J. Aust. Math. Soc. **87** (2009), 421–428
doi:10.1017/S1446788709000135

INVERSE SPECTRAL PROBLEMS FOR SINGULAR RANK-ONE PERTURBATIONS OF A HILL OPERATOR

KAZUSHI YOSHITOMI

(Received 2 August 2008; accepted 21 December 2008)

Communicated by J. J. Koliha

Abstract

We investigate an inverse spectral problem for the singular rank-one perturbations of a Hill operator. We give a necessary and sufficient condition for a real sequence to be the spectrum of a singular rank-one perturbation of the Hill operator.

2000 *Mathematics subject classification*: primary 34A55; secondary 47A55, 47A75.

Keywords and phrases: Hill operator, $\mathcal{H}_{-2}$-perturbation, inverse spectral problem.

1. Introduction

In this paper we discuss the spectrum of a singular rank-one perturbation of the Hill operator

$$A = -\frac{d^2}{dx^2} \quad \text{in } \mathcal{H} = L^2((0, 2\pi)),$$
$$\mathrm{Dom}(A) = \left\{ u \in H^2((0, 2\pi)) \,\middle|\, u(2\pi-) = u(0+),\ \frac{d}{dx}u(2\pi-) = \frac{d}{dx}u(0+) \right\}.$$

We recall the definition of a singular rank-one perturbation from the textbook of Albeverio and Kurasov [5, Ch. 1]. Let $\langle \cdot, \cdot \rangle$ stand for the inner product in $\mathcal{H}$, and $\| \cdot \|_{\mathcal{H}}$ for the norm in $\mathcal{H}$. For $s \geq 0$, let $\mathcal{H}_s(A)$ be the Hilbert space $\mathrm{Dom}(A^{s/2})$ equipped with the norm $\|\psi\|_s = \|(A+1)^{s/2}\psi\|_{\mathcal{H}}$. We designate by $\mathcal{H}_{-s}(A)$ the dual space of $\mathcal{H}_s(A)$. By $\langle \cdot, \cdot \rangle_{-s,s}$ we denote the dual coupling of $\mathcal{H}_{-s}(A)$ and $\mathcal{H}_s(A)$. Let φ be a vector in $\mathcal{H}_{-2}(A) \setminus \mathcal{H}$ satisfying the normalization condition

$$\|(A-i)^{-1}\varphi\|_{\mathcal{H}} = 1.$$

This work has been partially supported by Grant-in-Aid for Scientific Research (No. 20540182), Japan Society for the Promotion of Science.

The restriction A^0 of the operator A to the space

$$\{\psi \in \mathrm{Dom}(A) \mid \langle \varphi, \psi \rangle_{-2,2} = 0\}$$

is a densely defined symmetric operator with deficiency indices (1, 1), and the deficiency subspace $\mathrm{Ker}(A^{0^*} \mp i)$ is spanned by the vector $g_{\pm i} \equiv (A \mp i)^{-1}\varphi$. For $\gamma \in \mathbb{R}$, let $m(\gamma) = (\gamma + i)/(\gamma - i)$. Since $|m(\gamma)| = 1$, the von Neumann theory implies that there is a unique selfadjoint extension A_γ of A^0 for which

$$\mathrm{Dom}(A_\gamma) = \{u - c\, m(\gamma) g_i + c g_{-i} \mid u \in \mathrm{Dom}(A^0),\ c \in \mathbb{C}\}.$$

The map $\gamma \mapsto A_\gamma$ is a bijection from $\mathbb{R}$ onto the set of all selfadjoint extensions of A^0 except A. The operator A_γ is called a singular rank-one perturbation of A. If $\varphi \notin \mathcal{H}_{-1}(A)$, we say that the perturbation is form unbounded; otherwise we say that it is form bounded. In the form bounded case, the operator A_γ admits the representation $A + \alpha\langle \varphi, \cdot \rangle_{-1,1}\varphi$, where $\alpha \equiv (-\gamma - \langle \varphi, A(A^2 + 1)^{-1}\varphi \rangle_{-1,1})^{-1}$; see [5, Theorem 1.3.1]. In the form unbounded case, the operator A_γ possesses a similar representation; see [5, Theorem 1.3.2]. It is also useful to recall that the resolvent of A_γ is given by the Krein formula

$$\frac{1}{A_\gamma - z} = \frac{1}{A - z} - \frac{1}{-\gamma + \langle \varphi, \frac{1+zA}{A-z} \frac{1}{A^2+1} \varphi \rangle_{-2,2}} \left\langle \frac{1}{A - \bar{z}} \varphi, \cdot \right\rangle \frac{1}{A - z} \varphi, \tag{1.1}$$

where $z \in \rho(A_\gamma) \cap \rho(A)$.

The purpose of this paper is to investigate an inverse spectral problem for the singular rank-one perturbations of A. To this end, let us first describe basic spectral properties of A_γ. We introduce some notation. For $j \in \mathbb{N}$, we denote by $\mu_j \equiv \mu_j(\varphi, \gamma)$ the jth eigenvalue of A_γ counted with multiplicity. For $j \in \mathbb{N}$, we define

$$g_j = \mu_{2j+1} - \mu_{2j},$$

which we call the jth gap of $\sigma(A_\gamma)$. Let $\psi_j(x) = (2\pi)^{-1/2} e^{ijx}$. It is readily seen that $\{\psi_j\}_{j=-\infty}^{\infty}$ is a complete orthonormal system for $\mathcal{H}$, and $A\psi_j = j^2\psi_j$. We define the Fourier coefficients of φ as $\alpha_j = \langle \varphi, \psi_j \rangle_{-2,2}$ for $j \in \mathbb{Z}$. We also define

$$\beta_j = \beta_j(\varphi) = \frac{1}{j^4 + 1}(|\alpha_j|^2 + |\alpha_{-j}|^2) \quad \text{for } j \in \mathbb{N} \quad \text{and} \quad \beta_0 = \beta_0(\varphi) = |\alpha_0|^2.$$

Since $\|(A - i)^{-1}\varphi\|_{\mathcal{H}} = 1$, we have $\sum_{j=0}^{\infty} \beta_j = 1$. Let $K = -\gamma - \sum_{j=0}^{\infty} j^2\beta_j$. We note that $K = -\infty$ if and only if $\varphi \notin \mathcal{H}_{-1}(A)$. For the sake of simplicity, we henceforth consider the case where $\beta_j \neq 0$ for every $j \geq 0$. We have the following implication concerning basic spectral properties of A_γ, which we demonstrate in Section 2.

PROPOSITION 1.1. *If $\beta_j \neq 0$ for every $j \geq 0$, then the following statements hold true.*

(i) *If $K < 0$, then $\mu_1 < 0$, $(j - 1)^2 < \mu_{2j} < j^2$ for every $j \in \mathbb{N}$, and $\mu_{2j+1} = j^2$ for every $j \in \mathbb{N}$.*

https://doi.org/10.1017/S1446788709000135 Published online by Cambridge University Press

(ii) *If* $K \geq 0$*, then* $(j-1)^2 < \mu_{2j-1} < j^2$ *for every* $j \in \mathbb{N}$*, and* $\mu_{2j} = j^2$ *for every* $j \in \mathbb{N}$.

(iii) *The vector* φ *belongs to* $\mathcal{H}_{-1}(A)$ *if and only if* $\sum_{j=1}^{\infty} j^{-2} g_j < \infty$.

We note that j^2 is a simple eigenvalue of A_γ for $j = 1, 2, \ldots$, while j^2 is a doubly degenerate eigenvalue of A for $j = 1, 2, \ldots$.

We now concentrate our attention on the form unbounded perturbations. Our main result is stated as follows and proved in Section 3.

THEOREM 1.2. *Let* $\{\tau_j\}_{j=1}^{\infty}$ *be a real sequence satisfying* $\tau_j < (j-1)^2 < \tau_{j+1}$ *for every* $j \in \mathbb{N}$*. In order for there to exist a vector* φ *in* $\mathcal{H}_{-2}(A) \setminus \mathcal{H}_{-1}(A)$ *and a real number* γ *such that* $\|(A-i)^{-1}\varphi\|_{\mathcal{H}} = 1$*,* $\beta_j(\varphi) \neq 0$ *for every* $j \geq 0$*,* $\mu_1(\varphi, \gamma) = \tau_1$*, and* $\mu_{2j}(\varphi, \gamma) = \tau_{j+1}$ *for every* $j \in \mathbb{N}$*, it is necessary and sufficient that*

$$\sum_{j=1}^{\infty} \frac{\tau_{j+1} - (j-1)^2}{j^2} = \infty \quad \textit{and} \quad \sum_{j=1}^{\infty} \frac{j^2 - \tau_{j+1}}{j^2} = \infty.$$

We describe the background to our work here. There are numerous works concerning inverse spectral problems for the Hill operators with locally integrable coefficients; we refer to [6, 8, 9, 11, 18–20, 23] and the references therein. We also mention that the spectrum of the Hill operator is determined by the spectra of the periodic and antiperiodic problems; see [22, Theorem XIII.90]. Recently, the Hill operators with distributional coefficients have attracted much attention. Kappeler and Möhr [12] and Korotyaev [14] discuss the Hill operators whose coefficients belong to a Sobolev space of order -1. These remarkable works led to our interest in an inverse spectral problem for the Hill operators with more singular coefficients. One of the ways to realize such an operator is to employ the $\mathcal{H}_{-2}$-perturbation theory. We note that there are also a vast number of works on the $\mathcal{H}_{-2}$-perturbation theory; we refer to [2–5, 10, 13, 16, 21] and the references therein. The theory of $\mathcal{H}_{-2}$-perturbations is closely related to that of quantum Hamiltonians with singular interactions. Indeed, the one-dimensional Schrödinger operators with point interactions—especially δ- and δ'-interactions—have been vigorously studied by means of the $\mathcal{H}_{-2}$-perturbation theory; see [2, 5, 16] and the references therein. Our work here is motivated by this background.

In Section 4 we recall several known results on the Hill operator with the δ'-interaction for the sake of comparison.

It is worth mentioning that the proof of the results in this paper is fairly simple. Our work here is accessible to readers with a knowledge of elementary function theory and basic functional analysis; one can follow the proof without preliminary knowledge of the inverse Sturm–Liouville theory.

2. Proof of Proposition 1.1

We first give a momentum representation of (1.1). We have

$$
\begin{aligned}
F(z) &\equiv -\gamma + \left\langle \varphi, \frac{1+zA}{A-z}\frac{1}{A^2+1}\varphi \right\rangle_{-2,2} \\
&= -\gamma + \sum_{j=0}^{\infty} \beta_j \frac{1+j^2 z}{j^2-z}.
\end{aligned} \tag{2.1}
$$

We define $T : \mathcal{H} \to \mathcal{H}$ by $T = \langle (A-\bar{z})^{-1}\varphi, \cdot\rangle (A-z)^{-1}\varphi$. Then

$$
Tu = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{\alpha_m \overline{\alpha_n}}{(m^2-z)(n^2-z)} \langle \psi_m, u\rangle \psi_n. \tag{2.2}
$$

The Krein resolvent formula (1.1) is now written as

$$
\frac{1}{A_\gamma - z} = \frac{1}{A-z} - \frac{1}{F(z)} T. \tag{2.3}
$$

Proof of assertions (i) and (ii). First, we prove the following two claims.

(a) For $j = 1, 2, \ldots$, the number j^2 is a simple eigenvalue of A_γ.
(b) $0 \in \rho(A_\gamma)$.

Let $E(\cdot)$ be the projection-valued measure associated with A_γ. Let j be a positive integer. We infer from (2.1), (2.2), and (2.3) that

$$
\begin{aligned}
E((j^2-\epsilon, j^2+\epsilon)) &= -\frac{1}{2\pi i} \int_{|z-j^2|=\epsilon} \frac{1}{A_\gamma - z}\, dz \\
&= -\frac{1}{|\alpha_j|^2 + |\alpha_{-j}|^2} [-|\alpha_{-j}|^2 \langle \psi_j, \cdot\rangle \psi_j + \alpha_j \overline{\alpha_{-j}} \langle \psi_j, \cdot\rangle \psi_{-j} \\
&\qquad + \alpha_{-j}\overline{\alpha_j} \langle \psi_{-j}, \cdot\rangle \psi_j - |\alpha_j|^2 \langle \psi_{-j}, \cdot\rangle \psi_{-j}],
\end{aligned}
$$

provided ϵ is a sufficiently small positive number. Since $\beta_j \neq 0$ by assumption, we get $\dim \mathrm{Ran}(E((j^2-\epsilon, j^2+\epsilon))) = 1$ when ϵ is sufficiently small and positive. Thus, implication (a) follows. A similar argument also gives claim (b).

Next, we prove the following three claims.

(c) For $j = 0, 1, \ldots$, the operator A_γ admits a unique eigenvalue in $(j^2, (j+1)^2)$ counted with multiplicity.
(d) If $K \geq 0$, then A_γ has no eigenvalue in $(-\infty, 0)$.
(e) If $K < 0$, then A_γ has a unique eigenvalue in $(-\infty, 0)$ counted with multiplicity.

Let j be a nonnegative integer. Since $F'(x) > 0$ on $(j^2, (j+1)^2)$, we see that $F(x) \to -\infty$ as $x \to j^2+$, and since $F(x) \to \infty$ as $x \to (j+1)^2-$, we infer that $F(x)$ admits a unique zero in $(j^2, (j+1)^2)$ counted with multiplicity. Combining this with (2.3) and the fact that T is a rank-one operator, we obtain $\dim \operatorname{Ran}(E((j^2, (j+1)^2))) = 1$, which is assertion (c). By the monotone convergence theorem, $\lim_{x\to-\infty} F(x) = K$. This, together with $\lim_{x\to 0-} F(x) = \infty$, $F'(x) > 0$ for every $x < 0$, and formula (2.3), yields implications (d) and (e). By (a)–(e) we obtain (i) and (ii). □

PROOF OF ASSERTION (iii). First, we prove the assertion in the case where $K < 0$. We recall the formula

$$\operatorname{Tr}\left(\frac{1}{A-z} - \frac{1}{A_\gamma - z}\right) = \frac{d}{dz} \log F(z), \tag{2.4}$$

where $z \in \rho(A_\gamma) \cap \rho(A)$ and the branch of the logarithm can be fixed arbitrarily. Let $t = 2\mu_1$. We utilize assertion (i) and integrate (2.4) on $(-\infty, t]$ to obtain

$$\int_{-\infty}^{t} \left(-\frac{1}{x} - \frac{1}{\mu_1 - x}\right) dx + \sum_{j=1}^{\infty} \log\left(1 - \frac{g_j}{j^2 - t}\right) = \lim_{x\to-\infty} \log\left(\frac{F(t)}{F(x)}\right),$$

where we employ the principal branch of the logarithm here and in what follows. So $\sum_{j=1}^{\infty} g_j/(j^2 - t) < \infty$ if and only if $\lim_{x\to-\infty} F(x) > -\infty$. We thus have (iii) in the case where $K < 0$. Next, we consider the case where $K \geq 0$. As was noted in the Introduction, $\varphi \in \mathcal{H}_{-1}(A)$. In addition, an argument similar to that above gives $\sum_{j=1}^{\infty} j^{-2} g_j < \infty$. □

3. Proof of Theorem 1.2

First, we prove that the condition is sufficient. We consider the function

$$G(z) = -\frac{\prod_{j=1}^{\infty}(1 - z/\tau_j)}{z\prod_{j=1}^{\infty}(1 - z/j^2)}.$$

Let us show that

$$G(t) = o(-t) \quad \text{as } t \to -\infty. \tag{3.1}$$

To this end we introduce the function

$$H(z) = \frac{\prod_{j=2}^{\infty}(1 - z/\tau_{j+1})}{\prod_{j=2}^{\infty}(1 - z/(j-1)^2)}.$$

We put $k = 2\tau_1$ and $s_j = \tau_{j+1} - (j-1)^2$. For $t < k$,

$$\log H(t) - \log H(k) = \sum_{j=2}^{\infty}\left[\log\left(1 + \frac{s_j}{(j-1)^2 - t}\right) - \log\left(1 + \frac{s_j}{(j-1)^2 - k}\right)\right].$$

https://doi.org/10.1017/S1446788709000135 Published online by Cambridge University Press

Thus,

$$\lim_{t\to-\infty}\log\frac{H(t)}{H(k)}=-\sum_{j=2}^{\infty}\log\left(1+\frac{s_j}{(j-1)^2-k}\right)=-\infty,$$

where we have used the assumption $\sum_{j=1}^{\infty} j^{-2}s_j=\infty$ and the monotone convergence theorem. Therefore, we obtain $H(t)\to 0$ as $t\to-\infty$, from which (3.1) follows.

On the other hand, we infer from [17, Ch. VII, Section 1, Theorem 1] that $\operatorname{Im} G(z)>0$ whenever $\operatorname{Im} z>0$, since $\tau_j<(j-1)^2<\tau_{j+1}$ for $j=1,2,\ldots$. This, together with the Nevanlinna theorem (see [1, Ch. VI, Section 59, Theorem 2]), implies that the function G admits the representation

$$G(z)=az+b+\int_{-\infty}^{\infty}\frac{1+wz}{w-z}\,d\sigma(w), \tag{3.2}$$

where a is a nonnegative constant, b is a real constant, and $\sigma(w)$ is a real-valued function which is nondecreasing and continuous from the right. Now

$$\sigma(w_2)-\sigma(w_1)=\lim_{y\to0+}\frac{1}{\pi}\int_{w_1}^{w_2}\frac{\operatorname{Im} G(t+iy)}{1+t^2}\,dt=0,$$

provided that $\sigma(w)$ is continuous at $w=w_1,w_2$ and the closed interval between w_1 and w_2 contains no pole of G. Therefore, $\sigma'=0$ on $\mathbb{R}\setminus\{j^2\mid j=0,1,\ldots\}$. Since σ is of bounded variation, we obtain

$$\int_{-\infty}^{\infty}\frac{1+wx}{w-x}\,d\sigma(w)=o(-x)\quad\text{as } x\to-\infty,$$

so that $a=0$ by (3.1) and (3.2). Inasmuch as $S\equiv\sum_{j=1}^{\infty}(\sigma(j^2)-\sigma(j^2-))<\infty$, there exists a $\varphi\in\mathcal{H}_{-2}(A)$ such that $\beta_j=(\sigma(j^2)-\sigma(j^2-))/S$ for $j=0,1,\ldots$. We put $\gamma=-b/S$. By (3.2) and $a=0$,

$$F(z)=\frac{1}{S}G(z). \tag{3.3}$$

Let us show that $\varphi\notin\mathcal{H}_{-1}(A)$. Let $l_j=j^2-\tau_{j+1}$ for $j\geq1$. For $t<k$,

$$\begin{aligned}&\log(-G(t))-\log(-G(k))\\&\quad=\int_k^t\left(-\frac{1}{x}+\frac{1}{x-\tau_1}\right)dx+\sum_{j=1}^{\infty}\left[\log\left(1-\frac{l_j}{j^2-t}\right)-\log\left(1-\frac{l_j}{j^2-k}\right)\right].\end{aligned}$$

As $\sum_{j=1}^{\infty} j^{-2}l_j=\infty$ by assumption, $G(t)\to-\infty$ as $t\to-\infty$, which, with (3.3), yields $\varphi\notin\mathcal{H}_{-1}(A)$. It follows by (3.3) that $\mu_1(\varphi,\gamma)=\tau_1$ and $\mu_{2j}(\varphi,\gamma)=\tau_{j+1}$ for every $j\in\mathbb{N}$. Thus, the sufficiency part of the theorem holds.

Next, we show that the condition is necessary. By (2.1), $\operatorname{Im} F(z) > 0$ whenever $\operatorname{Im} z > 0$; from this and [17, Ch. VII, Section 1, Theorem 1] we deduce that there is a positive constant k for which

$$F(z) = -k \cdot \frac{(1 - z/\mu_1)\prod_{j=1}^{\infty}(1 - z/\mu_{2j})}{z\prod_{j=1}^{\infty}(1 - z/j^2)}.$$

On the other hand, we infer from (2.1) and $\sum_{j=0}^{\infty} \beta_j < \infty$ that

$$F(x) = o(-x) \quad \text{as } x \to -\infty.$$

Therefore, we obtain $\sum_{j=1}^{\infty} j^{-2}(\mu_{2j} - (j-1)^2) = \infty$ by an analogous argument to that above. This completes the proof of Theorem 1.2.

4. Remarks

By L we designate the operator $-d^2/dx^2$ in $\mathcal{H}$ subject to the transmission condition

$$\frac{d}{dx}u(\pi-) = \frac{d}{dx}u(\pi+), \quad u(\pi+) - u(\pi-) = \beta\frac{d}{dx}u(\pi-)$$

as well as the periodic boundary condition

$$u(2\pi-) = u(0+), \quad \frac{d}{dx}u(2\pi-) = \frac{d}{dx}u(0+),$$

where $\beta \in \mathbb{R} \setminus \{0\}$ is a parameter. Let δ_π stand for the Dirac delta function supported at the point π. We put $C_0 = 2\sum_{j=1}^{\infty} j^2(1 + j^4)^{-1}$ and $C_1 = 2\sum_{j=1}^{\infty}(1 + j^4)^{-1}$. We note that if

$$\varphi = \sqrt{\frac{2\pi}{C_0}}\frac{d}{dx}\delta_\pi \quad \text{and} \quad \gamma = \frac{1}{C_0}\left(\frac{2\pi}{\beta} + 1 + C_1\right),$$

then A_γ is equal to L; see [2, Ch. III.3, Formula (3.58)]. We also note that $(d/dx)\delta_\pi \in \mathcal{H}_s(A)$ for every $s < -3/2$, whereas $(d/dx)\delta_\pi \notin \mathcal{H}_{-3/2}(A)$. So the operator L is one of the form unbounded singular rank-one perturbations of A. Let us recall several known results on the spectrum of L. For $j \in \mathbb{N}$, let ν_j stand for the jth eigenvalue of L counted with multiplicity. Then $\nu_{2j-1} = (j-1)^2$, $j = 1, 2, \ldots,$ for $\beta > 0$, while $\nu_{2j+1} = j^2$, $j = 1, 2, \ldots,$ for $\beta < 0$. Furthermore,

$$\nu_{2j+1} - \nu_{2j} = j + O(1)$$

as $j \to \infty$; see [2, Ch. III.3, Theorem 3.6].

For results on the spectrum of the Hill operator with a more general point interaction, see [7, 15].

Acknowledgement

The author thanks the referee for valuable comments which improved the paper.

https://doi.org/10.1017/S1446788709000135 Published online by Cambridge University Press

References

[1] N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space* (Dover Publications, New York, 1993).

[2] S. Albeverio, F. Gesztesy, R. Høegh-Krohn and H. Holden, *Solvable Models in Quantum Mechanics*, 2nd edn, With an appendix by Pavel Exner (AMS Chelsea Publishing, Providence, RI, 2005).

[3] S. Albeverio and P. Kurasov, 'Rank one perturbations, approximations, and selfadjoint extensions', *J. Funct. Anal.* **148** (1997), 152–169.

[4] ———, 'Rank one perturbations of not semibounded operators', *Integral Equations Operator Theory* **27** (1997), 379–400.

[5] ———, *Singular Perturbations of Differential Operators. Solvable Schrödinger Type Operators*, London Mathematical Society Lecture Note Series, 271 (Cambridge University Press, Cambridge, 2000).

[6] G. Borg, 'Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte', *Acta Math.* **78** (1946), 1–96.

[7] P. Exner and H. Grosse, 'Some properties of the one-dimensional generalized point interactions (a torso)', arXiv:math-ph/9910029, 1999.

[8] J. Garnett and E. Trubowitz, 'Gaps and bands of one-dimensional periodic Schrödinger operators', *Comment. Math. Helv.* **59** (1984), 258–312.

[9] ———, 'Gaps and bands of one-dimensional periodic Schrödinger operators II', *Comment. Math. Helv.* **62** (1987), 18–37.

[10] S. Hassi and H. de Snoo, 'On rank one perturbation of selfadjoint operators', *Integral Equations Operator Theory* **29** (1997), 288–300.

[11] H. Hochstadt, 'On the determination of a Hill's equation from its spectrum', *Arch. Rational Mech. Anal.* **19** (1965), 353–362.

[12] T. Kappeler and C. Möhr, 'Estimates for periodic and Dirichlet eigenvalues of the Schrödinger operator with singular potentials', *J. Funct. Anal.* **186** (2001), 62–91.

[13] A. Kiselev and B. Simon, 'Rank one perturbations with infinitesimal coupling', *J. Funct. Anal.* **130** (1995), 345–356.

[14] E. Korotyaev, 'Characterization of the spectrum of Schrödinger operators with periodic distributions', *Int. Math. Res. Not.* **37** (2003), 2019–2031.

[15] P. Kurasov and J. Larson, 'Spectral asymptotics for Schrödinger operators with periodic point interactions', *J. Math. Anal. Appl.* **266** (2002), 127–148.

[16] S. T. Kuroda and H. Nagatani, 'Resolvent formulas of general type and its application to point interactions', *J. Evol. Equ.* **1** (2001), 421–440.

[17] B. Ja. Levin, *Distribution of Zeros of Entire Functions* (American Mathematical Society, Providence, RI, 1980).

[18] B. M. Levitan, *Inverse Sturm–Liouville Problems* (VNU Science Press, Utrecht, 1987).

[19] W. Magnus and S. Winkler, *Hill's Equation* (Dover Publications, New York, 2004).

[20] V. A. Marchenko and I. V. Ostrovskii, 'A characterization of the spectrum of the Hill operator', *Math. USSR-Sb.* **26** (1975), 493–554.

[21] A. Posilicano, 'A Krein-like formula for singular perturbations of self-adjoint operators and applications', *J. Funct. Anal.* **183** (2001), 109–147.

[22] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV: Analysis of Operators* (Academic Press, San Diego, CA, 1978).

[23] P. Ungar, 'Stable Hill equations', *Comm. Pure Appl. Math.* **14** (1961), 707–710.

KAZUSHI YOSHITOMI, Department of Mathematics and Information Sciences,
Tokyo Metropolitan University, Minami Ohsawa 1-1, Hachioji,
Tokyo 192-0397, Japan
e-mail: yositomi@tmu.ac.jp

https://doi.org/10.1017/S1446788709000135 Published online by Cambridge University Press