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On the absolute Nörlund summability factors of Fourier series

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The object of this paper is to give a general theorem which implies Izumi's Theorem and Kanno's Theorem on the absolute Nörlund summability factors of Fourier series and deduce to several known and new results from the theorem.

1.

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + \ldots + p_n \; ; \; P_{-k} = p_{-k} = 0 \text{ , for } k \geq 1 \text{ .}$$

The sequence $\{t_n\}$, given by

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k = \frac{1}{P_n} \sum_{k=0}^{n} P_k a_{n-k} \text{ , } \quad (P_n \neq 0) \text{ ,} \tag{1.1}$$

defines the Nörlund means of the sequence $\{s_n\}$ generated by the sequence of constants $\{p_n\}$.

The series $\sum a_n$ is said to be absolutely summable (N, p_n) , or summable $|N, p_n|$, if the series

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| \tag{1.2}$$

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is convergent.

In the special cases in which $p_n = \Gamma(n+\alpha)/\Gamma(\alpha)\Gamma(n+1)$, $\alpha > 0$, and $p_n = 1/(n+1)$, summability $|N, p_n|$ are the same as the summability $|C, \alpha|$ and the absolute harmonic summability, respectively.

2.

Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. We assume without any loss of generality that the Fourier series of $f(t)$ is given by

$$(2.1) \qquad \sum_{n=1}^{\infty} \left(a_n \cos nt + b_n \sin nt\right) = \sum_{n=1}^{\infty} A_n(t)$$

and $\int_{-\pi}^{\pi} f(t)dt = 0$. We write $\varphi_x(t) = \varphi(t) = \frac{1}{2}\{f(x+t)+f(x-t)\}$, $\lambda(n) = \lambda_n$, and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

Dealing with the absolute Nörlund summability of Fourier series, Izumi and Izumi [3] proved the following theorem, which is a generalization of theorems due to Bosanquet [1] and Mohanty [6, 7].

THEOREM A. *Let* $\{p_n\}$ *be non-negative and non-increasing and* $\lambda(t)$, $t > 0$, *be a positive non-decreasing function such that* $\{\lambda_n/(n+1)\}$ *is non-increasing,*

$$(2.2) \qquad \sum_{k=n}^{\infty} \frac{\lambda_k}{(k+1)P_k} = O\left(\frac{\lambda_n}{P_n}\right), \quad n = 0, 1, 2, \ldots$$

and

$$(2.3) \qquad \int_0^{\pi} \lambda(C/t)|d\varphi(t)| < \infty .$$

Then the series

$$\sum_{n=0}^{\infty} \lambda_n A_n(t)$$

is summable $|N, p_n|$, *at* $t = x$.

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Generalizing the theorem of Varshney [8], Kanno [4] proved the following theorem.

THEOREM B. *Let* $\{p_n\}$ *be non-negative and non-increasing. Let* $\lambda(t)$, $t > 0$ *, be a positive, non-decreasing function satisfying the condition* $\{\lambda_n/P_n\}$ *is non-increasing.*

If the conditions

$$\sum_{k=n}^{\infty} \frac{p_k\lambda_k}{P_k^2} = O\left(\frac{\lambda_n}{P_n}\right) , \quad n = 0, 1, 2, \ldots \tag{2.4}$$

and

$$\int_0^{\pi} \lambda(C/t)|d\varphi(t)| < \infty$$

for some constant $C > 0$ *hold, then the series*

$$\sum_{n=0}^{\infty} \frac{(n+1)p_n}{P_n} \lambda_n A_{n+1}(t)$$

is summable $|N, p_n|$ *at* $t = x$.

Also see Dikshit [2] for the proofs of these theorems.

We generalize these theorems in the following form.

THEOREM. *Let* $\{p_n\}$ *be non-negative and non-increasing. Suppose that* $\left\{\lambda_n^{(1)}\right\}$ *is a positive bounded sequence and* $\lambda^{(2)}(t)$, $t > 0$ *, is a positive non-decreasing function such that* $\left\{\lambda_n^{(1)}\lambda_n^{(2)}/(n+1)\right\}$ *is non-increasing,*

$$\sum_{k=n}^{\infty} \frac{\lambda_k^{(1)}\lambda_k^{(2)}}{kP_k} = O\left(\frac{\lambda_n^{(2)}}{P_n}\right), \quad n = 1, 2, \ldots \tag{2.6}$$

and

$$\int_0^{\pi} \lambda^{(2)}(C/t)|d\varphi(t)| < \infty \text{, for a constant } C\ (> 2\pi) . \tag{2.7}$$

Then the series

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$$\sum_{n=0}^{\infty} \lambda_n^{(1)}\lambda_n^{(2)}A_{n+1}(t)$$

is summable $|N, p_n|$ *, at* $t = x$.

If $\lambda_n^{(1)} = 1$ and $\lambda_n^{(2)} = \lambda_n$, our theorem reduces to Theorem A. If we put $\lambda_n^{(1)} = (n+1)p_n/P_n$ and $\lambda_n^{(2)} = \lambda_n$, we easily see that, under the same assumptions as those of Theorem B, the condition (2.6) is satisfied and the sequence $\left\{\lambda_n^{(1)}\lambda_n^{(2)}/(n+1)\right\}$ is non-increasing because

$$\sum_{k=n}^{\infty} \frac{\lambda_n^{(1)}\lambda_n^{(2)}}{kP_k} \le A \sum_{k=n}^{\infty} \frac{p_k\lambda_k^{(2)}}{P_k^2} = O\left(\frac{\lambda_n^{(2)}}{P_n}\right)$$

and

$$\frac{\lambda_n^{(1)}\lambda_n^{(2)}}{n+1} = \frac{p_n\lambda_n^{(2)}}{P_n} .$$

Therefore our theorem includes Theorem B.

3.

We shall require the following lemmas to prove the theorem.

LEMMA 1 [2]. *Let* $\{p_n\}$ *be a given sequence, then for any* x *, we have*

$$(1-x) \sum_{k=m}^{n} p_k x^k = p_m x^m - p_n x^{n+1} - \sum_{k=m}^{n-1} \Delta p_k x^{k+1}$$

where m *and* n *are integers such that* $n \ge m \ge 0$.

This lemma is easily obtained.

LEMMA 2 [5]. *If* $\{p_n\}$ *is non-negative, non-increasing, then for* $0 \le a \le b < \infty$, $0 \le t \le \pi$ *, and for any* n *, we have*

$$\left|\sum_{k=a}^{b} p_k \exp\left(i(n-k)t\right)\right| \le AP_{[1/t]} ,$$

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where A *is a positive constant and* $[x]$ *denotes the integral part of* x .

4.

Proof of the Theorem. By (1.1) we have

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k \lambda_{n-k}^{(1)} \lambda_{n-k}^{(2)} A_{n+1-k}(t) ,$$

where

$$A_k(x) = \frac{2}{\pi} \int_0^\pi \varphi(t) \cos kt dt .$$

Hence,

$$\begin{aligned} t_n - t_{n-1} &= \sum_{k=1}^{n} \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}}\right) \lambda_k^{(1)} \lambda_k^{(2)} A_{k+1}(t) \\ &= \frac{2}{\pi} \int_0^\pi \varphi(t) \left\{\frac{1}{P_n P_{n-1}} \sum_{k=1}^{n} \left(P_n p_{n-k} - P_{n-k} p_n\right) \lambda_k^{(1)} \lambda_k^{(2)} \cos(k+1)t\right\} dt \\ &= \frac{2}{\pi} \int_0^\pi d\varphi(t) \left\{\frac{1}{P_n P_{n-1}} \sum_{k=1}^{n} \left(P_n p_{n-k} - P_{n-k} p_n\right) \lambda_k^{(1)} \lambda_k^{(2)} \frac{\sin(k+1)t}{k+1}\right\} . \end{aligned}$$

Thus, to prove the theorem, it is enough to show that

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}|$$

$$\leq \frac{2}{\pi} \int_0^\pi |d\varphi(t)| \left| \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{k=1}^{n} \left(P_n p_{n-k} - P_{n-k} p_n\right) \lambda_k^{(1)} \lambda_k^{(2)} \frac{\sin(k+1)t}{k+1} \right| = O(1) .$$

Considering the condition (2.7), it suffices for our purpose to prove that uniformly in $0 < t \leq \pi$,

$$(4.1) \quad I = \sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} \frac{\left(P_n p_{n-k} - P_{n-k} p_n\right)}{P_n P_{n-1}} \lambda_k^{(1)} \lambda_k^{(2)} \frac{\sin(k+1)t}{k+1} \right| = O\left(\lambda^{(2)}(C/t)\right) .$$

Let us write $\tau = [C/2t]$, so we have

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$$I \le \sum_{n=1}^{2\tau+1} \left| \sum_{k=1}^{n} \frac{(P_n p_{n-k} - P_{n-k} p_n)}{P_n P_{n-1}} \lambda_k^{(1)} \lambda_k^{(2)} \frac{\sin(k+1)t}{k+1} \right|$$

$$+ \sum_{n=2\tau+2}^{\infty} \left| \sum_{k=1}^{\tau} \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \right) \lambda_k^{(1)} \lambda_k^{(2)} \frac{\sin(k+1)t}{k+1} \right|$$

$$+ \sum_{n=2\tau+2}^{\infty} \left| \sum_{k=\tau+1}^{n} \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \right) \lambda_k^{(1)} \lambda_k^{(2)} \frac{\sin(k+1)t}{k+1} \right|$$

$$= I_1 + I_2 + I_3 ,$$

say.

Since $\{p_n\}$ is non-negative and non-increasing, $\left\{\lambda_n^{(1)}\right\}$ is bounded, $\left\{\lambda_n^{(2)}\right\}$ is non-decreasing and $|\sin(k+1)t| \le (k+1)t$, we have

$$I_1 \le At \sum_{n=1}^{2\tau+1} \lambda_n^{(2)} \frac{1}{P_n} \sum_{k=1}^{n} p_{n-k} \le A\lambda^{(2)}(C/t)\cdot t\cdot 2\tau$$

$$= O\left(\lambda^{(2)}(C/t)\right) .$$

Also, since $\{P_{n-k}/P_n\}$ is monotonic non-decreasing and bounded for each fixed $k \ge 0$, we have

$$I_2 \le At \sum_{k=1}^{\tau} \lambda_k^{(2)} \sum_{n=2\tau+2}^{\infty} \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \right) \le At\lambda^{(2)}(C/t) \sum_{k=1}^{\tau} 1$$

$$= O\left(\lambda^{(2)}(C/t)\right) ,$$

by virtue of the hypotheses that $\left\{\lambda_n^{(1)}\right\}$ is bounded and $\left\{\lambda_n^{(2)}\right\}$ is non-decreasing.

In order to prove that $I_3 = O\left(\lambda^{(2)}(C/t)\right)$, we consider the sum

$$I_3^* = \sum_{n=2\tau+2}^{N} \left| \sum_{k=\tau+1}^{n} \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \right) \lambda_k^{(1)} \lambda_k^{(2)} \frac{\exp(ikt)}{k+1} \right| .$$

Then it is enough to prove that

$$I_3^* = O\left(\lambda^{(2)}(C/t)\right) \quad \text{as} \quad N \to \infty .$$

Now, we observe that

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$$I_3^* \le \sum_{n=2\tau+2}^{N} \left| \sum_{k=\tau+1}^{m} \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}}\right) \lambda_k^{(1)} \lambda_k^{(2)} \frac{\exp(ikt)}{k+1} \right|$$

$$+ \sum_{n=2\tau+2}^{N} \frac{1}{P_{n-1}} \left| \sum_{k=m+1}^{n} p_{n-k} \lambda_k^{(1)} \lambda_k^{(2)} \frac{\exp(ikt)}{k+1} \right|$$

$$+ \sum_{n=2\tau+2}^{N} \frac{p_n}{P_n P_{n-1}} \left| \sum_{k=m+1}^{n} P_{n-k} \lambda_k^{(1)} \lambda_k^{(2)} \frac{\exp(ikt)}{k+1} \right|$$

$$= I_{31}^* + I_{32}^* + K_{33}^* ,$$

say, where $m = [n/2]$.

Since $\{P_{n-k}/P_n\}$ is non-decreasing for each fixed $k \ge 0$ and $|1-\exp(it)|^{-1} = O(t^{-1})$, we obtain by an application of Lemma 1,

$$I_{31}^* = \sum_{n=2\tau+2}^{N} \left| \sum_{k=\tau+1}^{m} \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}}\right) \lambda_k^{(1)} \lambda_k^{(2)} \frac{\exp(ikt)}{k+1} \right|$$

$$\le A\tau \sum_{n=2\tau+2}^{N} \left(\frac{P_{n-\tau-1}}{P_n} - \frac{P_{n-\tau-2}}{P_{n-1}}\right) \lambda_{\tau+1}^{(1)} \lambda_{\tau+2}^{(2)} \frac{|\exp(i(\tau+1)t)|}{\tau+2}$$

$$+ A\tau \sum_{n=2\tau+2}^{N} \left(\frac{P_{n-m}}{P_n} - \frac{P_{n-m-1}}{P_{n-1}}\right) \lambda_m^{(1)} \lambda_m^{(2)} \frac{|\exp(i(m+1)t)|}{m+1}$$

$$+ A\tau \sum_{n=2\tau+2}^{N} \left| \sum_{k=\tau+1}^{m-1} \Delta\left(\frac{\lambda_k^{(1)} \lambda_k^{(2)}}{k+1}\right) \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}}\right) \exp(i(k+1)t) \right|$$

$$+ A\tau \sum_{n=2\tau+2}^{N} \left| \sum_{k=\tau+1}^{m-1} \left(\frac{p_{n-k}}{P_n} - \frac{p_{n-k-1}}{P_{n-1}}\right) \frac{\lambda_{k+1}^{(1)} \lambda_{k+1}^{(2)}}{k+2} \exp(i(k+1)t) \right| .$$

Since for each fixed k , $\{p_{n-k}/P_n\}$ is non-increasing while $\{P_{n-k}/P_n\}$ is non-decreasing, we obtain

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$$I^*_{31} \le A\lambda^{(1)}_{\tau+1}\lambda^{(2)}_{\tau+1}\left(\frac{P_{N-\tau-1}}{P_N} - \frac{p_\tau}{P_{2\tau+1}}\right) + A\tau \sum_{n=2\tau+2}^{N} \left(\frac{P_n p_{n-m} - p_n P_{n-m}}{P_n P_{n-1}}\right) \frac{\lambda^{(1)}_m \lambda^{(2)}_m}{m+1}$$

$$+ A\tau \sum_{k=\tau+1}^{[N/2]-1} \Delta\left(\frac{\lambda^{(1)}_k \lambda^{(2)}_k}{k+1}\right) \sum_{n=2k+1}^{N} \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}}\right)$$

$$+ A\tau \sum_{k=\tau+1}^{[N/2]-1} \frac{\lambda^{(1)}_{k+1}\lambda^{(2)}_{k+1}}{k+2} \sum_{n=2k+1}^{N} \left(\frac{p_{n-k-1}}{P_{n-1}} - \frac{p_{n-k}}{P_n}\right)$$

$$\le A\lambda^{(2)}(C/t) + A\tau \sum_{n=2\tau+2}^{N} \left(\frac{p_m}{P_m}\right) \frac{\lambda^{(1)}_m \lambda^{(2)}_m}{m+1}$$

$$+ A\tau \sum_{k=\tau+1}^{[N/2]-1} \Delta\left(\frac{\lambda^{(1)}_k \lambda^{(2)}_k}{k+1}\right) + A\tau \sum_{k=\tau+1}^{N} \frac{\lambda^{(1)}_{k+1}\lambda^{(2)}_{k+1}}{k+1} \frac{p_k}{P_{2k}}$$

$$\le A\lambda^{(2)}(C/t) + A\tau p_{\tau+1} \sum_{n=\tau+1}^{N} \frac{\lambda^{(1)}_n \lambda^{(2)}_n}{nP_n}$$

$$+ A\tau \frac{\lambda^{(1)}_{\tau+1}\lambda^{(2)}_{\tau+1}}{\tau+1} + A\tau p_{\tau+1} \sum_{k=\tau+1}^{N} \frac{\lambda^{(1)}_k \lambda^{(2)}_k}{kP_k}$$

$$\le A\lambda^{(2)}(C/t) + A\tau p_{\tau+1} \frac{\lambda^{(2)}_{\tau+1}}{P_{\tau+1}} + A\lambda^{(2)}_{\tau+1} + A\tau p_{\tau+1} \frac{\lambda^{(2)}_{\tau+1}}{P_{\tau+1}}$$

$$\le A\lambda^{(2)}(C/t)$$

as $N \to \infty$, by virtue of the hypotheses (2.6) and that $\left\{\lambda^{(1)}_k \lambda^{(2)}_k/(k+1)\right\}$ is non-increasing.

Since $\left\{\lambda^{(1)}_k \lambda^{(2)}_k/(k+1)\right\}$ is non-increasing, we have by Abel's Lemma and Lemma 2,

$$I_{32}^{*} = \sum_{n=2\tau+2}^{N} \frac{1}{P_{n-1}} \left| \sum_{k=m+1}^{n} p_{n-k} \frac{\lambda_k^{(1)}\lambda_k^{(2)}}{k+1} \exp(ikt) \right|$$

$$\leq A \sum_{n=2\tau+2}^{N} \frac{1}{P_{n-1}} \frac{\lambda_{m+1}^{(1)}\lambda_{m+1}^{(2)}}{m+2} \max_{m<l\leq n} \left| \sum_{k=m+1}^{l} p_{n-k} \exp(ikt) \right|$$

$$\leq AP_\tau \sum_{n=\tau}^{N} \frac{\lambda_n^{(1)}\lambda_n^{(2)}}{nP_n} \leq AP_\tau \cdot \frac{\lambda_\tau^{(2)}}{P_\tau}$$

$$= O\left(\lambda^{(2)}(C/t)\right)$$

as $N \to \infty$, by virtue of the hypothesis (2.6).

By a similar method, we have

$$I_{33}^{*} = \sum_{n=2\tau+2}^{N} \frac{p_n}{P_n P_{n-1}} \sum_{k=m+1}^{n} P_{n-k} \frac{\lambda_k^{(1)}\lambda_k^{(2)}}{k+1} \exp(ikt)$$

$$= \sum_{n=2\tau+2}^{N} \frac{p_n}{P_n P_{n-1}} \left| \sum_{k=m+1}^{n} P_{n-k} \frac{1}{p_{n-k}} p_{n-k} \frac{\lambda_k^{(1)}\lambda_k^{(2)}}{k+1} \exp(ikt) \right|$$

$$\leq A \sum_{n=2\tau+2}^{N} \frac{p_n}{P_n P_{n-1}} \frac{P_m}{p_{n-m}} \frac{\lambda_m^{(1)}\lambda_m^{(2)}}{m} \max_{m<l\leq n} \left| \sum_{k=m+1}^{l} p_{n-k} \exp(ikt) \right|$$

$$\leq AP_\tau \sum_{n=\tau+1}^{N} \frac{\lambda_n^{(1)}\lambda_n^{(2)}}{nP_n} \leq AP_\tau \cdot \frac{\lambda_\tau^{(2)}}{P_\tau}$$

$$= O\left(\lambda^{(2)}(C/t)\right) .$$

Collecting the above estimations I_{31}^{*}, I_{32}^{*}, and I_{33}^{*}, we prove that uniformly in $0 < t \leq \pi$, $I_3^{*} = O\left(\lambda^{(2)}(C/t)\right)$ and *a fortiori* that $I_3 = O\left(\lambda^{(2)}(C/t)\right)$. Therefore, by I_1, I_2, and I_3, we have

$$I = O\left(\lambda^{(2)}(C/t)\right) .$$

This completes the proof of the theorem.

5.

In this section, we consider some applications of our theorem.

Using a result of Das and Srivastava (*cf.* [4], Theorem B), it is shown

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by Kanno [4] that it is possible to deduce the following four corollaries from his theorem. However, if we apply our theorem, we need not appeal to the result of Das and Srivastava (*cf.* [4], Theorem B) for proofs of their corollaries.

COROLLARY 1 [6]. *If*

$$\int_0^{\pi} t^{-\alpha}|d\varphi(t)| < \infty ,$$

then the series $\sum_{n=1}^{\infty} n^{\alpha}A_n(t)$ *is summable* $|C, \beta|$ *at* $t = x$, *where* $0 \le \alpha < \beta < 1$.

COROLLARY 2. *If* $0 < \alpha < 1$, $\beta \ge 0$, *and*

$$\int_0^{\pi} (\log C/t)^{\beta}|d\varphi(t)| < \infty ,$$

then the series $\sum_{n=1}^{\infty} (\log n)^{\beta}A_n(t)$ *is summable* $|C, \alpha|$ *at* $t = x$.

This corollary coincides to Bosanquet [1] for $\beta = 0$, and Mohanty [7] for $\beta = 1$, respectively.

COROLLARY 3. *If* $1 > \alpha \ge 0$, $\beta \ge 0$, $\alpha + \beta < 1$, *and*

$$\int_0^{\pi} \left(\log \frac{C}{t}\right)^{\beta} |d\varphi(t)| < \infty ,$$

then the series

$$\sum_{n=0}^{\infty} \frac{A_n(t)}{\{\log(n+2)\}^{1-\beta}} \text{ is summable } |N, 1/(n+2)\{\log(n+2)\}^{\alpha}| .$$

For $\alpha = \beta = 0$, this corollary is due to Varshney [8].

COROLLARY 4. *If*

$$\int_0^{\pi} \left(\log \log \frac{C}{t}\right)^{\beta} |d\varphi(t)| < \infty \text{ for } 0 \le \beta < 1 ,$$

then the series

$$\sum_{n=0}^{\infty} \frac{A_n(t)}{\log(n+2)\{\log \log(n+2)\}^{1-\beta}} \text{ is summable } |N, 1/(n+2)\log(n+2)|$$

at $t = x$.

As these corollaries are analogously proved, we shall prove here only Corollary 4.

Proof of Corollary 4. In our theorem, we put $p_k = 1/(k+2)\log(k+2)$, $\lambda_k^{(1)} = 1/\log(k+2)\log \log(k+2)$, $\lambda_k^{(2)} = \{\log \log(k+2)\}^{\beta}$. Then

$$P_n = \sum_{k=0}^{n} 1/(k+2)\log(k+2) \simeq \log \log(n+2) .$$

Moreover it is easy to see that

$$\sum_{k=n}^{\infty} \frac{\lambda_k^{(1)}\lambda_k^{(2)}}{kP_k} = O\left(\frac{\{\log \log(n+2)\}^{\beta}}{\log \log(n+2)}\right) = O\left(\frac{\lambda_n^{(2)}}{P_n}\right) .$$

Hence all assumptions of our theorem hold. Therefore the proof is complete. Further, by our theorem, we obtain the following Corollaries 5 and 6, which correspond to Corollaries 3 and 4 for $\beta = 1$, respectively.

COROLLARY 5. *If*

$$\int_0^{\pi} \log\left(\frac{C}{t}\right) |d\varphi(t)| < \infty ,$$

then the series

$$\sum_{n=0}^{\infty} A_n(t) \text{ is summable } |N, \log(n+2)/(n+2)| ,$$

at $t = x$.

Proof. Putting

$$p_k = \log(k+2)/(k+2) , \quad \lambda_k^{(1)} = 1/\log(k+2) , \quad \lambda_k^{(2)} = \log(k+2) ,$$

then we have

$$P_n = \sum_{k=0}^{n} \frac{\log(k+2)}{k+2} \simeq \{\log(n+2)\}^2 .$$

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On the other hand, we have

$$\sum_{k=n}^{\infty} \frac{1}{kP_k} = O\left(\frac{\log(n+2)}{\{\log(n+2)\}^2}\right) = O\left(\frac{\lambda_n^{(2)}}{P_n}\right) .$$

Therefore, by our theorem, we see that Corollary 5 holds.

By a similar method, we can prove the following corollary.

COROLLARY 6. *If*

$$\int_0^{\pi} \left\{\log \log \frac{C}{t}\right\} |d\varphi(t)| < \infty ,$$

then the series

$$\sum_{n=0}^{\infty} A_n(t)/\log(n+2) \quad \textit{is summable} \quad |N, \log \log(n+2)/(n+2)\log(n+2)| ,$$

at $t = x$.

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