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Abstract

Building on coprincipal mesoprimary decomposition [Kahle and Miller, *Decompositions of commutative monoid congruences and binomial ideals*, Algebra and Number Theory **8** (2014), 1297–1364], we combinatorially construct an irreducible decomposition of any given binomial ideal. In a parallel manner, for congruences in commutative monoids we construct decompositions that are direct combinatorial analogues of binomial irreducible decompositions, and for binomial ideals we construct decompositions into ideals that are as irreducible as possible while remaining binomial. We provide an example of a binomial ideal that is not an intersection of irreducible binomial ideals, thus answering a question of Eisenbud and Sturmfels [*Binomial ideals*, Duke Math. J. **84** (1996), 1–45].

1. Introduction

An ideal in a commutative ring is irreducible if it is not expressible as an intersection of two ideals properly containing it. Irreducible ideals are primary, and any ideal I in a Noetherian ring is an intersection of irreducible ideals. These *irreducible decompositions* are thus special cases of primary decomposition, but likewise are hard to compute in general. If I is a monomial ideal, however, this task is much easier: any monomial ideal is an intersection of irreducible ideals that are themselves monomial ideals (see [MS05, Theorem 5.27] for polynomial rings and [Mil02, Theorem 2.4] for affine semigroup rings), and these *monomial irreducible decompositions* are heavily governed by combinatorics. The ease of monomial irreducible decomposition plus the existence of binomial primary decomposition in polynomial rings over algebraically closed fields [ES96, Theorem 7.1] motivated Eisenbud and Sturmfels to ask the following question.

Question 1.1 [ES96, Problem 7.5]. Does every binomial ideal over an algebraically closed field admit a binomial irreducible decomposition?

We answer Question 1.1 using the theory of mesoprimary decomposition [KM14]. Our response has three stages. First, congruences in Noetherian commutative monoids admit *soccular decompositions* (Theorem 4.2), which should be considered the direct combinatorial analogues of binomial irreducible decompositions. (*Soccular congruences* (Definition 3.2) fail to be irreducible for the same reason that prime congruences do; see the end of [KM14, § 2] for details.) Second, lifting to binomial ideals the method of constructing soccular congruences (but not lifting the construction itself; see Example 5.2) yields ideals that are, in a precise sense, as irreducible as possible while remaining binomial (Definition 5.1). The resulting notion

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of *binoccular decomposition* for binomial ideals (Theorem 5.7) proceeds as far as possible toward irreducible decomposition while remaining confined to the category of binomial ideals. Theorem 6.4 demonstrates, by example, that the confines of binomiality can prevent reaching all the way to irreducible decomposition by exhibiting a binomial ideal not expressible as an intersection of binomial irreducible ideals, thus solving Eisenbud and Sturmfels' problem in the negative. That said, our third and final stage produces irreducible decompositions of binomial ideals (Corollary 7.8) in a manner that is as combinatorial as mesoprimary decomposition: each coprincipal component has a canonical *irreducible closure* (Definition 7.1) that, while not itself an irreducible ideal, has a canonical primary decomposition all of whose components are irreducible (Theorem 7.5).

All three of the decompositions in this paper—soccular, binoccular, and irreducible—descend directly from coprincipal decomposition [KM14, Theorems 8.4 and 13.3] (see Theorems 2.6 and 2.11 for restatements of these results). This is true in two senses: (i) the components in all three types of decomposition are cogenerated by the same witnesses that cogenerate the corresponding coprincipal components; and (ii) the components themselves are constructed by adding new relations to the corresponding coprincipal components. To be more precise, soccular congruences are constructed by adding relations between all pairs of *protected witnesses* (Definition 3.11) for coprincipal congruences while maintaining their cogenerators (Theorem 3.14 and Corollary 3.15). Similarly, binoccular ideals are constructed by repeatedly throwing into a coprincipal ideal as many binomial socle elements as possible while maintaining a monomial cogenerator in the socle (Definitions 5.3 and 5.5). In contrast, irreducible closures allow arbitrary polynomials to be thrown in, not merely binomials. Although this concrete description of irreducible closure is accurate, the construction of irreducible closures (Definition 7.1) is accomplished with more abstract, general commutative algebra. Consequently, the reason why irreducible closures have canonical irreducible decompositions is particularly general, from the standpoint of commutative algebra, involving embeddings of rings inside of Gorenstein localizations (Remark 7.6).

Finally, it bears mentioning that for the proofs of correctness—at least for the decompositions in rings as opposed to monoids—we make explicit a unifying principle, in the form of equivalent criteria involving socles and monomial localization (Lemma 5.6), for when a binomial ideal in a monoid algebra equals a given intersection of ideals.

Note on prerequisites

Although the developments here are based on those in [KM14], the reader is not assumed to have assimilated the results there. The exposition here assumes familiarity only with the most basic monoid theory used in [KM14]. To make this paper self-contained, every result from [KM14] that is applied here is stated precisely in § 2 with prerequisite definitions. In fact, § 2 serves as a handy summary of [KM14], proceeding through most of its logical content as efficiently as possible.

2. Preliminaries

We need to briefly review some definitions and results from [KM14]. Following that paper, we first deal with monoid congruences (the combinatorial setting) and then the respective binomial ideal counterparts (the arithmetic setting). Throughout, let Q denote a commutative Noetherian monoid and $\Bbbk$ a field. We assume familiarity with basic notions from monoid theory; see §§ 2 and 3 of [KM14], which contain an introduction to the salient points with binomial algebra in mind.

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For an example of the kinds of concepts we assume, an element $q \in Q$ is *partly cancellative* if $q+a = q+b \neq \infty \Rightarrow a = b$ for all cancellative $a, b \in Q$, where $\infty \in Q$ is nil [KM14, Definition 2.9].

Definition 2.1. An equivalence relation $\sim$ on Q is a *congruence* if $a \sim b$ implies $a + c \sim b + c$ for all $a, b, c \in Q$. A *binomial* in $\Bbbk[Q]$ is an element of the form $\mathbf{t}^a - \lambda \mathbf{t}^b$ where $a, b \in Q$ and $\lambda \in \Bbbk$. An ideal $I \subset \Bbbk[Q]$ is *binomial* (respectively *monomial*) if it can be generated by binomials (respectively monomials).

Remark 2.2. A binomial ideal $I \subset \Bbbk[Q]$ induces a congruence $\sim_I$ on Q that sets $a \sim_I b$ whenever $\mathbf{t}^a - \lambda \mathbf{t}^b \in I$ for some nonzero $\lambda \in \Bbbk$. The quotient algebra $\Bbbk[Q]/I$ is finely graded by the quotient monoid $Q/\sim_I$. Conversely, each congruence on Q is of the form $\sim_I$ for some binomial ideal $I \subset \Bbbk[Q]$, although more than one I is possible: the nil class can be zero or not [KM14, Proposition 9.5], and the congruence forgets coefficients.

Definition 2.3 [KM14, Definitions 2.12, 3.4, 4.7, 4.10, 7.1, 7.2, 7.7, and 7.12]. Fix a congruence $\sim$ on Q and a prime $P \subset Q$. Write $\overline{Q}_P = Q_P/\sim$, where Q_P is the localization along P, and denote by $\overline{q}$ the image of $q \in Q$ in $\overline{Q} = Q/\sim$.

(i) An element $q \in Q$ is an *aide* for $w \in Q$ and a generator $p \in P$ if $\overline{w} \neq \overline{q}$, and $\overline{w} + \overline{p} = \overline{q} + \overline{p}$, and $\overline{q}$ is maximal in the set $\{\overline{q}, \overline{w}\}$. The element q is a *key aide* for w if q is an aide for w for each generator of P. An element $w \in Q$ is a *witness* for P if it has an aide for each $p \in P$, and a *key witness* for P if it has a key aide. A key witness w is a *cogenerator* of $\sim$ if $w + p$ is nil modulo $\sim$ for all $p \in P$.

(ii) The congruence $\sim$ is *P-primary* if every $p \in P$ is nilpotent in $\overline{Q}$ and every $f \in Q \smallsetminus P$ is cancellative in $\overline{Q}$. A P-primary congruence $\sim$ is *mesoprimary* if every element of the quotient $\overline{Q}$ is partly cancellative. The congruence $\sim$ is *coprincipal* if it is mesoprimary and every cogenerator for $\sim$ generates the same ideal in $\overline{Q}$.

(iii) The *coprincipal component $\sim_w^P$ of $\sim$ cogenerated by a witness $w \in Q$ for P* is the coprincipal congruence that relates $a, b \in Q$ if one of the following is satisfied:
- both $\overline{a}$ and $\overline{b}$ generate an ideal not containing $\overline{q}$ in $\overline{Q}_P$; or
- $\overline{a}$ and $\overline{b}$ differ by a unit in $\overline{Q}_P$ and $\overline{a} + \overline{c} = \overline{b} + \overline{c} = \overline{q}$ for some $\overline{c} \in \overline{Q}_P$.

A (key) witness for P may be called a (key) $\sim$-witness for P to specify $\sim$. Congruences may be called P-mesoprimary or P-coprincipal to specify P.

Definition 2.4 [KM14, Definitions 5.1 and 5.2]. Fix a congruence $\sim$ on a monoid Q, a prime ideal $P \subset Q$, and an element $q \in Q$ that is not nil modulo $\sim$.

(i) Let $G_P \subset Q_P$ denote the unit group of the localization Q_P, and write $K_q^P \subset G_P$ for the stabilizer of $\overline{q} \in \overline{Q}_P$ under the action of G_P.

(ii) If $\approx$ is the congruence on Q_P that sets $a \approx b$ whenever
- a and b lie in P_P or
- a and b lie in G_P and $a - b \in K_q^P$,

then the *P-prime congruence* of $\sim$ at q is $\ker(Q \to Q_P/\approx)$.

(iii) The P-prime congruence at q is *associated to* $\sim$ if q is a key witness for P.

Definition 2.5 [KM14, Definition 8.1]. An expression of a congruence $\sim$ on Q as a common refinement $\bigcap_i \approx_i$ of mesoprimary congruences is a *mesoprimary decomposition* of $\sim$ if, for each $\approx_i$ with associated prime $P_i \subset Q$, the P_i-prime congruences of $\sim$ and $\approx_i$ at each cogenerator for $\approx_i$ coincide. This decomposition is *key* if every cogenerator for every $\approx_i$ is a key witness for $\sim$.

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THEOREM 2.6 [KM14, Theorem 8.4]. *Each congruence $\sim$ on Q is the common refinement of the coprincipal components cogenerated by its key witnesses.*

The proof of Theorem 2.6 at the source [KM14, Theorem 8.4] yields the following corollary, which is necessary for Theorem 4.5.

COROLLARY 2.7. *Given a congruence $\sim$ on Q and elements $a, b \in Q$ with $a \not\sim b$, there exists a monoid prime $P \subset Q$ and an element $u \in Q$ such that (after possibly swapping a and b) the element $a + u$ is a key $\sim$-witness for P with key aide $b + u$.*

A few more definitions are required before a precise statement of the main existence result for binomial ideals from [KM14] can be made in Theorem 2.11.

DEFINITION 2.8 [KM14, Definitions 11.7, 11.11, and 12.1]. Let $I \subset \Bbbk[Q]$ be a binomial ideal. Fix a prime $P \subset Q$ and an element $q \in Q$ with $\mathbf{t}^q \notin I_P$.

- (i) Let $G_P \subset Q_P$ denote the unit group of Q_P, and write $K_q^P \subset G_P$ for the subgroup of G_P that fixes the class of q modulo $\sim_I$.
- (ii) Denote by $\rho : K_q^P \to \Bbbk^*$ the group homomorphism such that $\mathbf{t}^u - \rho(u)$ lies in the kernel of the $\Bbbk[G_P]$-module homomorphism $\Bbbk[G_P] \to \Bbbk[Q_P]/I_P$ taking $1 \mapsto \mathbf{t}^q$.
- (iii) The *P-mesoprime ideal of I at q* is the preimage I_q^P in $\Bbbk[Q]$ of $(I_q^P)_P = I_\rho + \mathfrak{m}_P$, where $I_\rho = \langle \mathbf{t}^u - \rho(u - v)\mathbf{t}^v \mid u - v \in K_q^P \rangle \subset \Bbbk[Q]_P$.
- (iv) An element $w \in Q$ is an *I-witness* for a monoid prime P if w is a $\sim_I$-witness for P or if $P = \emptyset$ is empty and I contains no monomials. $w \in Q$ is an *essential I-witness* if w is a key $\sim_I$-witness or some polynomial in $\Bbbk[Q_P]/I_P$ annihilated by $\mathfrak{m}_P$ has $\mathbf{t}^w$ minimal (under Green's preorder) among its nonzero monomials.
- (v) The mesoprime I_q^P is *associated to I* if q is an essential I-witness for P.

DEFINITION 2.9 [KM14, Definitions 10.4, 12.14, 12.18]. Fix a binomial ideal $I \subset \Bbbk[Q]$ and a prime $P \subset Q$.

- (i) The ideal I is *mesoprimary* (respectively *coprincipal*) if the congruence $\sim_I$ is mesoprimary (respectively coprincipal) and I is maximal among binomial ideals in $\Bbbk[Q]$ inducing this congruence.
- (ii) The *P-coprincipal component of I at $w \in Q$* is the preimage $W_w^P(I) \subset \Bbbk[Q]$ of the ideal $I_P + I_\rho + M_w^P(I) \subset \Bbbk[Q]_P$, where $M_w^P(I)$ is the ideal generated by the monomials $\mathbf{t}^u \in \Bbbk[Q]$ such that $w \notin \langle u \rangle \subset Q_P$.

DEFINITION 2.10 [KM14, Definition 13.1]. An expression $I = \bigcap_j I_j$ is a *mesoprimary decomposition* if each component I_j is P_j-mesoprimary and the P_j-mesoprimes of I and I_j at each cogenerator of I_j coincide. This decomposition is *combinatorial* if every cogenerator of every component is an essential I-witness. A mesoprimary decomposition is a *coprincipal decomposition* if every component is coprincipal.

THEOREM 2.11 [KM14, Theorem 13.3]. *Every binomial ideal $I \subset \Bbbk[Q]$ is the intersection of the coprincipal components cogenerated by its essential witnesses. In particular, every binomial ideal admits a combinatorial mesoprimary decomposition.*

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Theorem 2.11 produces a primary decomposition of any binomial ideal via the next result. Precise details about the primary components here can be found at the cited locations in [KM14].

Proposition 2.12 [KM14, Corollary 15.2 and Proposition 15.4]. *Fix a mesoprimary ideal $I \subset \Bbbk[Q]$. The associated primes of I are exactly the minimal primes of its associated mesoprime. Consequently, I admits a canonical minimal primary decomposition. When $\Bbbk = \overline{\Bbbk}$ is algebraically closed, every component of this decomposition is binomial.*

Theorem 2.13 [KM14, Theorems 15.6 and 15.12]. *Fix a binomial ideal $I \subseteq \Bbbk[Q]$. Each associated prime of I is minimal over some associated mesoprime of I. If $\Bbbk = \overline{\Bbbk}$ is algebraically closed, then refining any mesoprimary decomposition of I by canonical primary decomposition of its components yields a binomial primary decomposition of I.*

3. Soccular congruences

Although the condition to be a coprincipal quotient is strong, it does not imply that a binomial ideal inducing a coprincipal congruence has simple socle. Precisely, the socle of a coprincipal quotient has only one monomial up to units locally at the associated prime. While this suffices for irreducible decomposition of monomial ideals, modulo a binomial ideal the socle can have binomials and general polynomials. Our first step is soccular decomposition (Theorem 4.5), which parallels, at the level of congruences, the construction of irreducible decompositions of binomial ideals (Theorem 7.7). While it is the optimal construction in the combinatorial setting, soccular decomposition cannot yield irreducible decompositions of binomial ideals in general since these need not be binomial (Example 6.1). To start, here is a simple example of a primary coprincipal binomial ideal that is reducible, demonstrating that coprincipal decomposition of ideals is not irreducible decomposition.

Example 3.1. The congruence on $\mathbb{N}^2$ induced by the ideal $I = \langle x^2 - xy, xy - y^2, x^3 \rangle$ is coprincipal, but $x - y \in \mathrm{soc}_{\mathfrak{m}}(I)$ for $\mathfrak{m} = \langle x, y \rangle$. This is because x and y are both key witnesses and each is an aide for the other.

Definition 3.2. A congruence $\sim$ on Q is *soccular* if its key witnesses all generate the same principal ideal in the localized quotient $Q_P/{\sim}$.

Definition 3.3. Fix a monoid prime $P \subset Q$ and a P-coprincipal congruence $\sim$ on Q with cogenerator $w \in Q$. The *(first) soccular collapse of* $\sim$ is the congruence $\approx$ that sets $a \approx b$ if $a, b \notin \langle w \rangle$ and $a + p \sim b + p$ for all $p \in P$. The ith *soccular collapse of* $\sim$ is the soccular collapse of the $(i-1)$st soccular collapse of $\sim$.

Soccular collapses remove key witness pairs that are not Green's equivalent to the cogenerator of a coprincipal congruence. It is routine to check that the soccular collapse of a coprincipal congruence is a coprincipal congruence (see the following lemmas). The construction stabilizes since Q is a Noetherian monoid and consequently the iterated soccular collapse of a coprincipal congruence is a soccular congruence.

In general, to form a congruence from a set of relations, one takes monoid closure and then transitive closure. Lemma 3.4 says that for a soccular collapse of a coprincipal congruence, both of these operations are trivial.

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Lemma 3.4. *The soccular collapse of a P-coprincipal congruence $\sim$ is a congruence on Q that coarsens $\sim$.*

Proof. The soccular collapse $\approx$ is symmetric and transitive since $\sim$ is symmetric and transitive. Suppose $a, b \notin \langle w \rangle$ with $a + p \sim b + p$ for all $p \in P$. Then for all $q \in Q$, $a + q + p \sim b + q + p$ for all $p \in P$ since $q + p \in P$, so $a + q \approx b + q$. Therefore $\approx$ is a congruence on Q. Lastly, if $a \sim b$, then $a + p \sim b + p$ for all $p \in P$, so $\sim$ refines $\approx$. □

Lemma 3.5. *Resuming the notation from Definition 3.3, if $a \approx b$ and $a \not\sim b$, then neither a nor b is maximal in Q modulo Green's relation.*

Proof. Given Lemma 3.4, the definition of $\approx$ ensures that a and b both precede w modulo Green's relation, which ensures a and b are not maximal. □

Lemma 3.6 shows that taking the soccular collapse of a coprincipal congruence does not modify Green's classes.

Lemma 3.6. *Resuming the notation from Definition 3.3, if $a, b \in Q$ differ by a cancellative modulo $\sim$, then soccular collapse does not join them.*

Proof. Suppose $a \approx b$ and $a = b + f$ for some cancellative element f. For each $p \in P$, $a + p = b + p$ by Lemma 3.4, and each is non-nil by Lemma 3.5. Thus $b + f + p \sim b + p$, so $f = 0$ by the partly cancellative property of $b + p$ in Definition 2.3(ii). □

Proposition 3.7. *Fix a P-coprincipal congruence $\sim$ on Q with cogenerator w. The soccular collapse $\approx$ of $\sim$ is coprincipal with cogenerator w, and $\approx$ coarsens $\sim$. Moreover, the elements $a, b \in Q$ distinct under $\sim$ but identified under $\approx$ are precisely the key witnesses of $\sim$ lying outside the Green's class of w.*

Proof. The congruence $\approx$ coarsens $\sim$ by Lemma 3.4. As $\sim$ is mesoprimary, Lemma 3.6 ensures that $\approx$ is also mesoprimary, and by Lemma 3.5$\approx$ agrees with $\sim$ on the Green's class of w. The final claim follows upon observing that a and b are by definition key witnesses for $\sim$. □

Definition 3.8. Fix a P-coprincipal congruence $\sim$ on Q. Two distinct key witnesses $a, b \in Q$ for $\sim$ form a *key witness pair* if each is a key aide for the other.

Remark 3.9. If $a, b \in Q$ form a key witness pair under a coprincipal congruence $\sim$ and neither of them is Green's equivalent to the cogenerator w, then they are no longer a key witness pair under the soccular collapse $\approx$ of $\sim$ by Proposition 3.7. However, $\approx$ may still have key witnesses, as shown in Example 3.10.

Example 3.10. Let $I = \langle x^3 - x^2y, x^2y - xy^2, xy^3 - y^4, x^5 \rangle \subset \Bbbk[x, y]$. The congruence $\sim_I$ and its soccular collapse are shown in Figure 1. The monoid element xy is a key witness for $\sim_I$, where it is paired with y^2, as well as for the soccular collapse of $\sim_I$, where it is paired with x^2.

Definition 3.11. Fix a coprincipal congruence $\sim$ on Q with cogenerator w. An element $a \in Q$ is a *protected witness for* $\sim$ if it is a key witness for the ith soccular collapse of $\sim$ for some $i \geqslant 1$. Elements $a, b \in Q$ form a *protected witnesses pair* if they form a key witness pair for some iterated soccular collapse of $\sim$.

Definition 3.12. Fix a coprincipal congruence $\sim$ on Q. The *soccular closure* $\overline{\approx}$ of $\sim$ is the congruence refined by $\sim$ that additionally joins any a and b related under some soccular collapse of $\sim$.

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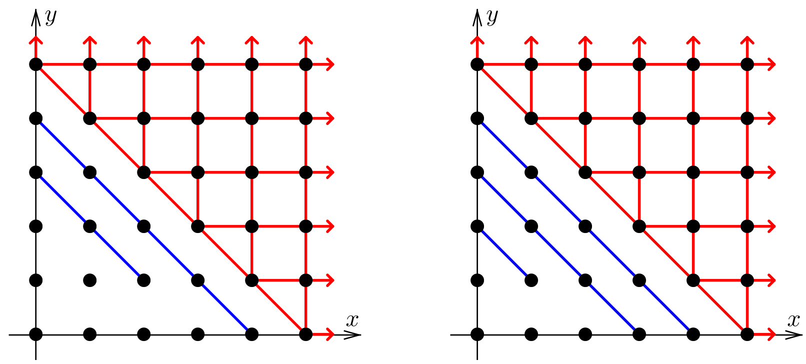

Figure 1. (Colour online) For $I = \langle x^3 - x^2y, x^2y - xy^2, xy^3 - y^4, x^5\rangle \subset \Bbbk[x, y]$, the congruence induced by I on $\mathbb{N}^2$ (left), and its soccular collapse (right). The monomial xy is a key witness for both $\sim_I$ and its soccular collapse.

Lemma 3.13. *Fix a coprincipal congruence $\sim$ on Q with cogenerator w. The soccular closure $\eqsim$ of $\sim$ is a soccular congruence, and its set of key witnesses is exactly the Green's class of w.*

Proof. By construction, the soccular closure has no key witnesses outside the Green's class of w. $\square$

We now characterize protected witnesses and give a non-iterative way to specify the soccular closure. To this end, let

$$(w :_{\sim} q) = \{p \in Q \mid \overline{q} + \overline{p} = \overline{w} \text{ in } Q_P/{\sim}\}.$$

Theorem 3.14. *Fix a P-coprincipal congruence $\sim$ on Q with cogenerator w, and write $\overline{Q} = Q/\sim$. Then $q, q' \in Q$ with distinct classes in $\overline{Q}$ are a protected witnesses pair for $\sim$ if and only if $(w :_{\sim} q) = (w :_{\sim} q')$.*

Proof. Let $\eqsim$ denote the soccular closure of $\sim$. Since passing to $\eqsim$ leaves the class of w under $\sim$ unchanged, $(w :_{\sim} q) = (w :_{\eqsim} q)$ for all $q \in Q$. Therefore, if q and q' are merged under $\eqsim$, the sets $(w :_{\sim} q)$ and $(w :_{\sim} q')$ coincide.

Now assume q and q' are not related under $\eqsim$. Pick an element $p \in P$ such that $q + p$ and $q' + p$ are distinct under $\eqsim$ and such that the image $\overline{p} \in \overline{Q}$ is maximal among images of elements in P with this property. Existence of p is guaranteed because $\sim$ is primary, whence $\overline{Q}_P$ has only finitely many Green's classes. Maximality of p implies that $q + p$ and $q' + p$ become merged in $\overline{Q}/\eqsim$ under the action of any element of P. Since $\eqsim$ has no key witness pairs, one of $q + p$ and $q' + p$ must be nil, and maximality of p implies the other is Green's equivalent to w. After possibly switching q and q', this gives $p \in (w :_{\sim} q)$ but $p \notin (w :_{\sim} q')$. $\square$

Corollary 3.15. *Fix a coprincipal congruence $\sim$ on Q cogenerated by w. The soccular closure $\eqsim$ of $\sim$ relates a and b if and only if $(w :_{\sim} a) = (w :_{\sim} b)$.* $\square$

4. Soccular decomposition of congruences

Every congruence can be expressed as a common refinement of soccular congruences. Our constructive proof first produces the decomposition in Corollary 4.3, which might not be a mesoprimary decomposition; see Remark 4.4. Theorem 4.5 removes unnecessary components and shows that the resulting decomposition is mesoprimary.

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Definition 4.1. Fix a P-coprincipal congruence $\sim$ on Q and a key witness $w \in Q$. The *soccular component* $\eqsim_w^P$ *of* $\sim$ *cogenerated by* w *along* P is the soccular closure of the coprincipal component $\sim_w^P$ cogenerated by w along P.

Theorem 4.2. *Any coprincipal congruence* $\sim$ *on* Q *is the common refinement of the soccular components cogenerated by its protected witnesses.*

Proof. Each soccular component coarsens $\sim$ by Lemma 3.4, so it suffices to show that their common refinement is $\sim$. Let $w \in Q$ denote a cogenerator of $\sim$ and fix distinct $a, b \in Q$. If the soccular component of $\sim$ at w (that is, the soccular closure of $\sim$) leaves a and b distinct, we are done. Otherwise, both a and b are protected witnesses, and the soccular component of $\sim$ at a joins b with the nil class. □

Corollary 4.3. *Any congruence* $\sim$ *on* Q *can be expressed as a common refinement of soccular congruences.*

Proof. Apply Theorem 2.6 to $\sim$, then Theorem 4.2 to each component. □

Remark 4.4. The decomposition in Corollary 4.3 is not necessarily a mesoprimary decomposition in the sense of Definition 2.5, since the associated prime congruence of a component $\approx$ cogenerated at a protected witness $q \in Q$ need not coincide with the prime congruence at q under $\sim$. The next theorem shows that the components in this decomposition cogenerated at protected witnesses that are not key $\sim$-witnesses are redundant, and the resulting decomposition is indeed a mesoprimary decomposition.

Theorem 4.5. *Any congruence* $\sim$ *is the common refinement of the soccular components cogenerated by its key witnesses.*

Proof. For elements $a, b \in Q$ with $a \not\sim b$, Corollary 2.7 produces, after possibly swapping a and b, a prime $P \subset Q$ and $u \in Q$ such that $a \not\sim_w^P b$ for a key witness $w = a + u$ with key aide $b + u$. Since $\eqsim_w^P$ has the same cogenerator and nil class as $\sim_w^P$, Corollary 3.15 ensures that $\eqsim_w^P$ does not relate a and b as well. □

5. Binoccular decomposition of binomial ideals

The binomial ideal analogue (Theorem 5.7) of soccular decomposition (Theorem 4.5) yields a decomposition into binoccular ideals (Definition 5.1), each of whose socles contains a monomial cogenerator and no two-term binomials other than linear combinations of monomial cogenerators. Due to the possibility of non-binomials in the socle, binoccular decomposition is not irreducible decomposition, but it is the best approximation that does not exit the class of binomial ideals. As with coprincipal decomposition, the relevant witnesses are essential witnesses rather than key witnesses.

For any monoid prime ideal $P \subset Q$, let $\mathfrak{m}_P = \langle \mathbf{t}^p : p \in P \rangle$. In general, a monoid prime ideal P in a subscript denotes monomial localization, which arises from inverting all monomials outside of $\mathfrak{m}_P$. (This notation was used in [KM14, § 11].)

Definition 5.1. Fix a binomial ideal $I \subset \Bbbk[Q]$ and a prime monoid ideal $P \subset Q$. The *P-socle of* I is the ideal

$$\mathrm{soc}_P(I) = \{f \in \Bbbk[Q]_P/I_P \mid \mathfrak{m}_P f = 0\} \subset \Bbbk[Q]_P/I_P.$$

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A binomial ideal $I \subset \Bbbk[Q]$ is *binoccular* if it is P-coprincipal and every monomial appearing in each binomial in $\mathrm{soc}_P(I)$ is a monomial cogenerator of $\Bbbk[Q]_P/I_P$.

Example 5.2. Binoccular ideals need not induce soccular congruences. The ideal $I = \langle x^2 - xy, xy + y^2\rangle$ is $\langle x, y\rangle$-coprincipal since it contains all monomials of degree 3. The monomials x and y form a key witness pair for $\sim_I$, but I is irreducible, so these monomials do not form a binomial socle element.

Example 5.2 implies that the witness protection program in § 3 cannot be expected to lift directly to the arithmetic setting, in the sense that collapsing the congruence $\sim_I$ combinatorially need not reflect an operation on I itself. Nonetheless, the analogous arithmetic collapse is easily defined and has the desired effect.

Definition 5.3. Fix a P-coprincipal binomial ideal $I \subset \Bbbk[Q]$ cogenerated by $w \in Q$. The *(first) binoccular collapse of* I is the ideal

$$I_1 = \langle \mathbf{t}^a - \lambda\mathbf{t}^b \mid \mathbf{t}^p(\mathbf{t}^a - \lambda\mathbf{t}^b) \in I \text{ for all } p \in P\rangle$$

and the ith *binoccular collapse* I_i *of* I is the binoccular collapse of I_{i-1}. The *binoccular closure of* I is the smallest ideal $\overline{I}$ containing all binoccular collapses of I.

Proposition 5.4. *Fix a P-coprincipal binomial ideal $I \subset \Bbbk[Q]$ cogenerated by $w \in Q$. The binoccular collapse I_1 of I is also a coprincipal ideal cogenerated by w, and for any binomial $\mathbf{t}^a - \lambda\mathbf{t}^b \in I_1$ outside of I, the elements a and b form a key witness pair for $\sim_I$.*

Proof. This follows from Definition 5.3 and Proposition 3.7 since $\sim_J$ coarsens $\sim_I$ and refines $\overline{\sim}_I$. □

Definition 5.5. Fix a binomial ideal $I \subset \Bbbk[Q]$, a prime $P \subset Q$, and $w \in Q$. The *binoccular component of I cogenerated by w* is the binoccular closure $\overline{W}{}_w^P(I)$ of the coprincipal component $W_w^P(I)$ of I cogenerated by w along P.

Lemma 5.6 is the core of the original proof of Theorem 2.11, but it was not stated explicitly in these terms. This unifying principle is also important as we construct binoccular decompositions of binomial ideals (Theorem 5.7) and irreducible decompositions of binomial ideals (Theorem 7.7).

Lemma 5.6. *Fix a binomial ideal $I \subset R = \Bbbk[Q]$ and (not necessarily binomial) ideals $W_1, \ldots, W_r$ containing I. The following are equivalent.*

(i) *We have $I = W_1 \cap \cdots \cap W_r$.*
(ii) *The natural map $R/I \to R/W_1 \oplus \cdots \oplus R/W_r$ is injective.*
(iii) *The natural map $\mathrm{soc}_P(I) \to R_P/(W_1)_P \oplus \cdots \oplus R_P/(W_r)_P$ is injective for every monoid prime $P \subset Q$ associated to $\sim_I$.*
(iv) *The natural map $\mathrm{soc}_\mathfrak{p}(I) \to R_\mathfrak{p}/(W_1)_\mathfrak{p} \oplus \cdots \oplus R_\mathfrak{p}/(W_r)_\mathfrak{p}$ is injective for every prime $\mathfrak{p} \in \mathrm{Ass}\,(I)$.*

Proof. The containments $I \subseteq W_1, \ldots, I \subseteq W_r$ induce a well-defined homomorphism

$$R/I \to R/W_1 \oplus \cdots \oplus R/W_r,$$

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whose kernel is $W_1 \cap \cdots \cap W_r$ modulo I. Thus $I = W_1 \cap \cdots \cap W_r$ holds if and only if this map is injective and therefore (i) $\iff$ (ii). Assume the homomorphism just constructed is injective. Exactness of localization produces an injective map

$$R_P/I_P \hookrightarrow R_P/(W_1)_P \oplus \cdots \oplus R_P/(W_r)_P$$

for each monoid prime $P \subset Q$. This proves (ii) $\Rightarrow$ (iii). Now assume (iii) holds and fix a prime $\mathfrak{p} \in \operatorname{Ass}(I)$. By Theorem 2.13, $\mathfrak{p}$ is minimal over some associated mesoprime of I. Since P is associated to $\sim_I$, the map

$$\operatorname{soc}_P(I) \to R_P/(W_1)_P \oplus \cdots \oplus R_P/(W_r)_P$$

is injective. Every monomial outside of $\mathfrak{m}_P$ also lies outside of $\mathfrak{p}$, so by inverting the remaining elements outside of $\mathfrak{p}$, we obtain the injection

$$\operatorname{soc}_P(I)_{\mathfrak{p}} \to R_{\mathfrak{p}}/(W_1)_{\mathfrak{p}} \oplus \cdots \oplus R_{\mathfrak{p}}/(W_r)_{\mathfrak{p}}.$$

Any element in $\operatorname{soc}_P(I)_{\mathfrak{p}}$ is annihilated by $\mathfrak{m}_P$, so $\operatorname{soc}_{\mathfrak{p}}(I) \subset \operatorname{soc}_P(I)_{\mathfrak{p}}$, yielding (iii) $\Rightarrow$ (iv). Finally, suppose (iv) holds. Fix a nonzero $f \in R/I$ and a prime $\mathfrak{p}$ minimal over the annihilator of f. The image $\overline{f} \in R_{\mathfrak{p}}/I_{\mathfrak{p}}$ of f is nonzero since $\mathfrak{p}$ contains the annihilator of f. Minimality of $\mathfrak{p}$ implies some power of $\mathfrak{p}$ annihilates $\overline{f}$, so $a\overline{f}$ is annihilated by $\mathfrak{p}$ for some $a \in \mathfrak{p}$. By assumption, $a\overline{f}$ has nonzero image in some $(R/W_i)_{\mathfrak{p}}$, meaning af has nonzero image in R/W_i. This proves (iv) $\Rightarrow$ (ii). □

Theorem 5.7. *For any binomial ideal $I \subset \Bbbk[Q]$, the intersection of the binoccular components cogenerated by its essential I-witnesses is a mesoprimary decomposition of I.*

Proof. Fix a monoid prime $P \subset Q$ associated to $\sim_I$ and a nonzero $f \in \operatorname{soc}_P(I)$. By Lemma 5.6, it suffices to show that f is nonzero modulo the localization along P of some binoccular component. By Definition 2.8.4, some nonzero monomial $\lambda \mathbf{t}^w$ of f is an essential I_P-witness for P. This means every monomial of f other than $\lambda \mathbf{t}^w$ that is nonzero modulo $W_w^P(I)_P$ is Green's equivalent to w, so f has nonzero image in the binoccular closure $\overline{W}_w^P(I)_P$. □

6. Nonexistence of binomial irreducible decomposition

The only binomials in the socle of a binoccular binomial ideal are binomials where both terms are monomial cogenerators. When the monomial ideal $\mathfrak{m}_P$ for the associated monoid prime P is a maximal ideal in $\Bbbk[Q]$, this means that in fact the socle has exactly one binomial, up to scale, namely the unique monomial cogenerator. However, even in that case the socle can contain non-binomial elements, too.

Example 6.1. Let $I = \langle x^2y - xy^2, x^3, y^3\rangle \subset \Bbbk[x, y]$. This ideal is binoccular, and its congruence is depicted in Figure 2. The binomial generator forces $x^2y^2 \in I$, so I is cogenerated by x^2y. The monomials x^2, xy and y^2 are all non-key witnesses, and $x^2 + y^2 - xy \in \operatorname{soc}_P(I)$ for $\mathfrak{m}_P = \langle x, y\rangle$. The expression $I = \langle x^2 + y^2 - xy, x^3, y^3\rangle \cap \langle x^3, y\rangle$ is an irreducible decomposition of I, and as we shall see in Theorem 6.4, every irreducible decomposition of I contains some non-binomial irreducible component.

Theorem 6.4 shows that the ideal in Example 6.1 cannot be written as the intersection of irreducible binomial ideals, answering Question 1.1 in the negative. Its proof uses an alternative characterization of irreducible ideals in terms of their socles (Lemma 6.3).

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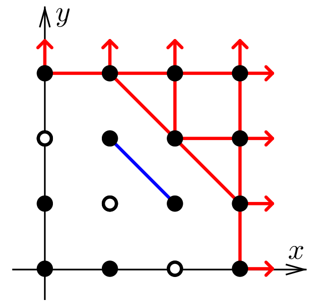

FIGURE 2. (Colour online) The congruence induced by $\langle x^2y - xy^2, x^3, y^3\rangle \subset \Bbbk[x, y]$ on $\mathbb{N}^2$. The non-binomial element $x^2 + y^2 - xy$ lies in the socle of I, and, as such, I does not admit a binomial irreducible decomposition.

DEFINITION 6.2. Fix an ideal I in a Noetherian ring R and a prime ideal $\mathfrak{p} \subset R$. The $\mathfrak{p}$-*socle of* I is

$$\mathrm{soc}_{\mathfrak{p}}(I) = \{f \in R_{\mathfrak{p}}/I_{\mathfrak{p}} \mid \mathfrak{p}f = 0\} \subseteq R_{\mathfrak{p}}/I_{\mathfrak{p}}.$$

The ideal I has *simple* socle if $\dim_{\Bbbk(\mathfrak{p})}(\mathrm{soc}_{\mathfrak{p}}(I)) = 1$, where $\Bbbk(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$ is the residue field at $\mathfrak{p}$.

LEMMA 6.3 [Vas98, Proposition 3.1.7]. *The number of components in any irredundant irreducible decomposition of a* $\mathfrak{p}$*-primary ideal* I *in a Noetherian ring* R *equals* $\dim_{\Bbbk(\mathfrak{p})} \mathrm{soc}_{\mathfrak{p}}(I)$.

THEOREM 6.4. *The ideal* $I = \langle x^2y - xy^2, x^3, y^3\rangle \subset \Bbbk[x, y]$ *cannot be expressed as an intersection of irreducible binomial ideals.*

Proof. Let $\mathfrak{m}_P = \langle x, y\rangle$. The $\Bbbk$-vector space $\mathrm{soc}_P(I)$ is spanned by $\alpha = x^2 + y^2 - xy$ and $\beta = x^2y$. Since $\dim_{\Bbbk}(\mathrm{soc}_P(I)) = 2$ and $\Bbbk = \Bbbk(\mathfrak{m}_P)$, any irredundant irreducible decomposition of I has exactly two components by Lemma 6.3. Suppose $I = I_1 \cap I_2$ with I_1 and I_2 irreducible. The equivalence of Lemma 5.6(i) and 5.6(ii) implies that the natural map $\Bbbk[x, y]/I \to \Bbbk[x, y]/I_1 \oplus \Bbbk[x, y]/I_2$ induces an injection $\mathrm{soc}_{\mathfrak{m}_P}(I) \hookrightarrow \mathrm{soc}_{\mathfrak{m}_P}(I_1) \oplus \mathrm{soc}_{\mathfrak{m}_P}(I_2)$ which is an isomorphism for dimension reasons. Possibly exchanging I_1 and I_2, assume $f = \alpha + \lambda\beta$ spans $\mathrm{soc}_{\mathfrak{m}_P}(I_1)$ for some $\lambda \in \Bbbk$. This implies $f \in I_2$ and $\mathrm{soc}_{\mathfrak{m}_P}(I + \langle f\rangle) = \mathrm{soc}_{\mathfrak{m}_P}(I_2)$, the latter by an explicit, elementary calculation. Lemma 5.6 yields $I_2 = I + \langle f\rangle$. □

Example 6.1 is the first example of a binomial ideal that does not admit a binomial irreducible decomposition. However, it is still possible to construct a (not necessarily binomial) irreducible decomposition from essentially combinatorial data, as Corollary 7.8 demonstrates.

Example 6.5 exhibits the difficulties in determining whether or not a given binomial ideal admits a binomial irreducible decomposition. This question is closely connected with understanding which components in a coprincipal decomposition are redundant.

Example 6.5. Consider the two ideals $I = \langle x^2y - xy^2, x^4 - x^3y, xy^3 - y^4, x^5\rangle$ and $J = \langle x^4y - x^3y^2, x^2y^3 - xy^4, x^6 - x^5y, xy^5 - y^6, x^7\rangle$, whose respective congruences are depicted in Figure 3.

The ideal I has three key witnesses aside from its cogenerator, and the binoccular decomposition produced in Theorem 5.7 has a component at each of these key witnesses. Any one of these three can be omitted, and omitting the component cogenerated by x^2y yields a binomial irreducible decomposition of I. In contrast, J has four non-maximal key witnesses, two of which cogenerate binoccular components that fail to admit binomial irreducible decompositions. Since only one can be omitted, J does not admit a binomial irreducible decomposition.

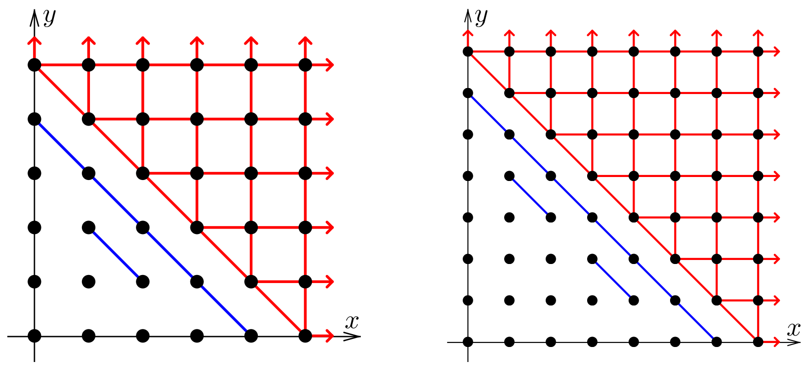

Figure 3. (Colour online) The congruences induced by $I = \langle x^2y - xy^2, x^4 - x^3y, xy^3 - y^4, x^5\rangle$ (left) and $J = \langle x^4y - x^3y^2, x^2y^3 - xy^4, x^6 - x^5y, xy^5 - y^6, x^7\rangle$ (right) on $\mathbb{N}^2$. The ideal I admits a binomial irreducible decomposition, but the ideal J does not.

Problem 6.6. Determine when all of the binoccular components without simple socle can be omitted from the decomposition in Theorem 5.7.

Question 6.7. Which binomial ideals admit binomial irreducible decompositions?

Question 6.7 is more general than Problem 6.6 but may involve primary decompositions that do not arise from mesoprimary decomposition.

7. Irreducible decomposition of binomial ideals

This section produces an irreducible decomposition of any given binomial ideal. We first define the irreducible closure of a coprincipal binomial ideal (Definition 7.1). Unlike a binoccular closure (Definition 5.3), which may have non-binomial elements in its socle, the cogenerators of coprincipal binomial ideals are the only socle elements that survive irreducible closure.

Definition 7.1. For a P-coprincipal binomial ideal $I \subset \Bbbk[Q]$ cogenerated by $w \in Q$, set $R_P = \Bbbk[Q_P]/I_P$ and let $G_P \subset Q_P$ denote the group of units. Write $\overline{w}^\perp$ for the unique graded $\Bbbk$-vector subspace of R_P such that $R_P = (\Bbbk[G_P] \cdot \mathbf{t}^{\overline{w}}) \oplus \overline{w}^\perp$. Let $\overline{w}^\perp_\infty$ denote the largest $\Bbbk[Q_P]$-submodule of R_P that lies entirely in $\overline{w}^\perp$ and set $\overline{R}_P = R_P/\overline{w}^\perp_\infty$. The *irreducible closure* of I is the ideal $\mathrm{Irr}\,(I) = \ker(\Bbbk[Q] \to \overline{R}_P)$.

Example 7.2. Let $I = \langle x^2y - xy^2, x^3, y^3, z^3\rangle$ and $\mathfrak{m}_P = \langle x, y, z\rangle$. Then $z^2(x^2 + y^2 - xy)$ lies in $\mathrm{soc}_P(I)$ and thus generates a $\Bbbk[x, y, z]$-submodule of $(x^2yz^2)^\perp$. On the other hand, the element $z(x^2 + y^2 - xy)$ lies in $\mathrm{soc}_P(\langle z^2(x^2 + y^2 - xy)\rangle + I)$ but outside of $\mathrm{soc}_P(I)$. Continuing yields the irreducible closure $\mathrm{Irr}\,(I) = \langle x^2 + y^2 - xy\rangle + I$ of I.

Recall the usual notion of essentiality from commutative algebra: a submodule N of a module M is *essential* if N intersects every nonzero submodule of M nontrivially.

Lemma 7.3. *If $I \subset \Bbbk[Q]$ is a P-coprincipal binomial ideal with monomial cogenerator $\mathbf{t}^w$, then $\langle \mathbf{t}^{\overline{w}}\rangle = \Bbbk[G_P] \cdot \mathbf{t}^{\overline{w}}$ is an essential $\Bbbk[Q_P]$-submodule of $\overline{R}_P$ that is isomorphic to a Gorenstein quotient of $\Bbbk[G_P]$.*

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Proof. The equality $\langle \mathbf{t}^{\overline{w}} \rangle = \Bbbk[G_P] \cdot \mathbf{t}^{\overline{w}}$ follows because $\mathbf{t}^{\overline{w}}$ is annihilated by $\mathfrak{m}_P$. The Gorenstein condition [Vas98, Appendix A.7] holds because the kernel of the surjection $\Bbbk[G_P] \to \langle \mathbf{t}^{\overline{w}} \rangle$ is, after faithfully flat extension to an algebraically closed coefficient field, generated by a binomial regular sequence [ES96, Theorem 2.1(b)]. To prove essentiality, first note that $\mathrm{soc}_P(\overline{R}_P)$ is an essential submodule of $\overline{R}_P$ because $\mathfrak{m}_P$ is nilpotent on $\overline{R}_P$, and then note that $\langle \mathbf{t}^{\overline{w}} \rangle = \mathrm{soc}_P(\overline{R}_P)$ by construction of $\overline{R}_P$.

Proposition 7.4. *Fix a P-coprincipal binomial ideal $I \subset \Bbbk[Q]$ with monomial cogenerator $\mathbf{t}^w$. The associated primes of R_P, $\overline{R}_P$, and $R_P/\mathfrak{m}_P$ coincide and are all minimal.*

Proof. The associated primes of R_P and $R_P/\mathfrak{m}_P$ coincide by Proposition 2.12. However, $R_P/\mathfrak{m}_P \cong \langle \mathbf{t}^{\overline{w}} \rangle \subseteq R_P$ maps isomorphically to an essential submodule of $\overline{R}_P$ by Lemma 7.3, so all three sets of associated primes coincide. □

Compare the next result to the coprincipal special case of Proposition 2.12.

Theorem 7.5. *The irreducible closure* $\mathrm{Irr}\,(I)$ *of any coprincipal ideal I has a unique minimal primary decomposition. Every primary component therein is irreducible.*

Proof. Minimality of all associated primes in Proposition 7.4 implies the first statement. Since localization preserves essentiality [Bass62, Corollary 1.3], the ordinary localization $\langle \mathbf{t}^{\overline{w}} \rangle_{\mathfrak{p}}$ at the prime $\Bbbk[Q]$-ideal $\mathfrak{p}$ is an essential submodule of $(\overline{R}_P)_{\mathfrak{p}}$ for every $\mathfrak{p} \in \mathrm{Ass}\,(\mathrm{Irr}\,(I))$ by Lemma 7.3. The same lemma implies that $\langle \mathbf{t}^{\overline{w}} \rangle_{\mathfrak{p}}$ is Gorenstein of dimension 0, so $\langle \mathbf{t}^{\overline{w}} \rangle_{\mathfrak{p}}$ has simple socle. Thus the quotient by $\mathrm{Irr}\,(I)_{\mathfrak{p}}$ has simple socle, whence $\mathrm{Irr}\,(I)_{\mathfrak{p}}$ is irreducible by Lemma 6.3. □

Remark 7.6. The proof of Theorem 7.5 via Lemma 7.3 and Proposition 7.4 shows, quite generally, that if a Noetherian ring is contained in a localization that has an essential submodule isomorphic to a Gorenstein ring, then the original ring has a unique minimal primary decomposition all of whose components are quotients modulo irreducible ideals.

We now extend Theorem 5.7 to irreducible closures before stating Corollary 7.8, our main result for this section.

Theorem 7.7. *Every binomial ideal $I \subset \Bbbk[Q]$ equals the intersection of the irreducible closures of the coprincipal components cogenerated by its essential witnesses.*

Proof. Fix a monoid prime $P \subset Q$ and nonzero $f \in \mathrm{soc}_P(I)$. By Definition 2.8.4, some nonzero monomial $\lambda \mathbf{t}^w$ of f is an essential I_P-witness for P. Every monomial of f that is nonzero modulo $\mathrm{Irr}\,(W_w^P(I))_P$ lies in the submodule $\langle \mathbf{t}^{\overline{w}} \rangle$ of $\Bbbk[Q]_P/\mathrm{Irr}\,(W_w^P(I))_P$, so f is nonzero modulo $\mathrm{Irr}\,(W_w^P(I))_P$. Lemma 5.6 completes the proof. □

Corollary 7.8. *Fix a binomial ideal $I \subset \Bbbk[Q]$. An irreducible decomposition of I results by intersecting the canonical primary components of the irreducible closures of the coprincipal components cogenerated by the essential I-witnesses.*

Proof. Apply Theorem 7.7, then Theorem 7.5. □

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