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The multiplicator of a regular product of groups

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It is shown that if G is an arbitrary regular product of its subgroups A_λ , $\lambda \in I$, then the multiplicator, $M(G)$, of G is the direct product of the $M(A_\lambda)$ together with a certain other group. This extends a calculation of $M(A_1 \times A_2)$ due to Schur. As an application, we find the multiplicator of a verbal wreath product $A \text{ wr}_V B$ where A is abelian. A representing group for a finite regular product is also constructed.

1. Regular products

In this section we define regular products and list some of their properties needed in Section 2. Properties (1.7) and (1.8) are of particular importance. Everything here can be found in Golovin [2].

DEFINITION 1.1. *G is a regular product of its subgroups A_λ , $\lambda \in I$, where I is an ordered set, if they generate G and $A_\lambda \cap \hat{A}_\lambda = E$ where $\hat{A}_\lambda = \text{sgp}\left(A_\mu^G \mid \mu \in I, \mu \neq \lambda\right)$. The A_λ are called regular factors of G .*

As usual A_μ^G denotes the normal closure of A_μ in G and E is the trivial group. From now on in this note we consider G to be a fixed group generated by its subgroups A_λ , $\lambda \in I$. The statement "G is a regular product of its subgroups A_λ " will be abbreviated to "G is a

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regular product".

If X_λ is an arbitrary subgroup of G $(\lambda \in I)$ then the notation $[X_\lambda]$ will be used for the subgroup,
$\mathrm{sgp}\{[x_\lambda, x_\mu] \mid x_\lambda \in X_\lambda;\ x_\mu \in X_\mu;\ \lambda, \mu \in I;\ \lambda \neq \mu\}$, called the *cartesian of the* X_λ .

The product $\prod_{\lambda \in I} X_\lambda$ will mean the set, in G , of all elements of the form $x_{\lambda_1} x_{\lambda_2} \ldots x_{\lambda_n}$ where $\lambda_j \in I$ and $\lambda_1 < \lambda_2 < \ldots < \lambda_n$. If each X_λ is normal in G then $\prod_{\lambda \in I} X_\lambda$ becomes the normal subgroup of G generated by the X_λ and any two orderings of I will yield the same product.

(1.2). $\left[A_\lambda^G\right]$ *is normal in* G .

(1.3). *If* G *is generated by its subgroups* A_λ *then*
$G = \left(\prod_{\lambda \in I} A_\lambda\right)\left[A_\lambda^G\right]$. *Furthermore* G *is a regular product if and only if each of its elements can be written uniquely as a product* $a_{\lambda_1} a_{\lambda_2} \ldots a_{\lambda_n} u$ *where* $a_{\lambda_i} \in A_{\lambda_i}$, $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ *and* $u \in \left[A_\lambda^G\right]$.

(1.4). *If* G *is a regular product and* X_λ *is a subgroup of* A_λ *for each* $\lambda \in I$, *then the subgroup of* G *generated by the* X_λ *is a regular product of the* X_λ .

(1.5). *If* G *is a regular product then* $G' = \left(\prod_{\lambda \in I} A'_\lambda\right)\left[A_\lambda^G\right]$.

DEFINITION 1.6. *If* G *is a regular product, a homomorphism* ϕ *of* G *to a group* $\bar{G}$ *is a regular homomorphism if* $\ker\phi$ *is a subgroup of* $\left[A_\lambda^G\right]$ *: hence the terms regular homomorphic image and regular quotient group.*

(1.7). *If* G *is a regular product and* $\phi : G \to \bar{G}$ *is a regular*

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homomorphism then ϕ *restricted to* A_λ *is an isomorphism for each* $\lambda \in I$ *and* $G\phi$ *is a regular product of the* $A_\lambda\phi$.

(1.8). *If* $\psi : \prod^*_{\lambda\in I} A_\lambda \to G$ $\left(\text{where } \prod^*_{\lambda\in I} A_\lambda \text{ denotes the free product of the } A_\lambda\right)$ *is the natural homomorphism induced by the identity map on each* A_λ , *then* G *is a regular product if and only if* ψ *is regular.*

This last result shows that the free product of a set of groups can be interpreted as the *largest* regular product of those groups. In the same way, their direct product is the *smallest* such product. For, if G is a regular product of the A_λ , then the quotient $G/\left[A_\lambda^G\right]$ is a direct product of copies of the A_λ .

2. Calculation of the multiplicator

We define the multiplicator of G to be $R\cap F'/[R, F]$ where F/R is a presentation for G . The most direct way of approaching $M(G)$, when G is a regular product, is to try to express $R\cap F'/[R, F]$ in terms of the groups $R\cap F_\lambda'/\left[(R\cap F_\lambda), F_\lambda\right]$ where F_λ is the group of elements in F which map onto A_λ . $\left(F_\lambda/R\cap F_\lambda\right.$ is a presentation for A_λ .$\left.\right)$ This is the sort of method previously applied to $M\left(A_1 \times A_2\right)$ (see [8]). The difficulty here lies in not knowing how the F_λ generate F . It can be avoided by taking a presentation, F_λ/R_λ , for each A_λ and constructing a presentation for G from the free product $\prod^*_{\lambda\in I} F_\lambda$.

In fact we begin with greater generality and let B_λ be a fixed group which maps epimorphically onto A_λ under ν_λ . This will allow us to construct a representing group for G when it is finite.

Suppose that C_λ is the kernel of ν_λ and that ν is the natural epimorphism from the free product $B = \prod^*_{\lambda\in I} B_\lambda$ onto $A = \prod^*_{\lambda\in I} A_\lambda$ induced

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by the ν_λ . Further, if ψ is the natural homomorphism from A onto G induced by the identity on each A_λ , let H be the kernel of ψ and H_0 the group in B which maps onto H under ν . We have that

$$\begin{aligned} H &= H \cap \left[A_\lambda^A\right] && \text{since} \quad H \le \left[A_\lambda^A\right] && \text{by (1.8)} \\ &= H_0\nu \cap \left[B_\lambda^B\right]\nu && \text{since} \quad \left[B_\lambda^B\right]\nu = \left[B_\lambda\nu^{B_\nu}\right] && \text{by definition of the cartesian} \\ &= \left(H_0\cap\left[B_\lambda^B\right]\right)\nu && \text{since} \quad \ker\nu \le H_0 && \text{by definition} \\ &= K\nu && \text{where} \quad K = H_0 \cap \left[B_\lambda^B\right] && \text{is normal in } B . \end{aligned}$$

LEMMA 2.1. *B/C is isomorphic to G where* $C = \left(\prod_{\lambda\in I} C_\lambda^B\right).K$.

Proof. (i) We first show that $\ker\nu = \prod_{\lambda\in I} C_\lambda^B$. (This is almost certainly well known but I cannot find it in the literature.) The kernel of ν contains $\prod_{\lambda\in I} C_\lambda^B$ since, by construction of ν , $C_\lambda\nu$ is trivial for each $\lambda \in I$. On the other hand,

$$\begin{aligned} B_\mu \cap \prod_{\lambda\in I} C_\lambda^B &= B_\mu \cap D_\mu C_\mu^B \qquad \text{where} \quad D_\mu = \prod_{\substack{\lambda\in I \\ \lambda\ne\mu}} C_\lambda^B \\ &= \left(B_\mu \cap D_\mu\left[C_\mu, B\right]\right)C_\mu \quad \text{since} \quad C_\mu \le B_\mu \\ &= C_\mu \end{aligned}$$

by Definition 1.1 and the fact that $B_\mu \cap D_\mu\left[C_\mu, B\right] \le B_\mu \cap \hat{B}_\mu$. Thus, if ρ is the canonical homomorphism from B onto $B / \prod_{\lambda\in I} C_\lambda^B$, then $B_\lambda\rho$ is isomorphic to B_λ/C_λ , and hence to A_λ .

Now $B\rho$ is generated by the $B_\lambda\rho$, $\lambda \in I$. Let η be the natural epimorphism from A onto B_ρ induced by the isomorphism from each A_λ to $B_\lambda\rho$. The following diagram commutes

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$$\begin{array}{ccc} B & \xrightarrow{\ \rho\ } & B/\prod_{\lambda\in I} C_\lambda^B \\ {\scriptstyle\nu}\downarrow & \nearrow{\scriptstyle\eta} & \\ A & & \end{array}$$

The kernel of ν is therefore contained in $\ker\rho = \prod_{\lambda\in I} C_\lambda^B$ and hence $\ker\nu = \prod_{\lambda\in I} C_\lambda^B$.

(ii) The product, $\nu\psi$, of mappings is an epimorphism from B onto G . Now,

$$\begin{aligned} b \in \ker\nu\psi &\iff b\nu\psi = 1 && \text{in } G \\ &\iff b\nu \in H && \text{since } H = \ker\psi \\ &\iff b \in \ker\nu.K && \text{since } K\nu = H . \end{aligned}$$

But

$$\begin{aligned} \ker\nu.K &= \left(\prod_{\lambda\in I} C_\lambda^B\right)K \quad \text{from (i)} \\ &= C . \end{aligned}$$

Thus $C = \ker\nu\psi$ from (i) and (ii).

We aim to find the structure of $C\cap B'/[C, B]$ which is isomorphic to $M(G)$ when B is free by Lemma 2.1.

LEMMA 2.2. $\prod_{\lambda\in I} C_\lambda^B = \left(\prod_{\lambda\in I} C_\lambda\right).D$ *in* B *where* $D = \prod_{\substack{\lambda,\mu\in I\\ \lambda\neq\mu}} [C_\lambda, B_\mu]^B$.

Proof. We prove that c_λ^b is in $C_\lambda D$ for all $c_\lambda \in C_\lambda$ and $b \in B$ by induction on the length of the normal form for b . This is trivial when b is of length one, for $c_\lambda^{b_\mu} = c_\lambda[c , b_\mu]$, $\lambda \neq \mu$, and $c_\lambda^{b_\lambda} \in C_\lambda$ since C_λ is normal in B . Suppose that $c_\lambda^b \in C_\lambda D$ for all elements b of length m and that bb_μ is a reduced word in B . Then

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$$c_\lambda^{bb_\mu} = (c'_\lambda d)^{b_\mu} \quad \text{where} \quad c'_\lambda \in C_\lambda,\ d \in D \quad \text{by the inductive hypothesis}$$

$$= c'_\lambda [c'_\lambda, b_\mu] d^{b_\mu} \in C_\lambda D \quad \text{since} \quad [c'_\lambda, b_\mu] \in D \quad \text{for} \quad \lambda \neq \mu ,$$

and if $\lambda = \mu$, then $[c'_\lambda, b_\lambda] \in C_\lambda$.

Thus C_λ^B is a subgroup of $C_\lambda D$. It follows that $\prod_{\lambda \in I} C_\lambda^B$ is a subgroup of $\prod_{\lambda \in I} (C_\lambda D)$. By definition, every element of $\prod_{\lambda \in I} (C_\lambda D)$ can be written in the form $c_{\lambda_1} d_{\lambda_1} c_{\lambda_2} d_{\lambda_2} \ldots c_{\lambda_n} d_{\lambda_n}$ where $c_{\lambda_i} \in C_{\lambda_i}$, $d_{\lambda_i} \in D$ and $\lambda_1 < \lambda_2 < \ldots < \lambda_n$. But

$$c_{\lambda_1} d_{\lambda_1} c_{\lambda_2} d_{\lambda_2} \ldots c_{\lambda_n} d_{\lambda_n}$$

$$= \left(c_{\lambda_1} c_{\lambda_2} \ldots c_{\lambda_n}\right) d_{\lambda_1}^{\left(c_{\lambda_2} c_{\lambda_3} \ldots c_{\lambda_n}\right)} d_{\lambda_2}^{\left(c_{\lambda_3} \ldots c_{\lambda_n}\right)} \ldots d_{\lambda_n}$$

which belongs to $\left(\prod_{\lambda \in I} C_\lambda\right) D$. Hence $\prod_{\lambda \in I} C_\lambda^B$ is a subgroup of $\left(\prod_{\lambda \in I} C_\lambda\right).D$. The reverse inclusion is trivial since $[C_\lambda, B_\mu]^B$ is a subgroup of C_λ^B.

(2.3). *(i)* $C \cap B' = \left(\prod_{\lambda \in I} (C_\lambda \cap B'_\lambda)\right).D.K$,

(ii) $[C, B] = \left(\prod_{\lambda \in I} [C_\lambda, B_\lambda]\right) D.[K, B]$.

Proof. *(i)* C consists of all elements of the form $c = c_{\lambda_1} c_{\lambda_2} \ldots c_{\lambda_m} dk$ where $c_{\lambda_i} \in C_{\lambda_i}$, $d \in D$, $k \in K$, by the preceding two lemmas. The fact that DK is in B' means that c belongs to B' if and only if $\bar{c} = c_{\lambda_1} c_{\lambda_2} \ldots c_{\lambda_m}$ belongs to B'. Now, by (1.5), every element of B' is of the form $b'_{\lambda_1} b'_{\lambda_2} \ldots b'_{\lambda_n} u$ where

$b'_\lambda \in B'_\lambda$, $u \in \left[B_\lambda^B\right]$. But both C_{λ_i} and B'_{λ_i} are subgroups of B_{λ_i} . Therefore $\bar{c}$ is in B' if and only if c_{λ_i} is in B'_{λ_i} by (1.2).

(ii)

$$(*)\qquad [C, B] = \left[\left(\prod_{\lambda\in I} C_\lambda^B\right)K, B\right]$$

$$= \left(\prod_{\lambda\in I}\left[C_\lambda^B, B\right]\right)[K, B] \quad \text{by [2], Lemma 4.3}$$

$$= \left(\prod_{\lambda\in I}[C_\lambda, B]^B\right)[K, B] .$$

Now $[C_\lambda, B]$ is generated by the elements $[c_\lambda, b]$ where $c_\lambda \in C_\lambda$, $b \in B$. We can prove that $[c_\lambda, b] \in [C_\lambda, B_\lambda]D$ by induction on the length of b precisely as in Lemma 2.2. The result is trivial for $b = b_\mu$. Suppose that $[c_\lambda, b] \in [C_\lambda, B_\lambda].D$ for b of length m , then $[c_\lambda, b] = x_\lambda d$ where $x_\lambda \in [C_\lambda, B_\lambda]$ and $d \in D$. If bb_μ is reduced then

$$[c_\lambda, bb_\mu] = [c_\lambda, b_\mu][c_\lambda, b]^{b_\mu}$$

$$= [c_\lambda, b_\mu]x_\lambda^{b_\mu}d^{b_\mu}$$

$$= [c_\lambda, b_\mu]x_\lambda[x_\lambda, b_\mu]d^{b_\mu} ,$$

which is an element of $[C_\lambda, B_\lambda].[C_\lambda, B_\mu].D$. Thus $[C_\lambda, B]$ is a subgroup of $[C_\lambda, B_\lambda]D$. Consequently,

$$[C_\lambda, B]^B \le [C_\lambda, B_\lambda]^B.D^B$$

$$= [C_\lambda, B_\lambda]\left[[C_\lambda, B_\lambda], B\right]D$$

$$\le [C_\lambda, B_\lambda][C_\lambda, B]D \quad \text{since } C_\lambda \trianglelefteq B_\lambda$$

$$\le [C_\lambda, B_\lambda]D .$$

Hence, from (*), $[C, B]$ is a subgroup of $\left(\prod_{\lambda\in I}([C_\lambda, B_\lambda]D)\right).[K, B]$.

Again, copying the argument of Lemma 2.2, every element of $\prod_{\lambda\in I}([C_\lambda, B_\lambda]D)$ is a product of the form $x_{\lambda_1}d_{\lambda_1}x_{\lambda_2}d_{\lambda_2}\ldots x_{\lambda_n}d_{\lambda_n}$ where $x_{\lambda_i}\in[C_{\lambda_i}, B_{\lambda_i}]$, $d_{\lambda_i}\in D$ and $\lambda_1<\lambda_2<\ldots<\lambda_n$. This product can be rearranged to show that it lies in $\left(\prod_{\lambda\in I}[C_\lambda, B_\lambda]\right)D$. That is, $[C, B]$ is a subgroup of $\left(\prod_{\lambda\in I}[C_\lambda, B_\lambda]\right)D.[K, B]$. The reverse inclusion is trivial.

To shorten the proof of the main theorem, a technical lemma is established.

LEMMA 2.4. $\left[C_\lambda, \left[B^B_\mu\right]\right]$ *is a subgroup of* D *for all* $\lambda\in I$.

Proof. It suffices to prove that $\left[C^B_\lambda, \left[B^B_\mu\right]\right]$ is a subgroup of D since $\left[C_\lambda, \left[B^B_\mu\right]\right]$ is a subgroup of $\left[C^B_\lambda, \left[B^B_\mu\right]\right]$. The cartesian $\left[B^B_\mu\right]$ is generated by the elements $[b_\mu, b_\rho]$ where $b_\mu\in B^B_\mu$; $b_\rho\in B^B_\rho$; $\mu, \rho\in I$; $\mu\neq\rho$. Thus, because C^B_λ is normal in B, $\left[C^B_\lambda, \left[B^B_\mu\right]\right]$ is generated by the elements $[b_\mu, b_\rho, c_\lambda]$, $b_\mu\in B^B_\mu$; $b_\rho\in B^B_\rho$; $c_\lambda\in C^B_\lambda$; $\mu\neq\rho$. Moreover,

$$\begin{aligned}\left[C^B_\lambda, B^B_\mu\right] &\leq \left[C_\lambda D, B^B_\mu\right] && \text{from Lemma 2.2}\\ &= \left[C_\lambda, B^B_\mu\right]\left[D, B^B_\mu\right] &&\\ &\leq D && \text{provided } \lambda\neq\mu .\end{aligned}$$

The lemma is therefore proved if $[b_\mu, b_\rho, c_\lambda]$, $\mu\neq\rho$, can be shown to lie in $\left[C^B_\lambda, B^B_\sigma\right]$ for some $\sigma\neq\lambda$.

The following identity comes from Hall [3], (10.2.1.3), page 150,

$$(**)\qquad [b_\mu, b_\rho, c_\lambda] = \left[c_\lambda, b^{-1}_\mu, b^{-1}_\rho\right]^{-b_\mu b^{-1}_\rho}\left[b^{-1}_\rho, c^{-1}_\lambda, b^{-1}_\mu\right]^{-c^{-1}_\lambda b^{-1}_{\mid}}$$

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Consider the first commutator. There are two cases. If $\lambda = \mu$ then $\left[c_\lambda, b_\mu^{-1}, b_\rho^{-1}\right]$ lies in $\left[C_\lambda^B, B_\rho^B\right]$ since C_λ is normal in B_λ ($\lambda \neq \rho$ since $\mu \neq \rho$ by hypothesis). If $\lambda \neq \mu$ then

$$\left[c_\lambda, b_\mu^{-1}, b_\rho^{-1}\right] = \left[c_\lambda, b_\mu^{-1}\right]^{-1}\left[c_\lambda, b_\mu^{-1}\right]^{b_\rho^{-1}}$$
$$= \left[c_\lambda, b_\mu^{-1}\right]^{-1}\left[c_\lambda^{b_\rho^{-1}}, b_\mu^{-b_\rho^{-1}}\right],$$

which is in $\left[C_\lambda^B, B_\mu^B\right]$. That is $\left[c_\lambda, b_\mu^{-1}, b_\rho^{-1}\right]^{-b_\mu b_\rho^{-1}}$ is in the normal subgroup $\left[C_\lambda^B, B_\rho^B\right]$. The second commutator $\left[b_\rho^{-1}, c_\lambda^{-1}, b_\mu\right] = \left[\left[c_\lambda^{-1}, b_\rho^{-1}\right]^{-1}, b_\mu\right]$ can be treated in the same fashion.

The result follows from (**).

THEOREM 2.5. *$C\cap B'/[C, B]$ is isomorphic to the direct product $\left(\prod_{\lambda\in I}^{\times} C_\lambda\cap B'/\left[C_\lambda, B_\lambda\right]\right) \times H/[H, A]$.*

Proof. The argument is split into three parts.

(i) Let ϕ be the canonical homomorphism from B onto B/D. Then ϕ is a regular homomorphism since $\ker\phi = D$ and D is clearly a subgroup of $\left[B_\lambda^B\right]$. It follows from (1.7) that ϕ restricted to B_λ is an isomorphism for each $\lambda \in I$ and that $B\phi$ is a regular product of the $B_\lambda\phi$.

Consider $N = \text{sgp}\left(C_\lambda\phi \mid \lambda \in I\right).K\phi$ in $B\phi$. Now $\left[C_\lambda, C_\mu\right]$, $\lambda \neq \mu$, is a subgroup of $D = \ker\phi$, thus

(a) $$\left[C_\lambda\phi, C_\mu\phi\right] = E \text{ in } B\phi .$$

Also

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$$\left[C_\lambda, K\right] \le \left[C_\lambda, \left[B_\lambda^B\right]\right] \text{ since } K \le \left[B_\lambda^B\right] \text{ by construction}$$
$$\le D \qquad \text{by Lemma 2.4.}$$

Thus

(b) $$\left[C_\lambda\phi, K\phi\right] = E .$$

In addition,

(c) $$C_\lambda\phi \cap \left(\prod_{\substack{\mu\in I\\ \mu\neq\lambda}} C_\mu\phi\right).K\phi \le B_\lambda\phi \cap \left(B_\lambda\phi\right) = E$$

and

(d) $$K\phi \cap \prod_{\lambda\in I} C_\lambda\phi \le \left[B_\mu\phi^{B\phi}\right] \cap \prod_{\lambda\in I} C_\lambda\phi$$
$$= E \text{ by (1.3).}$$

Conditions (a) to (d) are sufficient to make N a direct product of its subgroups $K\phi$ and $C_\lambda\phi$.

(ii)

$$C\cap B'/[C, B] \simeq \frac{C\cap B'/D}{[C,B]/D} \text{ since } D \le [C, B] \text{ from (2.3) (ii)}$$
$$= (C\cap B')\phi/[C, B]\phi .$$

From (i) and (2.3), we have that $(C\cap B')\phi$ is the direct product of its subgroups $K\phi$ and $\left(C_\lambda \cap B_\lambda'\right)\phi$, $\lambda \in I$, (since $D\phi = E$) and that $[C, B]\phi$ is the direct product of its subgroups $[K, B]\phi$ and $\left[C_\lambda, B_\lambda\right]\phi$, $\lambda \in I$. This, together with the facts that $[K, B]\phi \le K\phi$ and $\left[C_\lambda, B_\lambda\right]\phi \le \left(C_\lambda\cap B_\lambda'\right)\phi$, implies that

$$(C\cap B')\phi/[C, B]\phi \simeq \left(\prod_{\lambda\in I}^{\times} \left(C_\lambda\cap B_\lambda'\right)\phi/\left[C_\lambda, B_\lambda\right]\phi\right) \times K\phi/[K, B]\phi$$
$$\simeq \left(\prod_{\lambda\in I}^{\times} \left(C_\lambda\cap B_\lambda'\right)/\left[C_\lambda, B_\lambda\right]\right) \times K\phi/[K, B]\phi ,$$

since ϕ is an isomorphism on B_λ .

(iii) In conclusion, we prove that $K\phi/[K, B]\phi$ is isomorphic to

$H/[H, A]$. Now,

(1) $$K\phi/[K, B]\phi \simeq \frac{KD/D}{[K,B]D/D} \quad \text{since} \quad D = \ker\phi$$
$$\simeq KD/[K, B]D$$
$$= K[K, B]D/[K, B]D \quad \text{since} \quad K \le B$$
$$\simeq K/K \cap [K, B].D \ .$$

Similarly

(2) $$K\nu/[K, B]\nu \simeq K/K \cap \ker\nu.[K, B] \ .$$

But $K \cap \ker\nu[K, B] = K \cap \left(\prod_{\lambda\in I} C_\lambda^B\right)[K, B]$ since $\ker\nu = \prod_{\lambda\in I} C_\lambda^B$ from Lemma 2.1 (i). Thus

$$K \cap \ker\nu.[K, B] = K \cap \left(\prod_{\lambda\in I} C_\lambda\right)D.[K, B]$$
$$= K \cap D[K, B] \qquad \text{since} \quad K, D[K, B] \le \left[B_\lambda^B\right] .$$

Hence

$$K\phi/[K, B]\phi \simeq K\nu/[K, B]\nu \quad \text{from (1) and (2)}$$
$$= H/[H, A] \quad \text{since} \quad K\nu = H \quad \text{by construction.}$$

The main result is a corollary to Theorem 2.5.

THEOREM 2.6. *$M(G)$ is isomorphic to the direct product $\left(\prod_{\lambda\in I}^{\times} M(A_\lambda)\right) \times H/[H, A]$ when G is a regular product of the A_λ and $A/H \simeq G$ where $A = \prod_{\lambda\in I}^{*} A_\lambda$.*

Proof. If B_λ is a free group then B_λ/C_λ is a presentation for A_λ and $C_\lambda \cap B'/[C_\lambda, B_\lambda]$ equals $M(A_\lambda)$. The free product, B , will be a free group and, since B/C is isomorphic to G , $M(G)$ equals $C \cap B'/[C, B]$. The result follows immediately from Theorem 2.5.

Theorem 2.6 reduces the problem of finding $M(G)$ to that of finding $H/[H, A]$. When $G = A_1 \times A_2$, $A = A_1 * A_2$ and $H/[H, A] = [A_1, A_2] / \left[[A_1, A_2], A\right]$. Wiegold ([9], Lemma 3.9) shows that the latter group is isomorphic to $A_1 \otimes A_2$. So we have the well known result, first proved by Schur, that $M(A_1 \times A_2)$ is isomorphic to $M(A_1) \times M(A_2) \times (A_1 \otimes A_2)$.

3. Construction of a representing group

There is a useful characterisation of $M(G)$, when G is finite, due to Schur [7]. Of all the pairs of groups (L, M) such that

(i) L is finite,

(ii) $M \le L' \cap Z(L)$,

(iii) $L/M \simeq G$,

there exists an L of maximal order called a *representing group* for G . The M corresponding to such an L is isomorphic to $M(G)$. We will call the pair (L, M) a *representing pair* when L is a representing group.

If G is finite then it must be generated by a finite set of subgroups $A_1, A_2, \ldots, A_n$. Let $\left(L_i, M(H_i)\right)$ be a representing pair for A_i , $i = 1, 2, \ldots, n$. Then these can be used to form a representing pair for G when G is a regular product.

We use the same constructions as before. There is an epimorphism from L_i to A_i with kernel $M(A_i)$ by hypothesis. Let $L = \prod_{i=1}^{n}{}^{*} L_i$ and σ be the natural epimorphism from L onto A induced by these epimorphisms. Put $J = H_1 \cap \left[L_i^L\right]$, where H_1 is the group in L which maps onto H under σ , and let $N = \prod_{\substack{i,j=1 \\ i \ne j}}^{n} \left[M(A_i), L_j\right]^L$. Then $J\sigma = H$ is normal in L .

THEOREM 3.1. *Suppose* G *is finite and a regular product of its subgroups* $A_1, A_2, \ldots, A_n$. *Let* $\left(L_i, M(A_i)\right)$ *be a fixed representing pair for* A_i . *Then* $(L\tau, M\tau)$ *is a representing pair for* G *where*

$\tau : L \to L/N[J, L]$ *and* $M = \left(\prod_{i=1}^{n} M(A_i)\right).J$ *in* L . *The groups* L, N *and* J *are as defined above.*

Proof. (i) Apply Lemma 2.1 by putting $I = \{1, 2, \ldots, n\}$, $B_i = L_i$ and $C_i = M(A_i)$. Then $L = B$, $C = M$, $N = D$ and $J = K$. We have

$$G \simeq B/C \qquad \text{from Lemma 2.1}$$

$$\simeq \frac{B/D[K,B]}{C/D[K,B]} \quad \text{since } D[K, B] \leq C \text{ from (2.3) (i).}$$

Thus $G = \frac{L/N[J,L]}{M/N[J,L]} = L\tau/M\tau$ by definition of τ .

Now J is a subgroup of L' immediately from its definition, and $M(A_i)$ is a subgroup of L_i' by hypothesis, so that M is a subgroup of L' . That is $(M\tau)'$ is a subgroup of $(L\tau)'$.

The kernel of τ contains $[J, L]$ as a subgroup which means that $J\tau$ is central in $L\tau$. Also $[M(A_i), L_j]$, $i \neq j$, is a subgroup of N which is in $\ker\tau$. Thus $M(A_i)\tau$ commutes with $L_j\tau$ for $i \neq j$. If $i = j$ then $M(A_i)$ commutes with L_i by hypothesis. Hence $M\tau$, as a product of $J\tau$ and the $M(A_i)\tau$, is central in $L\tau$.

The pair $(L\tau, M\tau)$ will be a representing pair if $M\tau$ is isomorphic to $M(G)$.

(ii) Theorem 2.5 gives that

$$M \cap L'/[M, L] \simeq \left(\prod_{i=1}^{n}{}^{\times} \left(M(A_i) \cap L_i'\right) / \left[M(A_i), L_i\right]\right) \times H/[H, A] .$$

But $M(A_i) \leq L_i' \cap Z(L_i)$ so that $M(A_i) \cap L_i' = M(A_i)$ and $[M(A_i), L_i] = E$. Hence

$$M \cap L'/[M, L] \simeq \left(\prod_{i=1}^{n}{}^{\times} M(A_i)\right) \times H/[H, A]$$

$$\simeq M(G) \qquad \text{by Theorem 2.6.}$$

Looking at the quotient $M \cap L'/[M, L]$, we have from (2.3) (ii), that

$$[M, L] = \left(\prod_{\substack{i,j=1 \\ i \neq j}}^{n} [M(A_i), L_j]\right)[J, L]$$

$$= N[J, L] \qquad \text{by definition}$$

$$= \ker\tau .$$

Thus $M\tau = M \cap L'/[M, L] \simeq M(G)$.

Theorem 3.1 has already been established by Wiegold in [10] for the case where $G = A_1 \times A_2$.

4. An application of Theorem 2.6

Blackburn has calculated the multiplicator of a wreath product $A \text{ wr } B$ for arbitrary groups A and B in [1]. I can obtain a more general result for a verbal wreath product, $A \text{ wr}_V B$, using techniques similar to those of Section 2. However, when A is abelian, $A \text{ wr}_V B$ is isomorphic to a regular product of A and B and $M(A \text{ wr}_V B)$ can be found directly using Theorem 2.6.

Let A_b be an isomorphic copy of A for each $b \in B$ and denote by a_b the element of A_b mapped to $a \in A$. The V-verbal product $D = \prod_{b \in B}^{V} A_b$ corresponding to a set of words V is defined to be the quotient C/C_V where $C = \prod_{b \in B}^{*} A_b$ and $C_V = V(C) \cap \left[A_b^C\right]$ (see Moran [5]). The mapping $a_b \mapsto a_{bb'}$ for all $a \in A$, $b \in B$ and fixed $b' \in B$, induces automorphisms of both C and D .

DEFINITION 4.1. *The free wreath product of* A *by* B , $A \text{ wr}_* B$, *is the splitting extension of* C *by* B *under the above action of* B *on* C . *Similarly the* V-*verbal wreath product* $A \text{ wr}_V B$, *is the splitting extension of* D *by* B *under the action of* B *on* D .

Actually $A \text{ wr}_V B = A \text{ wr}_* B$ when V is the empty word.

The verbal wreath product is generated by A_1 and B since

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$a_b = b^{-1}a_1b$ for all $a \in A$, $b \in B$. It is therefore a quotient of $A * B$ by some group H . Now both $\left[A_b^C\right]$ and $V(C)$ are invariant under the action of B on C by definition of the cartesian and since verbal subgroups are characteristic. That is $C_V = V(C) \cap \left[A_b^C\right]$ is normal in $A \operatorname{wr}_* B$. It follows from Definition 4.1 that $A \operatorname{wr}_V B$ is isomorphic to $A \operatorname{wr}_* B/C_V$. But it is well known that the natural epimorphism from $A * B$ onto $A \operatorname{wr}_* B$ induced by the identity on B and the isomorphism $A \overset{\sim}{\to} A_1$, is also an isomorphism (Hall and Hartley [4]). Hence $H = C_V\phi^{-1}$ and,

(4.2). *$A \operatorname{wr}_V B$ is isomorphic to $A*B/C_V\phi^{-1}$.*

By (1.8) $A \operatorname{wr}_V B$ will be a regular product of A and B if $C_V\phi^{-1}$ is contained in the cartesian $[A, B]$, or equivalently, if C_V is contained in $[A, B]\phi$. Firstly

$$\begin{aligned}[A, B]\phi^{-1} &= \operatorname{sgp}\left([a, b]\phi^{-1} \mid a \in A, b \in B\right) \\ &= \operatorname{sgp}\left(a_1^{-1}a_b \mid a \in A, b \in B\right) \\ &= \operatorname{sgp}\left(a_b^{-1}a_{b'} \mid a \in A; b, b' \in B\right) .\end{aligned}$$

Also, by (1.2), $\left[A_b^C\right]$ is the normal closure in C of the commutators $[a_b, a'_{b'}]$, $a, a' \in A$; $b, b' \in B$. But

$$\begin{aligned}[a_b, a'_{b'}] &= a_b^{-1}a'^{-1}_{b'}a_ba'_{b'} \\ &= \left(a_b^{-1}a_{b'}\right)a_{b'}^{-1}a'^{-1}_{b'}a_ba'_b\left(a'^{-1}_b a'_{b'}\right) \\ &= \left(a_b^{-1}a_{b'}\right)[a_{b'}, a'_{b'}]a'^{-1}_{b'}a_{b'}^{-1}a_ba'_b\left(a'^{-1}_b a'_{b'}\right) \\ &= \left(a_b^{-1}a_{b'}\right)[a, a']_{b'}\left((aa')_{b'}\right)^{-1}(aa')_b\left(a'^{-1}_b a'_{b'}\right) \\ &= \left(a_b^{-1}a_{b'}\right)\left((aa')_{b'}\right)^{-1}(aa')_b\left(a'^{-1}_b a'_{b'}\right)\end{aligned}$$

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for all $a, a' \in A$ if and only if A is abelian. Thus $\left[A_b^C\right]$ is contained in $[A, B]\phi$ and $A \text{ wr}_V B$ is indeed a regular product of A and B .

THEOREM 4.3. *If A is abelian, the multiplicator of $A \text{ wr}_V B$ is isomorphic to $M(A) \times M(B) \times C_V/[C_V, C][C_V, B]$ where $C_V = V(C) \cap \left[A_b^C\right]$, $C = \prod_{b\in B}^* A_b$ and the quotient is evaluated in $A \text{ wr}_* B$.*

Proof. From Theorem 2.6 the multiplicator of $A*B/C_V\phi^{-1}$ is $M(A) \times M(B) \times C_V\phi^{-1}/\left[C_V\phi^{-1}, A*B\right]$. Also

$$C_V\phi^{-1}/\left[C_V\phi^{-1}, A*B\right] \simeq C_V/[C_V, A \text{ wr}_* B]$$
$$= C_V/[C_V, C.B] \text{ since } A \text{ wr}_* B \text{ is the splitting extension of } C \text{ by } B$$
$$= C_V/[C_V, C][C_V, B] .$$

The ordinary wreath product $A \text{ wr } B$ is obtained from $A \text{ wr}_V B$ when $V(C) = C'$ and D becomes the direct product $\prod_{b\in B}^{\times} A_b$.

COROLLARY 4.4. *If A is abelian, $M(A \text{ wr } B)$ is isomorphic to $M(A) \times M(B) \times \left(\prod_{b<b'}^{\times} A_b \otimes A_{b'}\right)/N$ where "<" is any fixed ordering of the elements of B and N is the subgroup generated by the elements*

$$\left.\begin{array}{ll}\left(a_b \otimes a'_{b'}\right)^{-1}\left(a_{bb''} \otimes a'_{b'b''}\right) & \text{when } bb'' < b'b'' \\ \left(a_b \otimes a'_{b'}\right)^{-1}\left(a'_{b'b''} \otimes a_{bb''}\right)^{-1} & \text{when } b'b'' < bb''\end{array}\right\}$$

for all $a, a' \in A$; $b, b', b'' \in B$.

Proof. When $V(C) = C'$, denote C_V by $C_\times$ and let α be the canonical homomorphism from $A \text{ wr}_* B$ onto $A \text{ wr}_* B/[C_\times, C]$; $([C_\times, C]$ is normal since C is normal). The quotient $C_V/[C_V, C][C_V, B]$ is isomorphic to $C_\times\alpha/[C_\times, B]\alpha$.

Now $C_\times = C' \cap \left[A_b^C\right] = \left[A_b^C\right]$, so that $C\alpha$ is the second nilpotent product, $\prod^{(2)} A$, of the A_b (see [6]). We have the following elementary facts:

(i) $\left[A_b^C\right]\alpha$ is central in $C\alpha$, by construction of α ;

(ii) $\left[A_b^C\right]\alpha = \prod_{b,b' \in B} [A_b, A_{b'}]\alpha$ by definition of the cartesian and (i);

(iii) $[A_b, A_{b'}]\alpha = [A_{b'}, A_b]\alpha$;

(iv) $\left[A_{b_1}, A_{b_1'}\right]\alpha \cap \operatorname{sgp}\left(\left[A_{b_2}, A_{b_2'}\right]\alpha \mid b_2, b_2' \in B; (b_2, b_2') \neq (b_1, b_1'), (b_1', b_1)\right)$

is trivial by Definition 1.1 and the associativity of nilpotent products [2];

(v) $[A_b, A_{b'}] \simeq A_b \otimes A_{b'}$ by the remark following Theorem 2.6.

Therefore $\left[A_b^C\right]\alpha$ is isomorphic to $\prod^\times_{b<b'} A_b \otimes A_{b'}$.

Finally,

$$[C_\times, B]\alpha = \operatorname{sgp}\left(c^{-1}c^{b''} \mid c \in \left[A_b^C\right], b'' \in B\right)\alpha$$

$$= \operatorname{sgp}\left(\left([a_b, a'_{b'}]^{-1}[a_b, a'_{b'}]^{b''}\right)\alpha \mid a \in A; b, b', b'' \in B\right)$$

by definition of $\left[A_b^C\right]$ and (i)

$$= \operatorname{sgp}\left(\left([a_b, a'_{b'}]^{-1}[a_{bb''}, a'_{b'b''}]\right)\alpha \mid a \in A; b, b', b'' \in B\right) .$$

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The quotient $C_\times\alpha/[C_\times, B]\alpha$ is clearly isomorphic to $\left(\prod_{b<b'}^\times A_b \otimes A_{b'}\right)/N$.

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