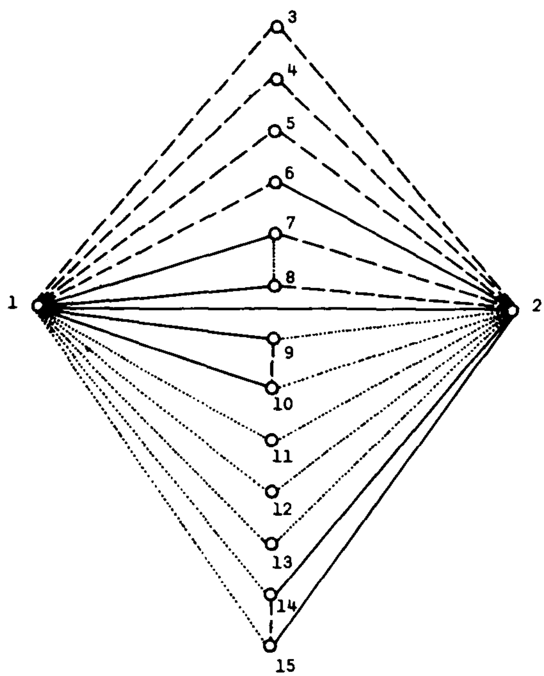


Figure 3.

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Consider the K_3 on $\{3,4,5\}$. There must be at least one R and one G edge, say $(3,4)$ R and $(4,5)$ G. As neither $(4,7)$ nor $(4,8)$ can be B and both cannot be G, at least one must be R, say $(4,7)$ is R (note that this choice is equivalent to choosing $(4,8)$ R). Hence $(3,7)$ is G and then $(3,8)$ is R. The K_5 on $\{3,4,5,7,8\}$ can now be uniquely completed. Consider the K_4 on $\{3,4,5,6\}$. Obviously $(5,6)$ must be R as in a proper 2-colouring of K_4 no vertex has degree three in either colour (see Figure 1). One of the edges $(8,9)$ or $(8,10)$ must be G and as 9 and 10 are interchangeable colour $(8,9)$ G and complete the K_5 on $\{2,7,8,9,10\}$. One of the edges $(11,12)$, $(12,13)$ and $(11,13)$ must be R so colour $(12,13)$ R. Either $(13,14)$ or $(12,14)$ is B so colour $(13,14)$ B. Then $(13,15)$ is R and $(12,15)$ is B. The K_5 on $\{11,12,13,14,15\}$ can now be completed. Looking at the K_5 on $\{9,10,11,12,13\}$ we see that $(9,11)$ and $(10,11)$ must both be R.

We have thus arrived at the partial colouring shown in Diagram 1a and the search is continued via the tree shown in Diagram 1b; no completion can be found.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1															
2	R														
3	B	B													
4	B	B	R												
5	B	B	G	G											
6	B	R	R/G	R/G	R										
7	R	B	G	R	R	B/G									
8	R	B	R	G	R	B/G	G								
9	R	G		R/B			B	G							
10	R	G	R/B				G	B	B						
11	G	G						R	R						
12	G	G							R/B	R/B	B				
13	G	G							R/B	R/B	B	R			
14	G	R				B/G			B/G	B/G	R	R	B		
15	G	R				B/G			B/G	B/G	R	B	R	B	

Diagram 1a.

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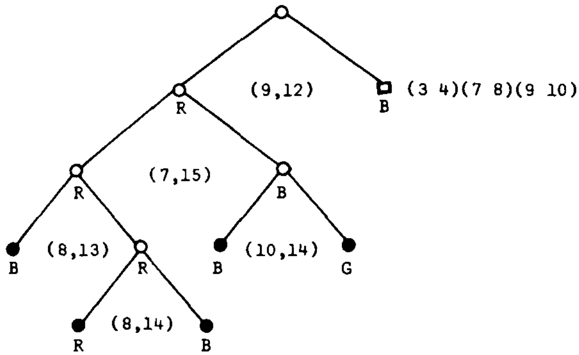

Diagram 1b.

CASE 3. Here there are three subcases to consider.

3.1. If five vertices are B-adjacent to both 1 and 2 then we can use a similar argument to that of *CASE* 1 to show we must have a monochromatic triangle.

3.2. If four vertices are B-adjacent to both 1 and 2 then vertices 1 and 2 must both have degree four in C_R by the same argument as was used in *CASE* 2.2. Thus C contains the subgraph drawn in Figure 4.

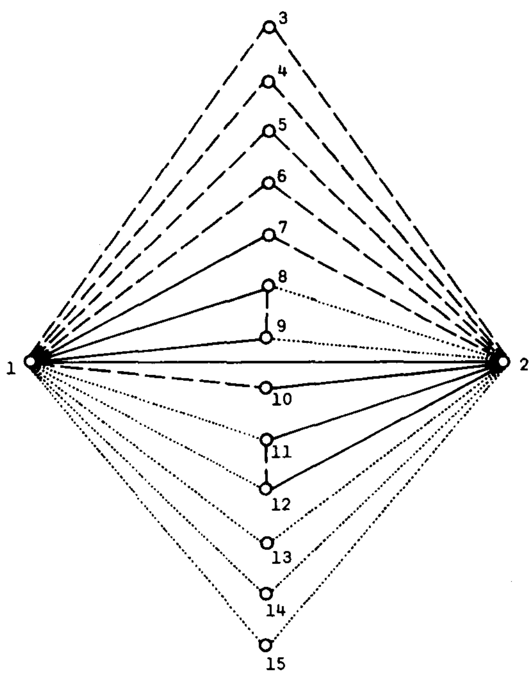

Figure 4.

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Obviously we can complete the K_5 on $\{3,4,5,6,7\}$ without any loss of generality and hence complete the K_5 on $\{3,4,5,6,10\}$ by colouring $(3,4)$, $(4,5)$, $(5,6)$, $(6,7)$, $(6,10)$, $(10,3)$ and $(7,3)$ R and the other edges G. One of the edges $(13,14)$, $(14,15)$ and $(13,15)$ is coloured R so colour $(13,14)$ R. Then, one of $(12,13)$ and $(11,12)$ is R so colour $(12,13)$ R and complete the K_5 on $\{11,12,13,14,15\}$. Now, one of $(8,13)$ and $(9,13)$ is coloured R so colour $(9,13)$ R and complete the K_5 on $\{8,9,13,14,15\}$. We see now that the edge $(7,10)$ must be B and the edges $(8,11)$ and $(9,12)$ must be G. The partial colouring is shown in Diagram 2a and the search uses Diagram 2b. No completion can be found.

3.3. Only three vertices, 3, 4, and 5, are B-adjacent to both 1 and 2. There are three subcases to consider.

3.3.1. Vertices 1 and 2 both have degree five in C_R. Hence C must contain the subgraph of Figure 5. This is seen by first colouring the edges incident with 1. Consider the K_5 on $\{8,9,2,10,11\}$. We can colour the edges

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1															
2	R														
3	B	B													
4	B	B	R												
5	B	B	G	R											
6	B	B	G	G	R										
7	R	B	R	G	G	R									
8	R	G					B/G								
9	R	G					B/G	B							
10	B	R	R	G	G	R	B								
11	G	R						G	B/G	B/G					
12	G	R						B/G	G	B/G	B				
13	G	G						B	R		B	R			
14	G	G						R	B		R	B	R		
15	G	G						R	R		R	R	B	B	

Diagram 2a.

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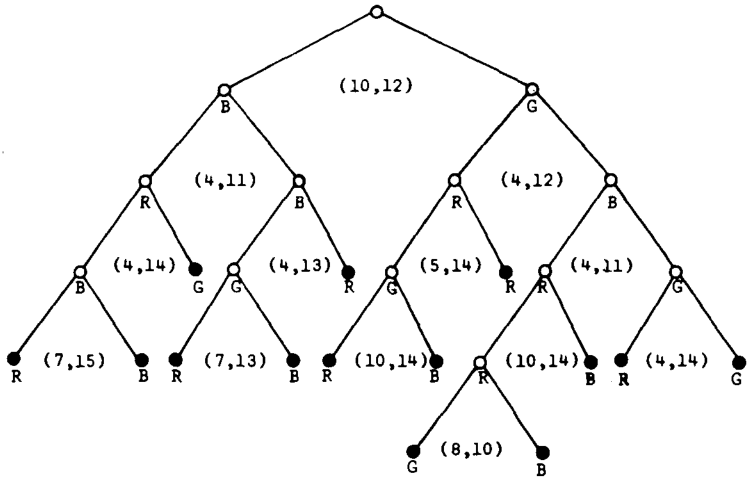

Diagram 2b.

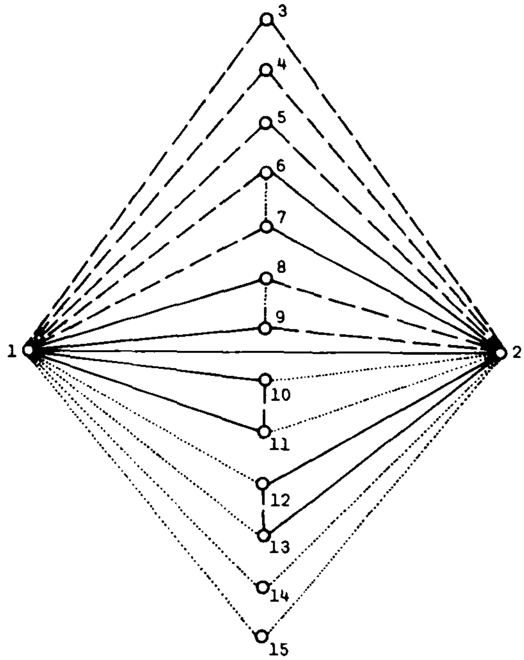

Figure 5.

(2,8) and (2,9) G, and (2,10) and (2,11) B. As vertex 2 has degree five in C_R it then follows that the edges (2,6), (2,7), (2,12) and (2,13) are coloured R.

In the K_3 on $\{3,4,5\}$ there must be at least one R and one G edge so colour (3,4) R and (4,5) G. Then colour (3,6) and (3,8) R and complete the

K_5 on $\{3,4,5,6,7\}$ and the K_5 on $\{3,4,5,8,9\}$. One of the edges $(9,10)$ and $(9,11)$ must be G to avoid a B triangle on $\{9,10,11\}$. Colour $(9,10)$ G and complete the K_5 on $\{2,8,9,10,11\}$. Similarly $(6,12)$ or $(6,13)$ must be G so colour $(6,12)$ G and complete the K_5 on $\{1,6,7,12,13\}$. As a 2-coloured K_4 has no vertex of degree three in either colour one of $(13,14)$ and $(13,15)$ must be R. Colour $(13,14)$ R. We must now colour both $(6,8)$ and $(7,9)$ B. Diagram 3a shows the partial colouring and Diagram 3b the tree used in the search. Again no completion can be found.

3.3.2. In C_R vertex 1 has degree five and vertex 2 degree four. There are two ways in which the edges from vertices 1 and 2 can be coloured and these are shown in Figure 6.

Note that if in case a we exchange the colours B and G then we have the colouring shown in Figure 3 which has already been dealt with. This leaves only case b to consider. Here we colour the edges $(3,4)$ R and $(4,5)$ G. Then colour $(3,6)$ R and complete the K_5 on $\{3,4,5,6,7\}$. One of $(7,8)$ and $(7,9)$ is

1															
2	R														
3	B	B													
4	B	B	R												
5	B	B	G	G											
6	B	R	R	G	R										
7	B	R	G	R	R	G									
8	R	B	R	G	R	B	B/G								
9	R	B	G	R	R	B/G	B	G							
10	R	G	R/B			R/G		B	G						
11	R	G		R/B			R/G	G	B	B					
12	G	R		R/B		G	B		R/G						
13	G	R	R/B			B	G	R/G				B			
14	G	G								R/B	R/B	R/B	R		
15	G	G								R/B	R/B	R/B	R/B	R/B	
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Diagram 3a.

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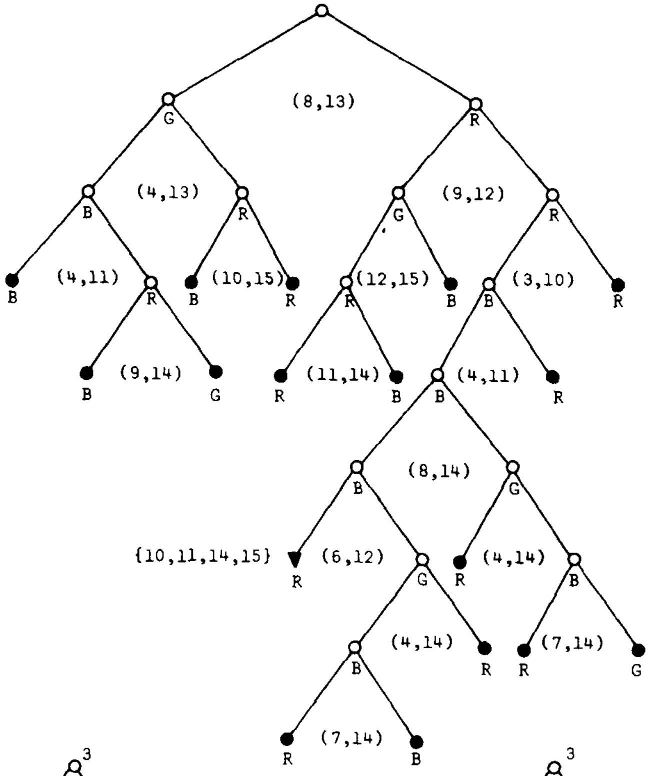

Diagram 3b.

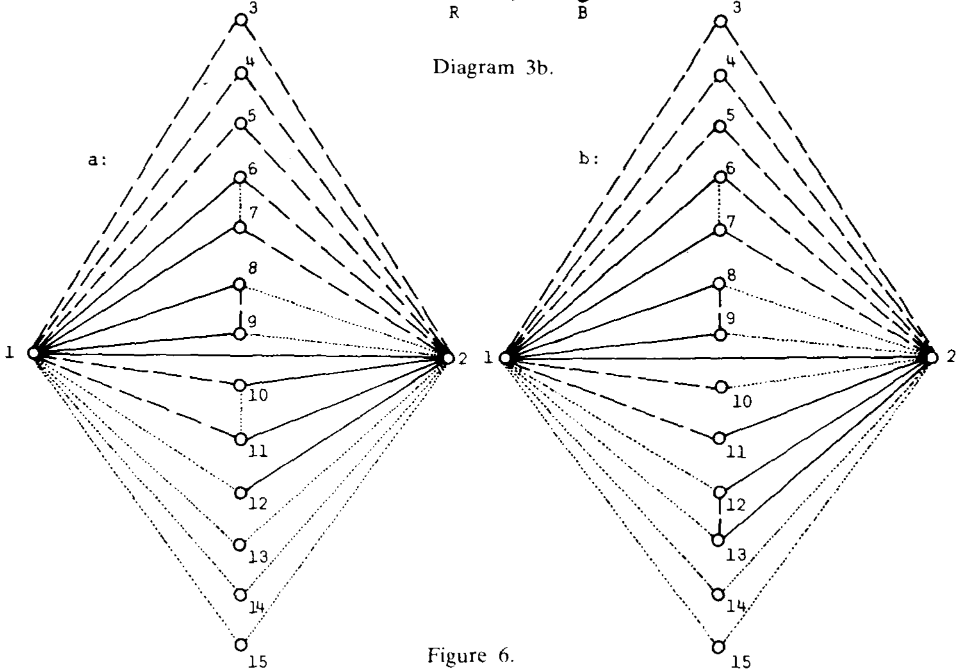

Figure 6.

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	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1															
2	R														
3	B	B													
4	B	B	R												
5	B	B	G	G											
6	R	B	R	G	R										
7	R	B	G	R	R	G									
8	R	G		R/B		G	B								
9	R	G	R/B			B	G	B							
10	B	G	R/G	R/G	R	B/G	B/G	R/B	R/B						
11	B	R	R/G	R/G	R	B/G	B/G			G					
12	G	R									B/G				
13	G	R									B/G	B			
14	G	G						R/B	R/B	R/B		R/B	R		
15	G	G						R/B	R/B	R/B		R/B	R/B	R/B	

Diagram 4a.

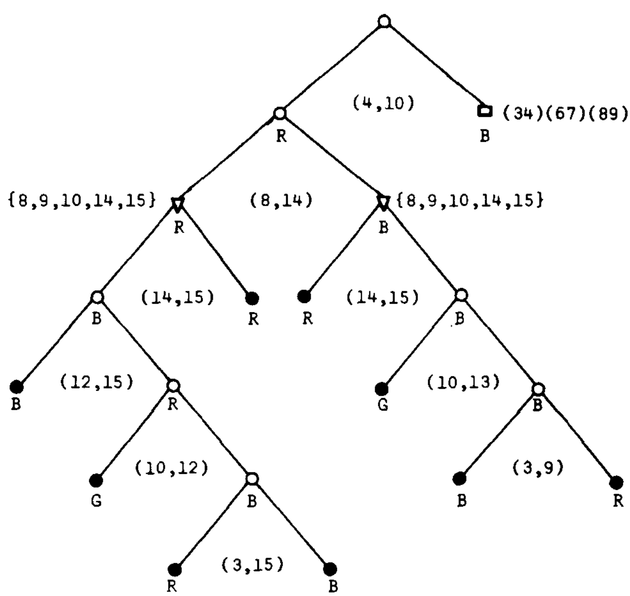

Diagram 4b.

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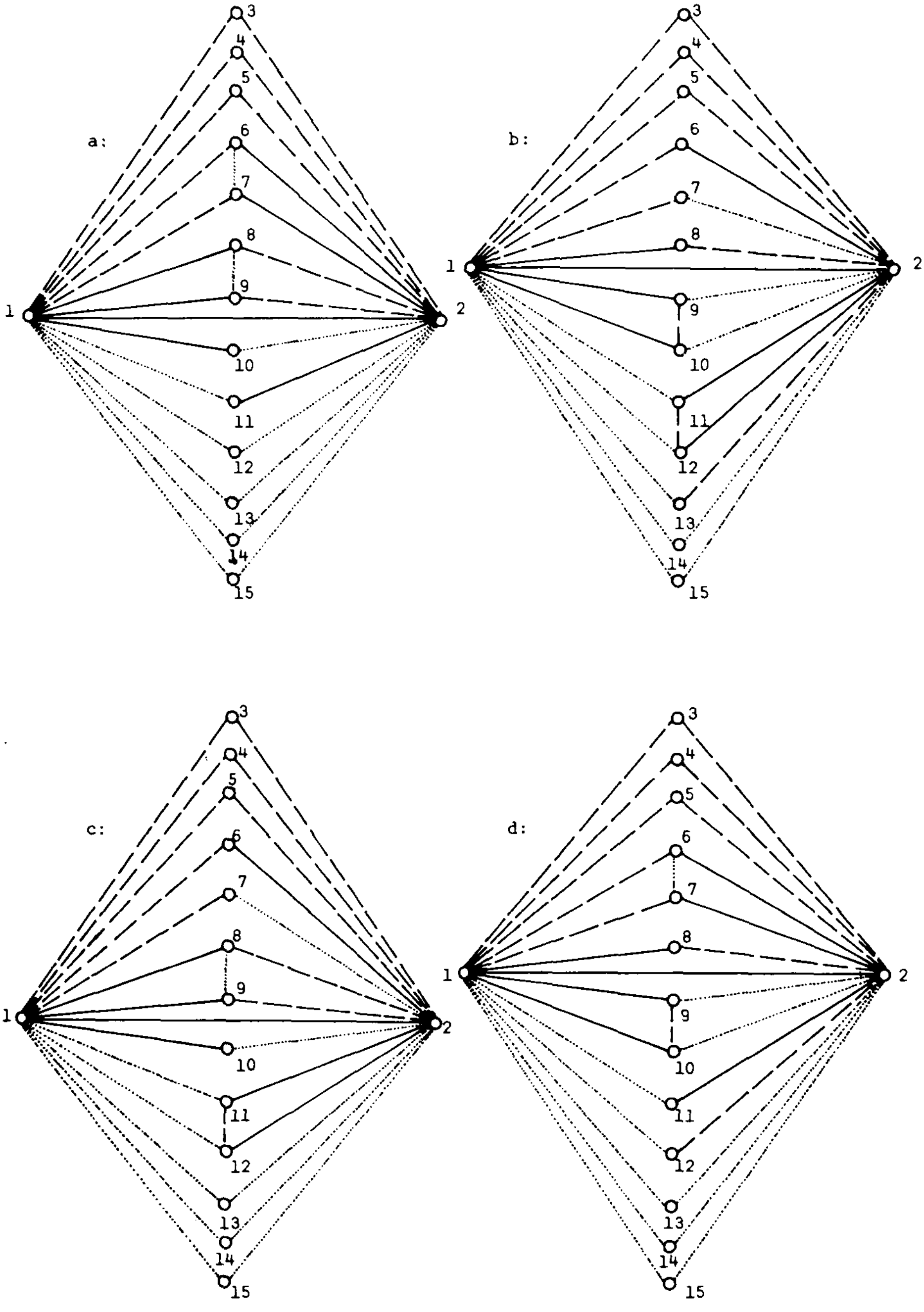

Figure 7.

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coloured G so colour $(7,9)$ G and complete the K_5 on $\{2,6,7,8,9\}$. As no vertex in a 2-coloured K_4 on $\{12,13,14,15\}$ has degree three in either colour, colour $(13,14)$ R. Looking at the K_5 on $\{3,4,5,10,11\}$ we see that $(5,3)$ and $(5,4)$ are G. Therefore we must have $(5,10)$ and $(5,11)$ R and hence $(10,11)$ G. We illustrate this partial colouring in Diagram 4a and in Diagram 4b the search tree used. Again, no completion can be found.

3.3.3. In C_R vertices 1 and 2 both have degree four. There are four ways in which the edges from vertices 1 and 2 can be coloured. These are shown in Figure 7.

a: This case is, by exchanging the colours B and G, the same colouring as that of Figure 4 and is immediately eliminated.

b: Colour the edges $(3,4)$ R and $(4,5)$ G. One of $(7,9)$ and $(7,10)$ is R so colour $(7,9)$ R. As one of $(7,14)$ and $(7,15)$ is B (see (*)) we can colour $(7,14)$ B. Colour $(12,13)$ R. The partial colouring is shown in Diagram 5a and the search tree in Diagram 5b. No colouring can be found.

1															
2	R														
3	B	B													
4	B	B	R												
5	B	B	R/G	G											
6	B	R	R/G	R/G	R/G										
7	B	G	R/G	R/G	R/G	R/G									
8	R	B	R/G	R/G	R/G										
9	R	G					R	B/G							
10	R	G					R/B	B/G	B						
11	G	R				B/G									
12	G	R				B/G					B				
13	G	B	R/G	R/G	R/G			R/G			R/B	R			
14	G	G					B		R/B	R/B	R/B	R/B	R/B		
15	G	G					R/B		R/B	R/B	R/B	R/B	R/B	R/B	
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Diagram 5a.

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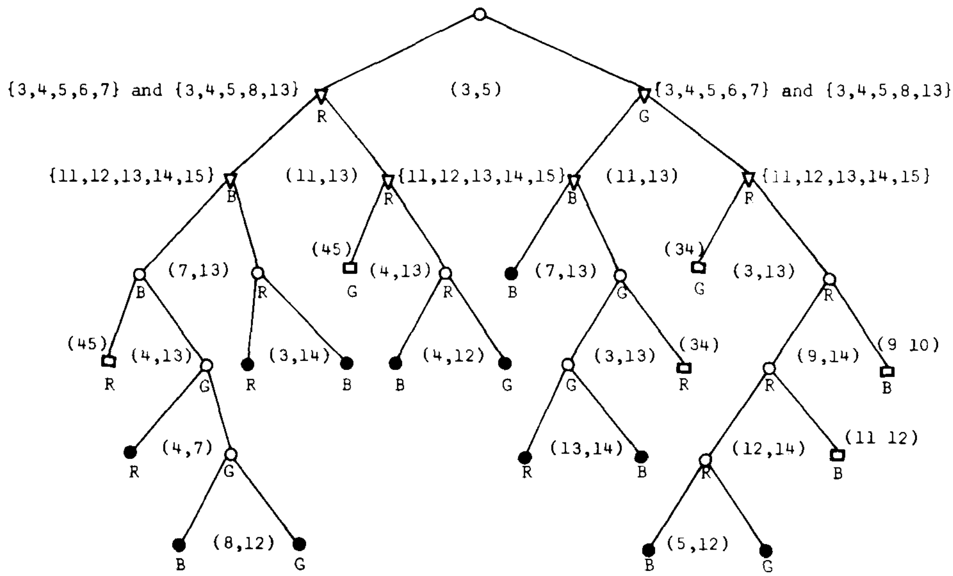

Diagram 5b.

c: Colour the edges $(3,4)$ R and $(4,5)$ G. Then colour $(3,8)$ R and complete the K_5 on $\{3,4,5,8,9\}$. Now colour $(13,14)$ R and $(14,15)$ B; then $(12,13)$ can be coloured R and the K_5 on $\{11,12,13,14,15\}$ can be completed. We must colour $(5,6)$ and $(5,7)$ R, as 5 is incident with two G edges in the K_5 on $\{3,4,5,6,7\}$, and hence $(6,7)$ is G. Similarly $(10,15)$ and $(7,15)$ must be R and hence $(7,10)$ is G. The partial colouring and search tree are given in Diagrams 6a and 6b. Again, no colouring can be found.

d: By exchanging the colours B and G it is immediately seen that this is the same case as c and as such has already been considered.

This completes the proof of the lemma.

THEOREM 1. *Any proper 3-colouring of* K_{15} *must have edge vector* $(35,35,35)$.

PROOF. The only possible edge vectors for a proper 3-colouring of K_{15} are

$$(37,37,31),\ (37,36,32),\ (37,35,33),\ (37,34,34),$$
$$(36,36,33),\ (36,35,34)\quad\text{and}\quad(35,35,35)$$

as the maximum number of edges in a monochromatic subgraph of K_{15} is 37.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1															
2	R														
3	B	B													
4	B	B	R												
5	B	B	G	G											
6	B	R	R/G	R/G	R										
7	B	G	R/G	R/G	R	G									
8	R	B	R	G	R	B/G	B/G								
9	R	B	G	R	R	B/G	B/G	G							
10	R	G					B	B/G	B/G						
11	G	R				B/G	B/G			B/G					
12	G	R				B/G	B/G			B/G	B				
13	G	G					B/R			B/R	B	R			
14	G	G					B/R			B/R	R	B	R		
15	G	G			B/G		R			R	R	R	B	B	

Diagram 6a.

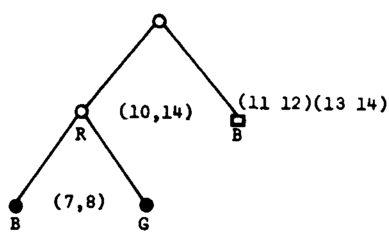

Diagram 6b.

We show that there is no monochromatic subgraph of C with either 37 or 36 edges. Only C_R need be considered. At least one vertex in C_R, say vertex 1, has degree four. Thus C must contain the subgraph of Figure 8 in which the remaining vertices have been labelled arbitrarily. Figure 9 shows the edges of this subgraph which are contained in C_R.

Recall that in a properly 3-coloured K_{15} each vertex has degree four or five in C_R. *Assume that* C_R *has* 37 *edges*. Then each vertex, except for vertex 1, has degree five. Each of the vertices 7, 8, 9, 10 is adjacent to two vertices in

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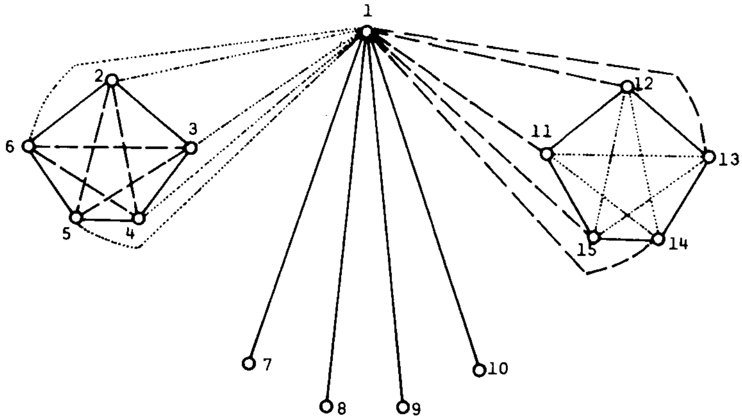

Figure 8.

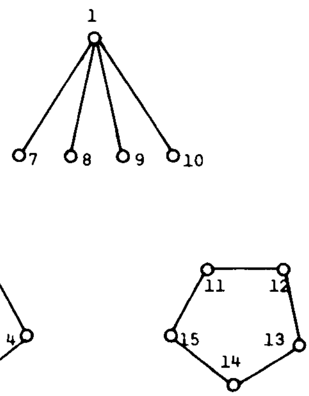

Figure 9.

both the sets $\{2, 3, 4, 5, 6\} = V$ and $\{11, 12, 13, 14, 15\} = W$. At most two of the vertices 7, 8, 9 and 10 can have two adjacent vertices in common. This follows from the lemma; as if (say) (7, 8) is B then there are at most two vertices R-adjacent to both 7 and 8 and at most two G-adjacent. Hence we can draw the edges (2, 7), (4, 7), (3, 8), (5, 8), (4, 9), (6, 9), (5, 10), (2, 10), (8, 12) and (8, 15) as in Figure 10.

All arguments are based on both avoiding a triangle and not contradicting the lemma.

Vertex 3 is adjacent to two of 11, 13, 14. As it must be adjacent to 11 and hence one of 13 and 14 we can draw the edges (3, 11) and (3, 14). Vertex 5 must now be adjacent to 13 so draw (5, 13). To avoid a triangle vertex 4 can only be adjacent to one of 12 and 15 but in both cases the lemma is

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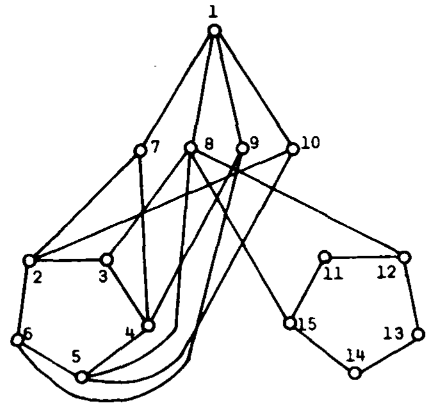

Figure 10.

contradicted. Thus 4 cannot have degree five and hence C_R cannot have 37 edges.

Assume that C_R *has* 36 *edges.* Then twelve vertices have degree five and three have degree four. Vertex 1 is already degree four and so there are only two others. There are three cases to consider.

CASE 1. None of 7, 8, 9 and 10 has degree four.

CASE 2. One of 7, 8, 9 and 10 has degree four.

CASE 3. Two of 7, 8, 9 and 10 have degree four.

CASE 1. If both vertices of degree four are in V then a simple count of the total degrees in V and W shows that we need to draw five edges from V to W but seven from W to V in order to have 36 edges This is obviously impossible. Therefore there is one vertex of degree four in V and one in W. Thus we can begin with the edges of Figure 10.

If one of the vertices 2 and 6 has degree four then the argument used is that of the case when C_R has 37 edges.

If one of the vertices 3 and 4 has degree four then as 5 cannot be adjacent to 11, 12 or 15 without contradicting the lemma we can draw (5, 14). Vertex 6 is now adjacent to 13. For the same reason 10 must also be adjacent to 13 but now the lemma is contradicted.

The only case remaining is when 5 has degree four. Here we have the edges (3, 11) and (3, 14), and hence (4, 13). But now 9 and 7 must both be adjacent to 12 and 14, contradicting the lemma.

CASE 2. We can assume that 8 has degree 5 and so we again begin with Figure 10. If the third vertex of degree four is in V, then six edges must be drawn from V to W but eight edges from W to V. Since this is impossible the third vertex of degree four must be in W. The argument now follows as in the case with 37 edges.

CASE 3. Let 7 and 10 be the vertices of degree four. Starting with Figure 9 there are three ways to draw the edges from 8 and 9 to V and W. The two vertices, 8 and 9, can have no adjacent vertex of V or W in common; they can have one of V (or equivalently one of W) in common; or one vertex of V and one of W in common. These are shown in Figure 11. We can assume that 7 is adjacent to two vertices in V and that 10 is adjacent to two vertices in W. If both 7 and 10 were adjacent to two vertices of V, say, we would need seven edges from V to W but nine from W to V.

3.1. Vertex 4 must be adjacent to two other vertices. It cannot be adjacent to both 7 and 10 as then there are three vertices R-adjacent to both 1

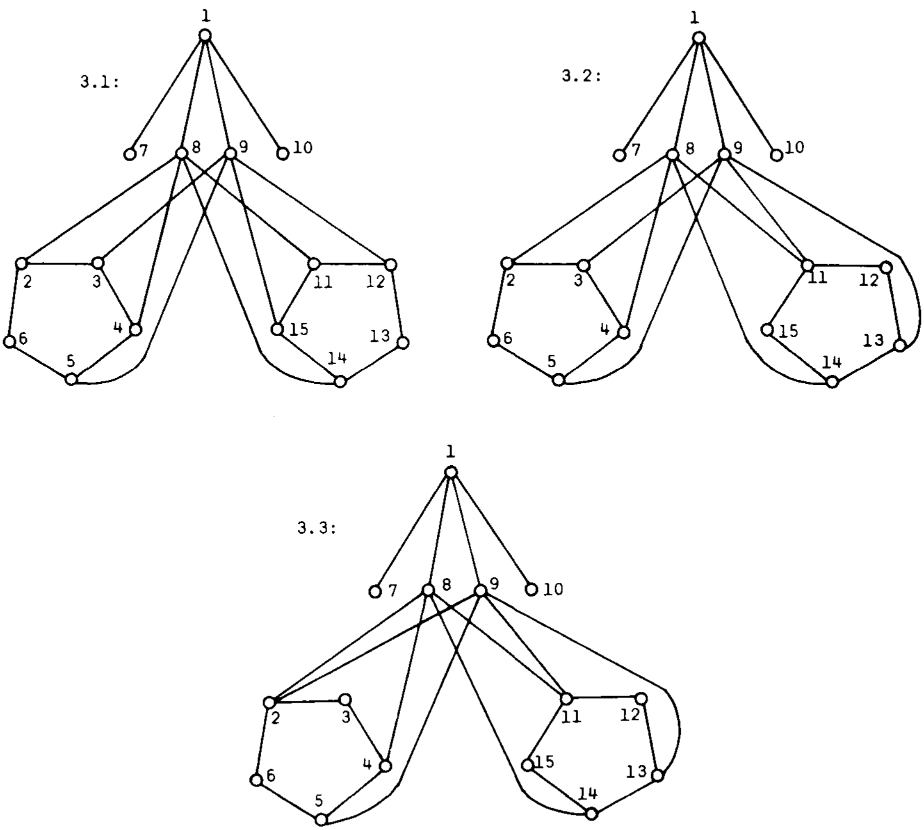

Figure 11.

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and 4. Hence we must have the edge (4, 13). The same argument when applied to vertex 3 gives the edge (3, 13) but we now have a triangle on the vertices 3, 4 and 13.

3.2. As no distinction has yet been made between vertices 8 and 9 we see that vertex 7 is adjacent to either vertices 6 and 3 or 2 and 5. If we draw (6, 7) and (3, 7) then we must have the edge (3, 15) and hence (4, 10) and (4, 12). It follows that as 5 is to have degree five it must be adjacent to both vertices 14 and 15 but this results in a triangle. If, instead, we draw (2, 7) and (5, 7) then 6 must be adjacent to 10, as it cannot be adjacent to three of B, but then 3 can only be adjacent to 15 and thus only has degree four.

3.3. Here we see that the lemma is contradicted.

Thus any proper 3-colouring of K_{15} must have edge-vector (35, 35, 35). This completes the proof of Theorem 1.

As was pointed out there are exactly two proper 3-colourings of K_{16}. From these, two proper 3-colourings of K_{15} can be obtained by deleting, in each of the K_{16}, a vertex and the edges incident with it. On deleting vertex 1 in each case we have the K_{15} as shown in Diagrams 7 and 8.

2															
3	B														
4	G	B													
5	G	G	B												
6	B	R	R	B											
7	G	B	R	R	G										
8	B	G	B	R	R	G									
9	R	B	G	B	R	R	G								
10	R	R	B	G	G	R	R	G							
11	R	G	G	R	R	G	B	B	G						
12	B	R	G	G	G	R	G	B	B	R					
13	R	B	R	G	B	G	R	G	B	B	R				
14	G	R	B	R	B	B	G	R	G	B	B	R			
15	G	G	R	B	G	B	B	G	R	R	B	B	R		
16	R	R	R	R	B	B	B	B	B	G	G	G	G	G	
	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Diagram 7.

2															
3	G														
4	B	B													
5	G	B	G												
6	B	R	R	B											
7	G	B	R	R	G										
8	B	B	G	R	R	G									
9	R	G	B	B	R	R	G								
10	R	R	B	G	G	R	R	G							
11	R	G	G	R	R	G	B	B	G						
12	G	R	G	B	G	R	B	G	B	R					
13	R	B	R	G	B	G	R	G	B	B	R				
14	G	R	B	R	B	B	G	R	G	B	B	R			
15	B	G	R	G	G	B	G	B	R	R	B	B	R		
16	R	R	R	R	B	B	B	B	B	G	G	G	G	G	
	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Diagram 8.

THEOREM 2. *There are exactly two proper 3-colourings of K_{15} and each can be embedded in a proper 3-colouring of K_{16}.*

PROOF. From Theorem 1 we know that any proper 3-colouring of K_{15} has edge-vector (35, 35, 35). We look at the monochromatic subgraph C_R which must contain the edges shown in Figure 12. (The B and G edges of Figure 8 are also assumed.)

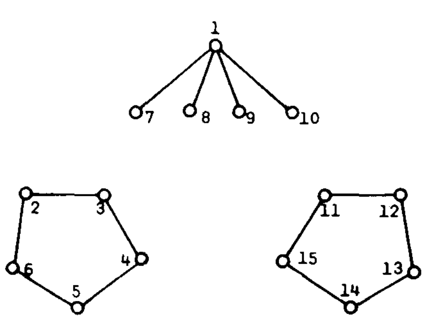

Figure 12.

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There are five cases to consider as C_R has five vertices of degree four (including vertex 1) and ten of degree five. The cases are as follows.

CASE 1. Vertices 7, 8, 9 and 10 have degree four.

CASE 2. Vertices 7, 8 and 9 have degree four whilst vertex 10 has degree five.

CASE 3. Vertices 7 and 8 have degree four and vertices 9 and 10 have degree five.

CASE 4. Vertex 7 has degree four whilst the other three, 8, 9 and 10, have degree five.

CASE 5. The four vertices 7, 8, 9 and 10 have degree five.

Each case will be considered in turn.

CASE 1. There must be two edges from each of 7 and 8, and one from each of 9 and 10, to the vertex set $V = \{2, 3, 4, 5, 6\}$. Also, there must be two edges from each of 9 and 10, and one from each of 7 and 8, to the set $W = \{11, 12, 13, 14, 15\}$. This yields the three possibilities shown in Figure 13.

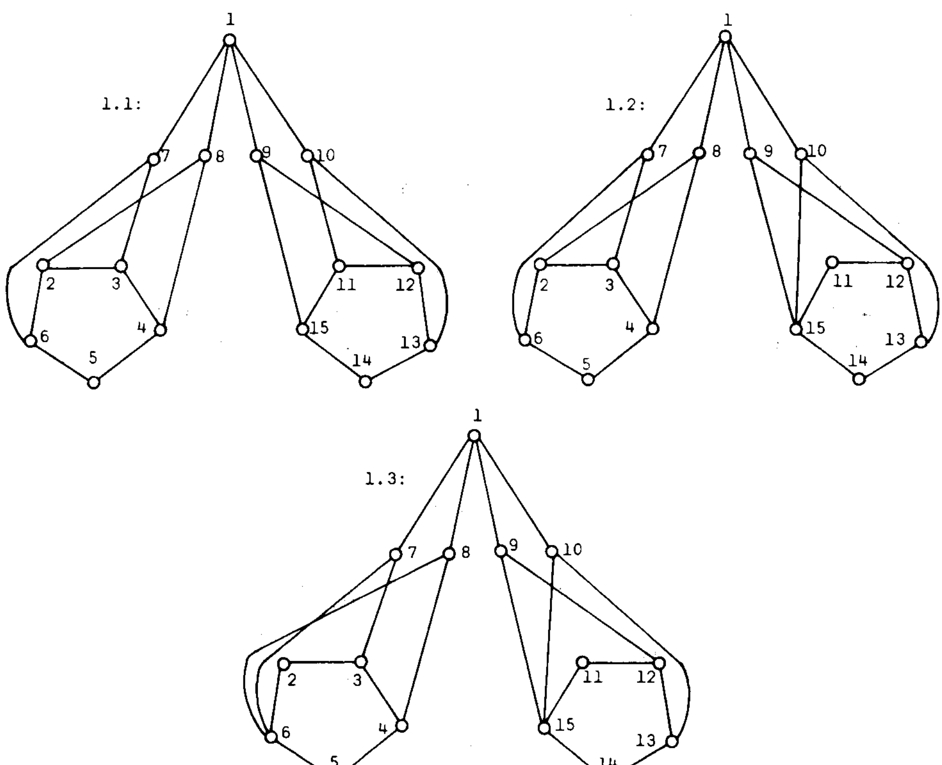

Figure 13.

As no vertex in V can be adjacent to more than two vertices in W, and vice versa, we can add the edges (8, 14) and (5, 9) in 1.1, (8, 14) and (7, 11) in 1.2 and (8, 11) and (7, 14) in 1.3.

1.1. Vertex 5 must be adjacent to vertices 10 and 14. Hence we have the edges (4, 13) and (4, 15), but now vertex 3 cannot be adjacent to any two of the vertices in W.

1.2. Vertex 14 cannot be adjacent to any two vertices in V without contradicting the lemma or forming a triangle.

1.3. Here the argument of 1.2 applies to vertex 11.

CASE 2. We are required to draw two edges from each of 8, 9, 10 and one from 7 to W; two edges from each of 7 and 10 and one from each of 8 and 9 to V. The fifth vertex of degree four must be in V. Draw the edges from 10 to V and W. There are then four ways that the edges from 8 and 9 can be drawn to W. These are shown in Figure 14. In both cases 2.1 and 2.4 the edge from 7 to W can be drawn uniquely.

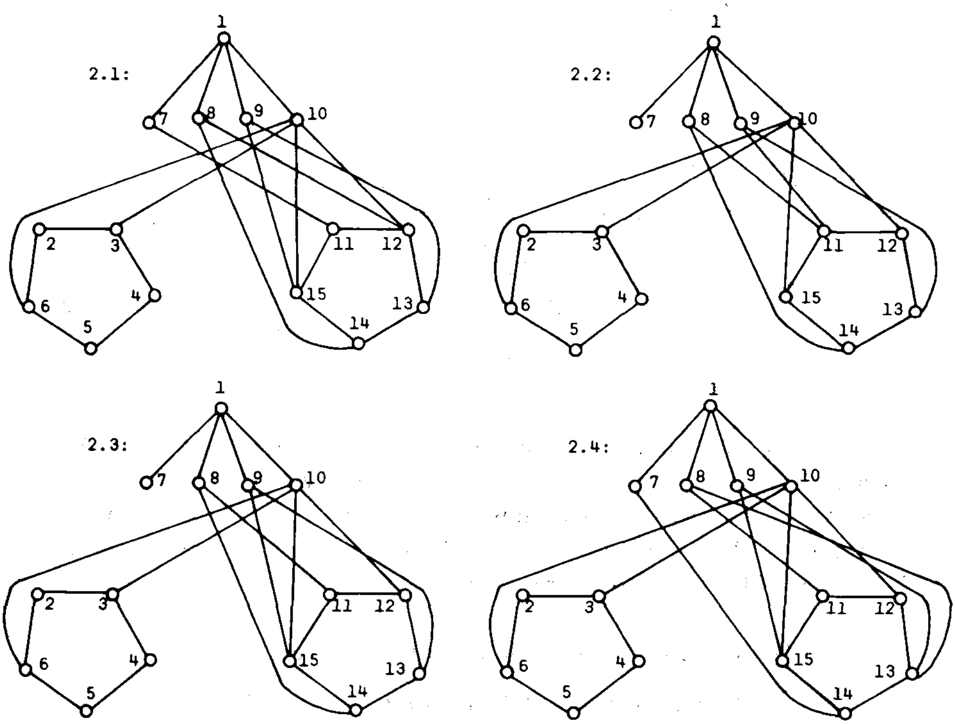

Figure 14.

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2.1. Vertex 11 is adjacent to two of 2, 4 and 5 so draw the edges (2, 11) and (4, 11). But now vertex 7 cannot be adjacent to any two of V without forming a triangle or contradicting the lemma.

2.2. Vertices 12 and 15 must both be adjacent to one of 4 and 5 and to vertex 7. But this is impossible.

2.3. Again vertex 12 is adjacent to 7 and one of 4 and 5. Hence we can draw the edges (2, 11), (4, 11) and then (5, 12) and (7, 12). But now vertex 7 cannot be adjacent to two vertices of V.

2.4. Vertex 12 cannot be adjacent to any two vertices of V.

CASE 3. Here we require that vertices 9 and 10 are each adjacent to two of V and two of W. Two cases arise and these are shown in Figure 15. Either V and W both have one vertex of degree four or one of them has two vertices of degree 4.

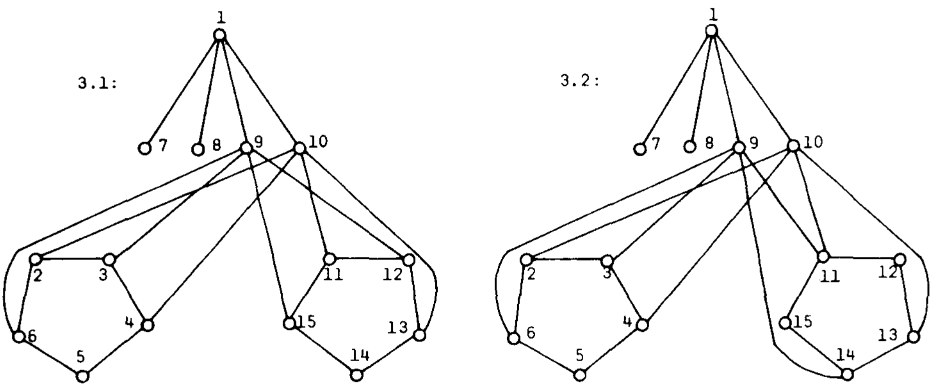

Figure 15.

3.1. As yet we have made no distinction between the sets V and W and so can assume that vertices 3 and 4 have degree five. Thus we have the edges (3, 8) and (3, 14). Vertex 2 can now only be adjacent to vertex 7 and so has degree four. But now vertex 4 has only degree four, a contradiction as we have assumed that it has degree five.

3.2. There are three possibilities to be considered.

3.2.1. Both other vertices of degree four are in V. Then we have the edge (5, 11) and hence (7, 15), (8, 15); one of (2, 15) and (4, 15); (7, 12), (8, 12) and one of (3, 12) and (16, 12). But the edges now drawn contradict the lemma.

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3.2.2. There is a vertex of degree four in each of V and W. We can assume that vertices 3 and 4 have degree five. We have the edges (3 8), (3, 12) and (4, 7) and 4 is adjacent to one of 14 and 15. Vertex 2 can be adjacent to only one of 14 and 15 and so has degree four. Thus we need to have (5, 8) as 5 has degree five. To ensure the same number of edges from W to V as from V to W vertices 7 and 8 cannot be adjacent to any more vertices in V. But then vertex 6 cannot be adjacent to two vertices in W without forming a triangle or contradicting the lemma.

3.2.3. If two vertices in W have degree four then all vertices in V have degree five. We then have the edges (3, 12), (3, 7), (2, 15) and (2, 8). However vertex 4 now has maximum degree four as it can only be adjacent to vertex 14.

CASE 4. Vertices 8, 9 and 10 are all adjacent to two vertices in both V and W. Let vertex 7 be adjacent to two vertices in V and one in W. There must then be one vertex of degree four in V and two in W. The two possibilities are shown in Figure 16.

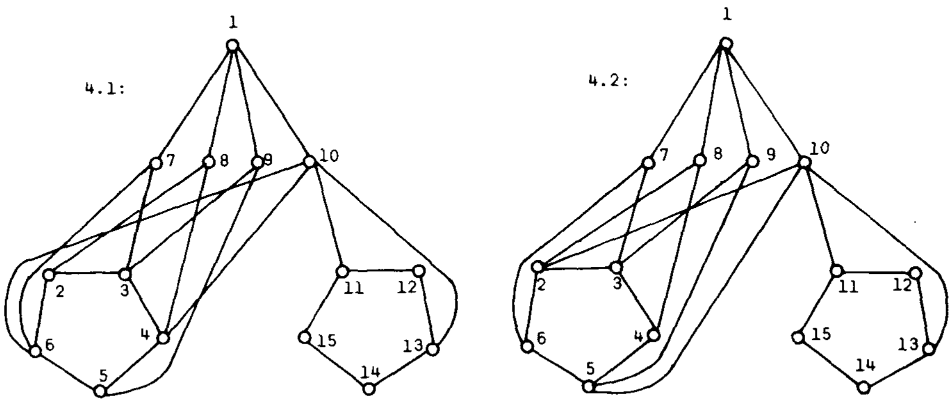

Figure 16.

4.1. We can add the edges (8, 12) and (8, 15). Vertex 2 can be adjacent to 13 or 14, but only one of them. Therefore vertex 2 has degree four and we have the edges (4, 14), (2, 13) and (3, 11). We now need to have (5, 12) and (5, 15) so that 5 has degree five but this contradicts the lemma.

4.2. Add the edges (8, 12), (8, 15), (9, 12) and (9, 14). Vertex 4 must have degree four as the only vertex it can be adjacent to is vertex 13. This then gives the edges (5, 15) and (3, 11). Vertex 6 now cannot be adjacent to any two of W, a contradiction.

CASE 5. Here 7, 8, 9 and 10 are all adjacent to two vertices in V and in W as in Figure 17.

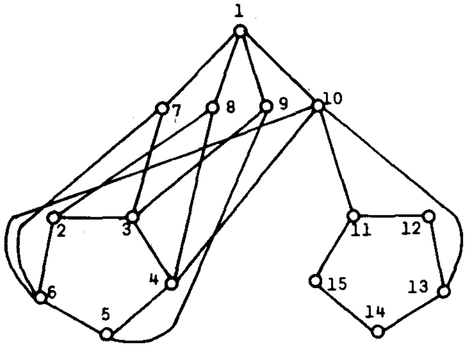

Figure 17.

There are two vertices of degree four in both V and W. Add the edges (8, 12), (8, 15), (7, 12), (7, 14), (9, 15) and (9, 13). The following edges must also be added: (4, 14), (3, 11), (2, 13), (5, 12) and (6, 15). This is the required monochromatic subgraph, C_R.

The incidence matrix is given in Diagram 9; the vertices have been

10															
9															
8															
7															
11		R	R												
13			R	R	G										
15				R	R	G									
12	R				R	R	G								
14	R	R			G	R	R	G							
6	R			R	R										
5		R				R				R					
4	R		R				R			B	R				
3		R		R				R		B	B	R			
2			R						R	R	B	B	R		
1	R	R	R	R	B	B	B	B	B	G	G	G	G	G	
	10	9	8	7	11	13	15	12	14	6	5	4	3	2	1

Diagram 9.

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rearranged to exhibit the isomorphism between C_R and the R edges in Diagrams 7 and 8. The B and G edges of Figure 8 are also included.

The tree of Diagram 10 shows that there are exactly six ways to colour all the remaining edges of the graph whose partial incidence matrix is given in Diagram 9.

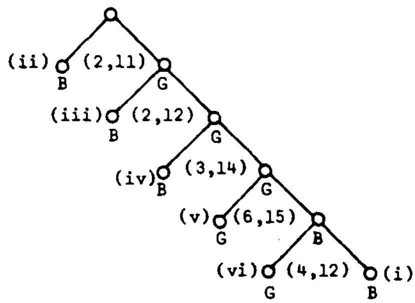

Diagram 10.

Look at each of the six cases in turn. The incidence matrices of the completed colourings appear in Diagram 11.

(i) It is seen that the edges $(2, 10)$, $(2, 9)$, $(5, 8)$ and $(5, 7)$ must be G and the colouring can then be completed.

(ii) Here we have that $(5, 7)$, $(5, 8)$ and $(6, 9)$ must be G whilst $(10, 15)$ must be B. The colouring is then completed.

(iii) The edges $(6, 8)$, $(6, 9)$, $(3, 10)$ and $(7, 10)$ must all be G. Then complete the colouring.

(iv) We have that $(2, 9)$, $(2, 10)$ and $(3, 8)$ must be G whereas $(7, 12)$ must be coloured B.

(v) In this case $(2, 9)$, $(2, 10)$, $(5, 7)$ and $(5, 8)$ must all be G and the colouring can then be completed.

(vi) As above the edges $(2, 9)$, $(2, 10)$, $(5, 7)$ and $(5, 8)$ must all be G and then we can complete the colouring.

It is immediately seen that (iii) and (vi) are identical colourings to those of Diagrams 8 and 7 respectively. A simple check shows that the remaining four colourings can all be extended to proper 3-colourings of K_{16} and so must be isomorphic to the colourings of Diagrams 7 and 8.

This completes the proof of the theorem.

The author would like to thank Professor W. D. Wallis for his assistance during the writing of this paper.

(i)

10															
9	B														
8	G	G													
7	G	B	B												
11	B	R	R	B											
13	G	B	R	R	G										
15	B	G	B	R	R	G									
12	R	B	B	G	R	R	G								
14	R	R	G	B	G	R	R	G							
6	R	B	G	R	R	G	B	G	B						
5	B	R	G	G	G	R	G	B	B	R					
4	R	G	R	G	B	G	R	B	G	B	R				
3	G	R	B	R	B	B	G	R	G	B	B	R			
2	G	G	R	B	G	B	B	G	R	R	B	B	R		
1	R	R	R	R	B	B	B	B	B	G	G	G	G	G	
	10	9	8	7	11	13	15	12	14	6	5	4	3	2	1

(ii)

10															
9	G														
8	G	B													
7	B	G	B												
11	G	R	R	B											
13	B	B	R	R	G										
15	B	G	B	R	R	G									
12	R	B	G	B	R	R	G								
14	R	R	B	G	G	R	R	G							
6	R	G	G	R	R	G	B	B	G						
5	B	R	G	G	G	R	G	B	B	R					
4	R	G	R	B	G	B	R	G	B	B	R				
3	G	R	B	R	B	B	G	R	G	B	B	R			
2	G	B	R	G	B	G	B	G	R	R	B	B	R		
1	R	R	R	R	B	B	B	B	B	G	G	G	G	G	
	10	9	8	7	11	13	15	12	14	6	5	4	3	2	1

Diagram 11.

(iii)

10															
9	G														
8	B	B													
7	G	B	G												
11	B	R	R	B											
13	G	B	R	R	G										
15	B	B	G	R	R	G									
12	R	G	B	B	R	R	G								
14	R	R	B	G	G	R	R	G							
6	R	G	G	R	R	G	B	B	G						
5	G	R	G	B	G	R	B	G	B	R					
4	R	B	R	G	B	G	R	G	B	B	R				
3	G	R	B	R	B	B	G	R	G	B	B	R			
2	B	G	R	G	G	B	G	B	R	R	B	B	R		
1	R	R	R	R	B	B	B	B	B	G	G	G	G	G	
	10	9	8	7	11	13	15	12	14	6	5	4	3	2	1

(iv)

10															
9	B														
8	G	B													
7	B	G	G												
11	B	R	R	G											
13	G	B	R	R	G										
15	B	G	B	R	R	G									
12	R	B	G	B	R	R	G								
14	R	R	B	B	G	R	R	G							
6	R	G	G	R	R	G	B	B	G						
5	G	R	B	G	B	R	G	B	G	R					
4	R	B	R	G	B	G	R	G	B	B	R				
3	B	R	G	R	G	B	G	R	B	B	B	R			
2	G	G	R	B	G	B	B	G	R	R	B	B	R		
1	R	R	R	R	B	B	B	B	B	G	G	G	G	G	
	10	9	8	7	11	13	15	12	14	6	5	4	3	2	1

Diagram 11 (cont)

(v)

10															
9	B														
8	B	G													
7	G	G	B												
11	B	R	R	B											
13	B	G	R	R	G										
15	G	B	B	R	R	G									
12	R	B	G	B	R	R	G								
14	R	R	B	G	G	R	R	G							
6	R	G	B	R	R	B	G	B	G						
5	B	R	G	G	G	R	G	B	B	R					
4	R	B	R	G	B	G	R	G	B	B	R				
3	G	R	G	R	B	G	B	R	G	B	B	R			
2	G	G	R	B	G	B	B	G	R	R	B	B	R		
1	R	R	R	R	B	B	B	B	B	G	G	G	G	G	
	10	9	8	7	11	13	15	12	14	6	5	4	3	2	1

(vi)

10															
9	B														
8	G	B													
7	G	G	B												
11	B	R	R	B											
13	G	B	R	R	G										
15	B	G	B	R	R	G									
12	R	B	G	B	R	R	G								
14	R	R	B	G	G	R	R	G							
6	R	G	G	R	R	G	B	B	G						
5	B	R	G	G	G	R	G	B	B	R					
4	R	B	R	G	B	G	R	G	B	B	R				
3	G	R	B	R	B	B	G	R	G	B	B	R			
2	G	G	R	B	G	B	B	G	R	R	B	B	R		
1	R	R	R	R	B	B	B	B	B	G	G	G	G	G	
	10	9	8	7	11	13	15	12	14	6	5	4	3	2	1

Diagram 11 (cont.)

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