More variations on Nagel and Gergonne analogues of the Steiner-Lehmus theorem

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1. *Introduction*

The celebrated Steiner-Lehmus theorem states that if the internal bisectors of two angles of a triangle are equal then the corresponding sides have equal lengths. That is to say if P is the incentre of $\triangle ABC$ and if BP and CP meet the sides AC and AB at B' and C', respectively, then

$$BB' = CC' \Rightarrow AB = AC.$$

An elegant proof of this theorem appeared in [1] and is reproduced in [2].

Several variations of the Steiner-Lehmus theorem have been considered in the literature. For example, in [2, Theorem 2], we considered the figure in which the internal angle bisectors of B and C of $\triangle ABC$ meet AC and AB at B' and C' and meet the external angle bisectors of C and B at D and E, respectively.

In what follows, we consider, in Theorems 1, 2, 3 and 4, more variations on the Steiner-Lehmus theme. In Theorem 1 the cevians from B and C through the Nagel centre of $\triangle ABC$ meet AC and AB at B' and C' and the external angle bisectors of C and B at D and E, respectively, as shown in Figures 1(a) and 1(b). In Theorems 2 and 3, which are illustrated in Figures 2 and 3, the same thing is done for the Gergonne centre. Also, in Theorem 3 we answer a question raised in section 5 of [3]. In Theorem 4 we consider the case where the external angle bisectors of B and C of $\triangle ABC$ meet the line drawn from A parallel to BC at E and D, respectively.

We shall denote the side lengths and angles of $\triangle ABC$ by a, b, c, A, B and C in the standard order. For convenience, we denote the area and perimeter of $\triangle ABC$ by $[ABC]$ and $\text{per}(ABC)$, respectively.

2. *Nagel analogue and stronger forms*

Let A', B' and C' be the points where the three excircles of $\triangle ABC$ touch BC, AC and AB and so the cevians AA', BB' and CC' meet at the Nagel centre N. Let $s = \frac{1}{2}(a + b + c)$. Then it is clear that

$$B'C = s - a = C'B, \qquad C'A = s - b = A'C, \qquad A'B = s - c = B'A.$$

It is also clear that

$$\begin{aligned} & \text{the external angle bisector of } B \text{ is parallel to } CC' \\ \Leftrightarrow \quad & \text{the external angle bisector of } C \text{ is parallel to } BB' \\ \Leftrightarrow \quad & s - a = a \Leftrightarrow 3a = b + c. \end{aligned}$$

Thus we shall assume that $3a \neq b + c$. Letting D and E be the points where the external angle bisectors of C and B meet the lines BB' and CC',

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respectively, it is easy to see that

$$D \text{ and } E \text{ are positioned as shown in Figure 1(a)} \quad \Leftrightarrow \quad b + c < 3a, \qquad (1)$$
$$D \text{ and } E \text{ are positioned as shown in Figure 1(b)} \quad \Leftrightarrow \quad b + c > 3a. \qquad (2)$$

Theorem 1: In ΔABC, where $3a \neq b + c$, the lines from B and C through the Nagel centre N meet AC and AB at B' and C' and meet the external angle bisectors of C and B at D and E, respectively, as shown in Figures 1(a) and 1(b).

(a) The statement $AB = AC$ is equivalent to each of the statements

(i) $BB' = CC'$, (ii) $[BCB'] = [CBC']$, (iii) $\text{per}(BCB') = \text{per}(CBC')$,
(iv) $BD = CE$, (v) $[BCD] = [CBE]$, (vi) $\text{per}(BCD) = \text{per}(CBE)$. (3)

(b) The statement $AB > AC$ is equivalent to each of the statements

(i) $BB' > CC'$, (ii) $[BCB'] > [CBC']$, (iii) $\text{per}(BCB') > \text{per}(CBC')$,
(iv) $BD > CE$, (v) $[BCD] > [CBE]$, (vi) $\text{per}(BCD) > \text{per}(CBE)$. (4)

Proof: We refer to Figures 1(a) and 1(b). Clearly, if $AB = AC$, then the equalities in (3) hold by symmetry. So it is enough to show that $AB > AC$ implies that all inequalities in (4) hold and the rest of (a) and (b) follow by contradiction. To see this, notice that an implication such as

$$AB > AC \qquad \Rightarrow \qquad BB' > CC' \qquad (5)$$

does indeed yield the converse implication

$$BB' > CC' \qquad \Rightarrow \qquad AB > AC. \qquad (6)$$

For if $BB' > CC'$, then AB can neither be equal to AC (because this would imply that $BB' = CC'$ by symmetry), nor less than AC (because this would imply that $BB' < CC'$ by (5)). Thus (5) yields (6). Similarly (5) implies that if $BB' = CC'$, then $AB = AC$.

So we assume that $AB > AC$ (i.e. $c > b$ whence $C > B$), and we are to prove that the inequalities in (4) hold. We let

$$\pi - B = 2\beta', \qquad \pi - C = 2\gamma'.$$

In what follows and in view of (1) and (2), refer to Figure 1(a) for the case $b + c < 3a$, and to Figure 1(b) for the case $b + c > 3a$.

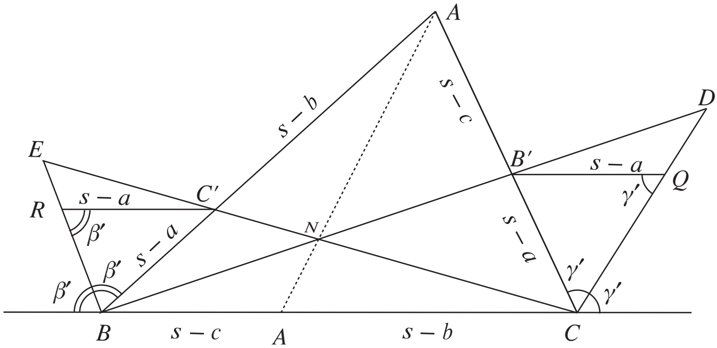

FIGURE 1(a): The case $b + c < 3a$ of Theorem 1

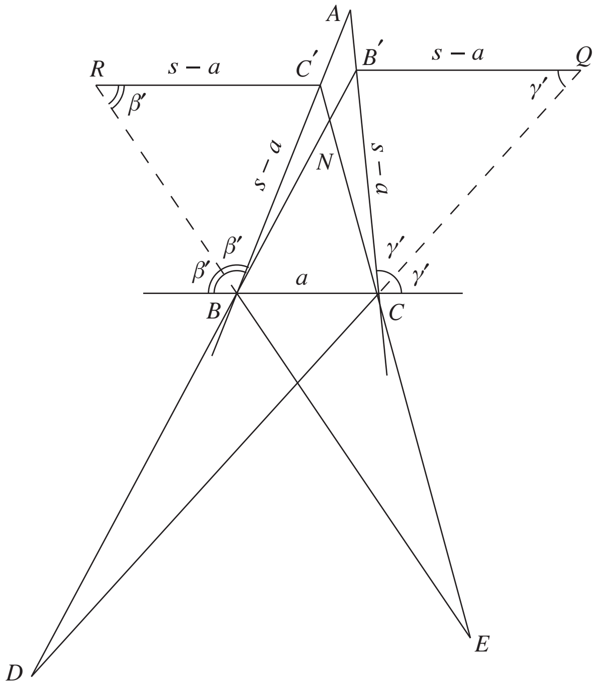

FIGURE 1(b): The case $b + c > 3a$ of Theorem 1

(b)(i): Since in $\Delta B'CB$ and $\Delta C'BC$ we have

$$B'C = s - a = C'B, \qquad CB = a = BC, \qquad C > B,$$

it follows by the 'open mouth theorem' that $BB' > CC'$ as desired.

(b)(ii): Since $c > b$ and $\dfrac{c}{\sin C} = \dfrac{b}{\sin B}$, it follows that $\sin C > \sin B$. Then

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$[BCB'] = \frac{1}{2}a(s-a)\sin C$ and $[CBC'] = \frac{1}{2}a(s-a)\sin B$, so that $[BCB'] > [CBC']$ as desired.

(b)(iii): Since $\text{per}(BCB') = s + BB'$ and $\text{per}(CBC') = s + CC'$, and since by (b)(i) $BB' > CC'$, it follows that $\text{per}(BCB') > \text{per}(CBC')$ as wanted.

(b)(iv): Draw $B'Q$ and $C'R$ parallel to BC (in either case). Note that $BB' = |BD - B'D|$ and $CC' = |CE - C'E|$ (in either case), so that

$$\frac{BB'}{BD} = \left|1 - \frac{B'D}{BD}\right| = \left|1 - \frac{s-a}{a}\right| = \left|1 - \frac{C'E}{CE}\right| = \frac{CC'}{CE}.$$

Since $BB' > CC'$, it follows that $BD > CE$ as required.

(b)(v): First, note that $[BCB'] = |[BCD] - [B'CD]|$ and $|CBC'| = |[CBE] - [C'BE]|$ (in either case). Then

$$\begin{aligned}\frac{[BCB']}{[BCD]} &= \left|1 - \frac{[B'CD]}{[BCD]}\right| \\ &= \left|1 - \frac{B'D}{BD}\right| \text{ (since the triangles have the same height)} \\ &= \left|1 - \frac{s-a}{a}\right|.\end{aligned}$$

Likewise

$$\frac{[CBC']}{[CBE]} = \left|1 - \frac{[C'BE]}{[CBE]}\right| = \left|1 - \frac{C'E}{CE}\right| = \left|1 - \frac{s-a}{a}\right|.$$

Thus $\dfrac{[BCB']}{[BCD]} = \dfrac{[CBC']}{[CBE]}$; but $[BCB'] > [CBC']$, so $[BCD] > [CBE]$ as required.

(b)(vi): Since in the $\Delta CB'Q$ and $\Delta BC'R$, we have (in either case)

$$CB' = B'Q = BC' = C'R = s - a, \qquad \angle CB'Q = C > B = \angle BC'R,$$

it follows by open mouth theorem that $CQ > BR$. Note also that $CQ = |CD - DQ|$ and $BR = |BE - ER|$ (in either case). So

$$\frac{CQ}{CD} = \left|1 - \frac{DQ}{CD}\right| = \left|1 - \frac{s-a}{a}\right| = \left|1 - \frac{ER}{BE}\right| = \frac{BR}{BE}.$$

But $CQ > BR$, so also $CD > BE$, and we conclude that

$$\text{per}(BCD) = BC + CD + BD > CB + BD + CE = \text{per}(CBE)$$

as required.

3. *Gergonne analogue and stronger forms*

Let A', B' and C be the points where the incircle of triangle ABC touches BC, AC and AB, respectively, and so the cevians AA', BB' and CC' meet at the Gergonne centre G.

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A Gergonne analogue of the Steiner-Lehmus theorem is given in Theorem 1 of [3]. In the next theorem we give a new proof and stronger forms of this theorem and in Theorem 3 we answer a question raised in [3].

Theorem 2: In ΔABC, the cevians from B and C through the Gergonne centre G meet AC and AB at B' and C', respectively, as shown in Figure 2.
(a) The statement $AB = AC$ is equivalent to each of the statements
(i) $BB' = CC'$, (ii) $[CBC'] = [BCB']$, (iii) $\text{per}(CBC') = \text{per}(BCB')$.
(b) The statement $AB > AC$ is equivalent to each of the statements
(i) $BB' > CC'$, (ii) $[CBC'] > [BCB']$, (iii) $\text{per}(CBC') > \text{per}(BCB')$.

Proof: Let $s = \frac{1}{2}(a + b + c)$. Then it is clear that

$$B'A = s - a = C'A, \qquad C'B = s - b = A'B, \qquad A'C = s - c = B'C,$$

as shown in Figure 2. As in Theorem 1, it is sufficient to prove (b). So let $AB > AC$. Draw $CM \parallel B'C'$ and join MB', as shown in Figure 2, and let

$$\angle AB'C' = \alpha, \qquad \angle C'CB' = \delta, \qquad \angle C'BB' = \mu.$$

From Figure 2, $\angle AC'B' = \angle AMC = \alpha$, $C'B'CM$ is cyclic, $\angle C'MB' = \delta$ and $MB' = CC'$. So $\mu < \delta < \alpha < \frac{\pi}{2}$. Since $\delta < \frac{\pi}{2}$, it follows that $\angle B'MB$ is obtuse, so that $BB' > MB' = CC'$, which proves (b)(i).

Note also that (b)(i) can be proved by applying the sine rule to $\Delta ABB'$ and $\Delta ACC'$ and using the fact that $\mu < \delta < \frac{\pi}{2}$.

(b)(ii): Since $[CBC'] = [CBM] + [CMC']$ and $\Delta CMC' \cong \Delta MCB'$, it follows that

$$[CBC'] = [CBM] + [MCB'] = [BCB'] + [BB'M] > [BCB']$$

as required.

(b)(iii):

$$\begin{aligned}\text{per}(CBC') &= BC + CC' + C'M + MB \\ &= BC + CB' + B'M + MB \text{ (since } CC' = B'M \text{ and } C'M = CB') \\ &= BC + CB' + B'B \text{ (by the triangle inequality)} \\ &= \text{per}(BCB')\end{aligned}$$

as required.

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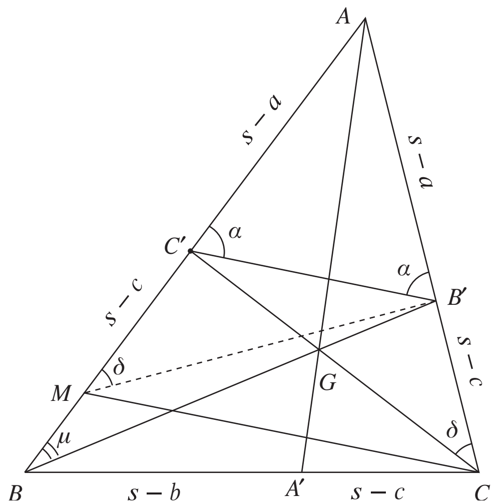

FIGURE 2: Theorem 2

Theorem 3: In triangle ABC, the cevians from B and C through the Gergonne centre G meet AC and AB at B' and C', respectively, and meet the external angle bisectors of C and B at D and E, respectively, as shown in Figure 3.

(a) The statement $AB = AC$ is equivalent to each of the statements
(i) $BE = CD$, (ii) $CE = BD$, (iii) $[CBE] = [BCD]$,
(iv) $\text{per}\,(CBE) = \text{per}\,(BCD)$.

(b) The statement $AB > AC$ is equivalent to each of the statements
(i) $BE > CD$, (ii) $CE > BD$, (iii) $[CBE] > [BCD]$,
(iv) $\text{per}\,(CBE) > \text{per}\,(BCD)$.

Proof: As in Theorem 1, it is sufficient to prove (b). So let $AB > AC$.
(b)(i): Put $\beta' = \frac{1}{2}(\pi - B)$ and $\gamma' = \frac{1}{2}(\pi - C)$. So $\beta' > \gamma'$. Applying the exterior angle theorem to $\Delta A'BC'$, we have $\angle C'A'B + \angle A'C'B = 2\beta'$; but $BA' = BC'$, so $\angle C'A'B = \angle A'C'B$, and therefore $\angle C'A'B = \beta'$. Similarly $\angle B'A'C = \gamma'$. Next, $\angle A'B'C' = \beta'$ and $\angle A'C'B' = \gamma'$, by the alternate segment theorem, so $\angle A'B'C' > \angle A'C'B'$, whence $A'C' > A'B'$. Then $A'C' \parallel BE$ and $A'B' \parallel CD$, so that

$$\frac{BE}{A'C'} = \frac{a}{s - c} \text{ and } \frac{CD}{A'B'} = \frac{a}{s - b}, \text{ whence } \frac{BE}{CD} = \frac{(s - b)A'C'}{(s - c)A'B'} > 1,$$

and thus $BE > CD$, as required.

(b)(ii): We have

$$\frac{CE}{CC'} = \frac{a}{s - c} \text{ and } \frac{BD}{BB'} = \frac{a}{s - b},$$

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whence

$$\frac{CE}{BD} = \frac{(s - b)(CC')}{(s - c)(BB')} = \frac{(a + c - b)(CC')}{(a + b - c)(BB')}.$$

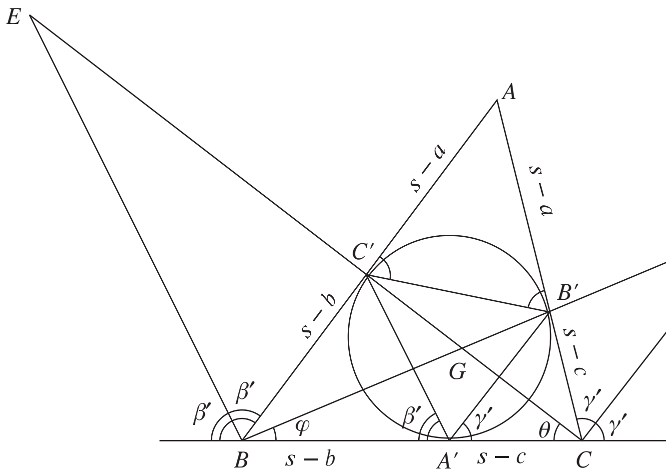

FIGURE 3: Theorem 3

Applying the cosine rule to $\Delta B'BA$, $\Delta C'CA$ and to ΔABC, we have

$$(BB')^2 = (s - a)^2 + c^2 - 2c(s - a)\cos A$$

$$(CC')^2 = (s - a)^2 + b^2 - 2b(s - a)\cos A$$

$$2bc\cos A = b^2 + c^2 - a^2.$$

But $2(s - a) = b + c - a$, $2(s - b) = a + c - b$ and $2(s - c) = a + b - c$. Therefore

$$4c(CC')^2 = c(b + c - a)^2 + 4b^2c - 2(b + c - a)(b^2 + c^2 - a^2),$$

$$4b(BB')^2 = b(b + c - a)^2 + 4bc^2 - 2(b + c - a)(b^2 + c^2 - a^2).$$

Thus we have

$$\frac{(CE)^2}{(BD)^2} = \frac{b(a + c - b)^2\left(c(b + c - a)^2 + 4b^2c - 2(b + c - a)(b^2 + c^2 - a^2)\right)}{c(a + b - c)^2\left(b(b + c - a)^2 + 4bc^2 - 2(b + c - a)(b^2 + c^2 - a^2)\right)}.$$

Put $\dfrac{(CE)^2}{(BD)^2} = \dfrac{m}{n}$, where

$$m = b(a + c - b)^2\left(c(b + c - a)^2 + 4b^2c - 2(b + c - a)(b^2 + c^2 - a^2)\right),$$

$$n = c(a + b - c)^2\left(b(b + c - a)^2 + 4bc^2 - 2(b + c - a)(b^2 + c^2 - a^2)\right),$$

and prove that $m > n$ (i.e. $m - n$ is positive), set $m - n = p + q + r$, where

$$p = bc(b + c - a)^2\left((a + c - b)^2 - (a + b - c)^2\right) = 4abc(c - b)(b + c - a)^2,$$

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$$q = 4bc\left(b^2(a+c-b)^2 - c^2(a+b-c)^2\right)$$
$$= 4bc(c-b)(b+c-a)^2\left(a(b+c)-(c-b)^2\right),$$
$$r = 2(b+c-a)\left(b^2+c^2-a^2\right)\left(c(a+b-c)^2 - b(a+c-b)^2\right)$$
$$= 2(c-b)(b+c-a)\left(b^2+c^2-a^2\right)\left(a^2-2a(b+c)+(c-b)^2\right).$$

Then we have

$$p + q = -4bc(c-b)(b+c-a)\left(a^2 - 2a(b+c) + (c-b)^2\right),$$

whence

$$m - n = 2(b+c-a)(c-b)\left(a^2-2a(b+c)+(c-b)^2\right)\left((c-b)^2-a^2\right)$$
$$= 2(b+c-a)(c-b)\left(a^2-(c-b)^2\right)\left(a(b+c)-a^2+a(b+c)-(c-b)^2\right)$$
$$= 32(c-b)(s-a)(s-b)(s-c)\left(a(s-a)+b(s-b)+c(s-c)\right),$$

which is clearly positive and hence $CE > BD$ as required.

Note that (b)(ii) answers the question raised in [3].

(b)(iii): Since $BC' = s-b > s-c = CB'$, $BE > CD$ by (b)(i), and $\beta' > \gamma'$, it follows that

$$[BC'E] = \tfrac{1}{2}(BC')(BE)\sin\beta' > \tfrac{1}{2}(CB')(CD)\sin\gamma' = [CB'D].$$

But $[CBC'] > [BCB']$ by (b)(ii) of Theorem 2. Therefore

$$[CBE] = [CBC'] + [BC'E] > [BCB'] + [CB'D] = [BCD]$$

as required.

Finally, (b)(iv) follows from (b)(i) and (b)(ii) where we have

$$\text{per}(CBE) = CB + BE + CE > BC + CD + BD = \text{per}(CBD).$$

4. *Another variation of the Steiner-Lehmus theorem*

In this section, we consider the case where the external angle bisectors of B and C meet the line drawn from A parallel to BC at E and D, respectively.

But first, a lemma:

Lemma: Given parallel line segments BC and EF, let the line from the midpoint N perpendicular to BC meet EF in M. Then, for P and Q on the line EF, we have that $\text{per}(PBC) > \text{per}(QBC)$ if $PM > QM$.

Proof: Join M, P and Q to R, the reflection of C in EF, as shown in Figure 4. Then M is the midpoint of BR, and by reflection in EF, we have $MC = MR$, $PC = PR$ and $QC = QR$. Now, if Q is between M and P, then by applying Euclid I.21 to ΔBPR, we have $BP + PR > BQ + QR$; that is, $BP + PC > BQ + QC$, whence $\text{per}(PBC) > \text{per}(QBC)$ as required. On the other hand, if M is between P and Q, then let Q' be the reflection of Q in

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MN, so that Q' is between M and P. Thus $\text{per}(PBC) > \text{per}(Q'BC)$; but $\Delta Q'BC \cong \Delta QCB$ by reflection in MN, so that once again $\text{per}(PBC) > \text{per}(QBC)$ as required.

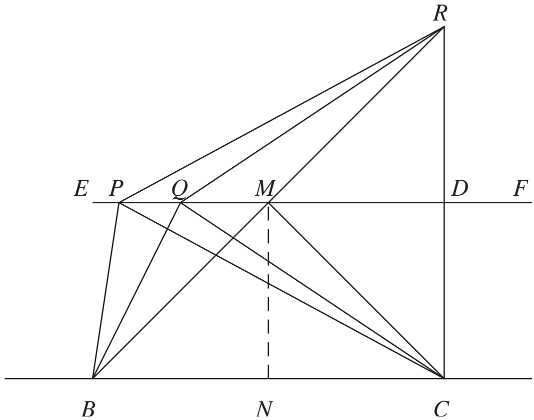

FIGURE 4: The lemma needed for Theorem

Theorem 4: In ΔABC, the external angle bisectors of B and C meet the line drawn from A parallel to BC at E and D, respectively, and BD and CE intersect AC and AB at B' and C', as shown in Figure 5.

(a) The statement $AB = AC$ is equivalent to each of the statements

(i) $BD = CE$, (ii) $CD = BE$, (iii) $\text{per}(BCD) = \text{per}(CBE)$,

(iv) $BB' = CC'$, (v) $\text{per}(CBB') = \text{per}(BCC')$, (vi) $[CBB'] = [BCC']$.

(b) The statement $AB > AC$ is equivalent to each of the statements

(i) $BD > CE$, (ii) $CD > BE$, (iii) $\text{per}(BCD) > \text{per}(CBE)$,

(iv) $BB' > CC'$, (v) $\text{per}(CBB') > \text{per}(BCC')$, (vi) $[CBB'] > [BCC']$.

Proof: As in Theorem 1, it is sufficient to prove (b). So let $AB > AC$. Put $\beta' = \frac{1}{2}(\pi - B), \gamma' = \frac{1}{2}(\pi - C)$.

(b)(i): Since $ED \| BC$, it follows that $\angle ADC = \gamma'$ and $\angle AEB = \beta'$. So in ΔBAD and ΔCAE, we have $AB = AE = c$, $AD = AC = b$ and $\angle BAD = A + C > A + B = \angle EAC$. So, by the open mouth theorem, we have that $BD > CE$ as required.

(b)(ii): Applying the sine rule to ΔCAD and ΔBAE we have

$$\frac{CD}{\sin C} = \frac{b}{\sin \gamma'} \text{ and } \frac{BE}{\sin B} = \frac{c}{\sin \beta'} \text{ and whence}$$

$$\frac{CD}{BE} = \frac{b \sin C \sin \beta'}{c \sin B \sin \gamma'} = \frac{\sin \beta'}{\sin \gamma'} > 1.$$

So $CD > BE$ as required.

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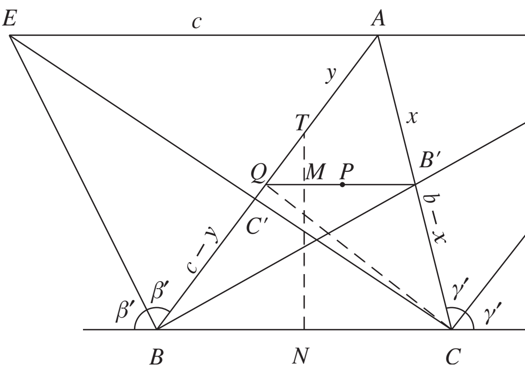

FIGURE 5: Theorem 5

(b)(iii): This follows, directly, from (b)(i) and (b)(ii).

(b)(iv): We deduce from $DE \parallel BC$ that

$$\frac{DB'}{BB'} = \frac{b}{a} \text{ and } \frac{EC'}{CC'} = \frac{c}{a}.$$

So we have

$$\frac{DB}{BB'} = \frac{a + b}{a} \text{ and } \frac{CE}{CC'} = \frac{a + c}{a}; \text{ whence } \frac{BB'}{CC'} = \frac{(a + c)BD}{(a + b)CE}.$$

But $BD > CE$ by (b)(i). Therefore $BB' > CC'$ as required.

(b)(v): Put $x = AB'$ and $y = AC'$. Then $CB' = b - x$ and $BC' = c - y$. Since $ED \parallel BC$, it follows that

$$\frac{x}{b - x} = \frac{b}{a} \text{ and } \frac{y}{c - y} = \frac{c}{a} \text{ and whence } x = \frac{b^2}{a + b} \text{ and } y = \frac{c^2}{a + c}.$$

Let the line through B' parallel to BC meet AB at Q, and let the perpendicular bisector of BC meet BC at N, $B'Q$ at M and AB at T. Then

$$\frac{AQ}{c} = \frac{x}{b} = \frac{b}{a + b} \text{ and } \frac{AC'}{c} = \frac{y}{c} = \frac{c}{a + c}.$$

But $b < c$, whence $\dfrac{b}{a + b} < \dfrac{c}{a + c}$, so that $AQ < AC'$, and C' is between B and Q. Then $C'C < C'Q + QC$, and, adding $BC' + BC$ to each side, we have that $\text{per}(BCC') < \text{per}(BCQ)$. So it remains to show that $\text{per}(BCQ) < \text{per}(CBB)$. But this will follow from the Lemma provided we show that $MQ < MB'$. This is obvious if Q is between M and B'. If, on the other hand, M is between Q and B', then let the reflection TC of TB in TN meet $B'Q$ at P. Then $MQ = MP$ and P is between M and B', so $MP < MB'$, whence $MQ < MB'$ again, and we are finished.

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(b)(vi): Since $BQ \parallel BC$, we have $[CBB'] = [CBQ]$, and since C' is between B and Q, we have $[CBQ] > [BCC']$, whence the result.

Note that other variations on Steiner-Lehmus theme can be obtained by taking centres in the above configurations other than Nagel and Gergonne centres, such as the centroid, the circumcentre, the orthocentre or the Fermat-Torricelli centre.

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