Solution with Axial Symmetry of Einstein's Equations of Teleparallelism

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§ 1. *Introduction.*

In a recent paper Dr G. C. McVittie[1] discussed the solution with axial symmetry of Einstein's new field-equations in his Unified Field Theory of Gravitation and Electricity.[2] Owing to an error in his calculation of the field equations, Dr McVittie did not obtain the general solution, which we discuss in the present paper.

It is assumed that the field is symmetrical about an axis and that the field variables depend on a single variable only, viz., the coordinate which is measured along the axis of symmetry.

We show that there is a general solution, of which a particular case agrees to a first approximation with the gravitational field of a constant electric force on the 1916 theory.

§ 2. *Theory of Teleparallelism.*

The x^μ are Gaussian coordinates, and the metric is given by

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu.$$

Four orthogonal unit vectors, whose covariant components are ${}^s h_\mu$ are constructed at each point of space. The condition that the ${}^s h_\mu$ should be orthogonal unit vectors is

$$g^{\mu\nu}\, {}^s h_\mu\, {}^t h_\nu = e_s\, \delta^s_t,$$

where[3] $$e_1 = e_2 = e_3 = -1\,, \quad e_4 = 1.$$

The equation

$${}_s h^\mu = (\text{minor of } {}^s h_\mu \text{ in } |\, {}^s h_\mu \,|\,) \,/\, |\, {}^s h_\mu \,| \tag{1}$$

defines the corresponding set of contravariant vectors. It may be deduced that

$$g^{\mu\nu} = e_s\, {}_s h^\mu\, {}_s h^\nu, \quad g_{\mu\nu} = e_s\, {}^s h_\mu\, {}^s h_\nu. \tag{2}$$

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[1] G. C. McVittie, *Proc. Edin. Math. Soc.*, (2) **2** (1930), 140.

[2] A. Einstein, *Berlin Akad. Sitz.*, **1** (1930) 18.

[3] The summation convention applies to all indices which appear twice, except in formulae containing the coefficients e_s. If e_s appears, a summation is to be understood only if the index s appears three times; thus $e_s\, \delta^s_t$ is not to be summed, but $e_s\, {}^s h_\mu\, {}^s h_\nu$ is.

The condition for teleparallelism is

$$\frac{\partial\, {}_s h^\lambda}{\partial x^\nu} + {}_s h^\mu\, \Delta^\lambda_{\mu\nu} = 0,$$

where the $\Delta^\lambda_{\mu\nu}$ are unique coefficients of affine connection. Hence

$$\Delta^\lambda_{\mu\nu} = {}_s h^\lambda\, \frac{\partial\, {}^s h_\mu}{\partial x^\nu}. \tag{3}$$

The quantities $\Lambda^\lambda_{\mu\nu}$, ϕ_μ are defined by the equations

$$\Lambda^\lambda_{\mu\nu} = \Delta^\lambda_{\mu\nu} - \Delta^\lambda_{\nu\mu}, \tag{4}$$

$$\phi_\mu \;= \Lambda^\alpha_{\mu\alpha}. \tag{5}$$

Einstein's field-equations are then

$$G^{\mu\nu} \equiv g^{\mu\alpha}\, g^{\beta\gamma}\, (\Lambda^\nu_{\alpha\beta\,;\,\gamma} - \Lambda^\delta_{\alpha\beta}\, \Lambda^\nu_{\delta\gamma}) = 0, \tag{6}$$

$$F_{\mu\nu} \equiv \frac{\partial\phi_\mu}{\partial x^\nu} - \frac{\partial\phi_\nu}{\partial x^\mu} = 0, \tag{7}$$

where the semicolon in equation (6) denotes that the covariant derivative with respect to the connection (3) is to be taken.

The ${}_s h^\mu$ have the following property of "rotation invariance":—let

$${}_s\bar{h}^\mu = a^t_s\, {}_t h^\mu \tag{8}$$

where the a^t_s are constants and

$$e_s\, a^s_k\, a^s_l = e_k\, \delta^k_l. \tag{9}$$

Then if $\bar{g}^{\mu\nu}$, $\overline{\Delta}^\lambda_{\mu\nu}$ are the quantities obtained by writing ${}_s\bar{h}^\mu$ instead of ${}_s h^\mu$ in equations (2) and (3) it may be proved that

$$\bar{g}^{\mu\nu} = g^{\mu\nu}, \qquad \overline{\Delta}^\lambda_{\mu\nu} = \Delta^\lambda_{\mu\nu}.$$

$g^{\mu\nu}$ and $\Delta^\lambda_{\mu\nu}$ are therefore invariant with respect to a constant orthogonal rotation of the fundamental vectors. Hence if any set of fundamental vectors satisfies equations (6) and (7), any other set obtained from these by a constant orthogonal rotation will also be a solution of the field equations, since these equations consist of functions of the $g^{\mu\nu}$ and $\Delta^\lambda_{\mu\nu}$ only.[1]

[1] For a fuller discussion of rotation invariance etc., see T. Y. Thomas, *Proc. Nat. Acad. (Washington)*, **16** (1930), 761.

§ 3. *Form of the $_s h^\mu$ for Axial Symmetry.*

We shall choose coordinates such that at the origin

$$ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (dx^4)^2;$$

since the fundamental vectors can be rotated to coincide in direction with the axes at any point, we may assume that at the origin

$$_s h^\mu = \delta^\mu_s.$$

McVittie[1] has proved that under these conditions all the vector components will be zero everywhere except $_1h^1$, $_2h^2$, $_3h^3$, $_4h^4$, $_1h^4$ and $_4h^1$. We can further make $_1h^4$ zero everywhere by a transformation of coordinates of the form

$$\bar{x}^1 = x^1, \qquad \bar{x}^4 = x^4 + \phi(x^1),$$

which gives

$$_1\bar{h}^4 = {}_1h^4 + \frac{\partial \phi}{\partial x^1}\, {}_1h^1.$$

Hence if we take

$$\frac{\partial \phi}{\partial x^1} = -\, {}_1h^4 / {}_1h^1,$$

we have $_1h^4 = 0$ everywhere.

There are thus five remaining components, $_1h^1$, $_2h^2$, $_3h^3$, $_4h^4$, and $_4h^1$, and of these $_2h^2 = {}_3h^3$ from symmetry. The covariant components which do not vanish are

$$^1h_1,\ ^2h_2 = {}^3h_3,\ ^4h_4 \text{ and } ^1h_4,$$

satisfying the condition

$$^s h_\mu = \delta^s_\mu \text{ at the origin.}$$

§ 4. *Solution of the Field Equations.*

The $_s h^\mu$ may be calculated in terms of the $^s h_\mu$ by equation (1). This gives

$$_1h^1 = 1/{}^1h_1,\quad {}_2h^2 = {}_3h^3 = 1/{}^2h_2,$$
$$_4h^4 = 1/{}^4h_4,\quad {}_4h^1 = -\,{}^1h_4/({}^1h_1\,{}^4h_4).$$

The values of the $\Delta^\lambda_{\mu\nu}$ which are not zero are, by equation (3),

$$\begin{aligned}
\Delta^4_{41} &= {}_4h^4 \frac{d\,{}^4h_4}{dx^1} = \frac{d}{dx^1}\log {}^4h_4, \\
\Delta^2_{21} &= \Delta^3_{31} = {}_2h^2 \frac{d\,{}^2h_2}{dx^1} = \frac{d}{dx^1}\log {}^2h_2, \\
\Delta^1_{11} &= {}_1h^1 \frac{d\,{}^1h_1}{dx^1} = \frac{d}{dx^1}\log {}^1h_1, \\
\Delta^1_{41} &= {}_4h^1 \frac{d\,{}^4h_4}{dx^1} + {}_1h^1 \frac{d\,{}^1h_4}{dx^1} = \left({}^4h_4 \frac{d\,{}^1h_4}{dx^1} - {}^1h_4 \frac{d\,{}^4h_4}{dx^1}\right) \Big/ ({}^1h_1\,{}^4h_4).
\end{aligned} \tag{12}$$

[1] G. C. McVittie, *loc. cit.*

Hence we obtain the following values for the $\Lambda^\lambda_{\mu\nu}$ which do not vanish identically

$$\Lambda^4_{41} = -\Lambda^4_{14} = \Delta^4_{41},$$
$$\Lambda^3_{31} = \Lambda^2_{21} = -\Lambda^3_{13} = -\Lambda^2_{12} = \Delta^2_{21},$$
$$\Lambda^1_{41} = -\Lambda^1_{14} = \Delta^1_{41}.$$

Similarly for the functions ϕ_μ, by equation (5)

$$\phi_1 = -\Delta^2_{21} - \Delta^3_{31} - \Delta^4_{41}, \quad \phi_4 = \Delta^1_{41}, \quad \phi_2 = \phi_3 = 0.$$

The equations (7) reduce to a single one, namely,

$$\frac{\partial \phi_4}{\partial x^1} = \frac{d}{dx_1}\Delta^1_{41} = 0.$$

Hence Δ^1_{41} is constant.

Equation (6) may be written out in full as

$$G^\nu_\mu \equiv g^{\alpha\beta}\left(\frac{\partial}{\partial x^\beta}\Lambda^\nu_{\mu\alpha} - \Delta^\gamma_{\mu\beta}\Lambda^\nu_{\gamma\alpha} - \Delta^\gamma_{\alpha\beta}\Lambda^\nu_{\mu\gamma} + \Delta^\nu_{\beta\gamma}\Lambda^\gamma_{\mu\alpha}\right) = 0.$$

Therefore

$$G^1_1 \equiv -g^{14}\left[\frac{d}{dx^1}\Delta^1_{41} - \Delta^1_{41}\Delta^4_{41}\right] - g^{44}(\Delta^1_{41})^2 = 0, \tag{13}$$

$$G^1_4 \equiv g^{11}\left[\frac{d}{dx^1}\Delta^1_{41} - \Delta^1_{41}\Delta^4_{41}\right] + g^{14}(\Delta^1_{41})^2 = 0, \tag{14}$$

$$G^4_1 \equiv -g^{14}\left[\frac{d}{dx^1}\Delta^4_{41} - (\Delta^4_{41})^2 - \Delta^1_{11}\Delta^4_{41}\right] - g^{44}\Delta^1_{41}\Delta^4_{41} = 0, \tag{15}$$

$$G^4_4 \equiv g^{11}\left[\frac{d}{dx^1}\Delta^4_{41} - (\Delta^4_{41})^2 - \Delta^1_{11}\Delta^4_{41}\right] + g^{14}\Delta^1_{41}\Delta^4_{41} = 0, \tag{16}$$

$$G^2_2 = G^3_3 \equiv g^{11}\left[\frac{d}{dx^1}\Delta^2_{21} - (\Delta^2_{21})^2 - \Delta^2_{21}\Delta^1_{11}\right] - g^{14}\Delta^1_{41}\Delta^2_{21} = 0. \tag{17}$$

The other ten G^ν_μ are identically zero.[1]

From (13) and (14) it follows that either

$$\begin{vmatrix} g^{11} & g^{14} \\ g^{14} & g^{44} \end{vmatrix} = 0$$

or else that

$$\left.\begin{aligned} \frac{d}{dx_1}\Delta^1_{41} - \Delta^1_{41}\Delta^4_{41} &= 0 \\ (\Delta^1_{41})^2 &= 0 \end{aligned}\right\}.$$

[1] In Dr McVittie's paper Δ^4_{41} occurred instead of Δ^1_{41} in the third term of G^1_1 and the first term of G^1_4. Also in equation (28) of McVittie's paper Δ^1_{41} was zero, so that it was not correct to deduce that either $g^{14} - g^{44} = 0$ or $\Delta^4_{41} = 0$.

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But the first assumption is impossible since the coordinates are to be approximately Galilean; therefore

$$\Delta^1_{41} = 0.$$

Hence the five equations, (13)-(17), reduce to the three following:—

$$\frac{d}{dx^1}\Delta^4_{41} - (\Delta^4_{41})^2 - \Delta^1_{11}\Delta^4_{41} = 0 \qquad (18)$$

$$\frac{d}{dx^1}\Delta^2_{21} - (\Delta^2_{21})^2 - \Delta^1_{11}\Delta^2_{21} = 0 \qquad (19)$$

$$\Delta^1_{41} = 0. \qquad (20)$$

From (12) and (20)

$${}^4h_4 \frac{d\,{}^1h_4}{dx^1} - {}^1h_4 \frac{d\,{}^4h_4}{dx^1} = 0;$$

therefore ${}^1h_4$ is a constant multiple of ${}^4h_4$. But at the origin ${}^4h_4 = 1$, ${}^1h_4 = 0$; therefore we have ${}^1h_4 = 0$ everywhere.

We may take ${}^1h_1 = 1$ everywhere without loss of generality, since, ${}^1h_1$ being a function of x^1 alone, a transformation

$$d\bar{x}^1 = {}^1h_1\, dx^1$$

will make ${}_1\bar{h}^1 = 1$.

Hence, by (12), $\Delta^1_{11} = 0$, and (18) reduces to

$$\frac{d}{dx^1}\Delta^4_{41} - (\Delta^4_{41})^2 = 0,$$

or

$$\frac{d^2}{(dx^1)^2}\log {}^4h_4 - \left(\frac{d}{dx^1}\log {}^4h_4\right)^2 = 0.$$

The solution of this equation, with a suitable choice of constants, is

$${}^4h_4 = \frac{1}{1+\alpha x^1},$$

where α is a constant.

Similarly from equation (19)

$${}^2h_2 = {}^3h_3 = \frac{1}{1+\beta x^1}.$$

Thus the complete solution is

$${}^1h_1 = 1, \quad {}^2h_2 = {}^3h_3 = \frac{1}{1+\beta x^1}, \quad {}^4h_4 = \frac{1}{1+\alpha x^1}, \qquad (22)$$

all the other ${}^{s}h_{\mu}$ being zero everywhere. The metric is therefore determined by

$$ds^2 = -(dx^1)^2 - \frac{(dx^2)^2 + (dx^3)^2}{(1+\beta x^1)^2} + \frac{(dx^4)^2}{(1+ax^1)^2}. \tag{23}$$

(22) is the general solution in its simplest form. Other forms of the solution may be obtained from (22) by a constant orthogonal rotation of the fundamental vectors

$${}^{t}\bar{h}_{\mu} = a^{t}_{s}\, {}^{s}h_{\mu}$$

combined with a transformation of the coordinates, in which the ${}^{s}h_{\mu}$ are transformed by the ordinary law for covariant vectors.

§ 5. *Comparison with the* 1916 *Theory.*

McVittie's solution[1] for the gravitational field of a constant electric force on the 1916 theory of Relativity was

$$ds^2 = e^{kz^1}(dz^4)^2 - e^{-2kz^1}(dz^1)^2 - e^{-kz^1}[(dz^2)^2 + (dz^3)^2] \tag{24}$$

where the electric force at the origin was equal to $k/4\sqrt{\pi}$. If in equation (23) we write

$$dx^1 = e^{-kz^1}dz^1, \quad x^1 = \frac{1}{k}(1 - e^{-kz^1}),$$

$$x^2 = z^2, \quad x^3 = z^3, \quad x^4 = z^4,$$

and neglect second and higher powers of k, a and β, it becomes

$$ds^2 = -e^{-2kz^1}(dz^1)^2 - (1-2\beta z^1)[(dz^2)^2 + (dz^3)^2] + (1-2az^1)(dz^4)^2.$$

This will agree with (24) as far as the first power of k, provided

$$2\beta = k, \quad 2a = -k.$$

Thus the field

$$ds^2 = -(dx^1)^2 - \frac{(dx^2)^2 + (dx^3)^2}{(1-ax^1)^2} + \frac{(dx^4)^2}{(1+ax^1)^2} \tag{25}$$

corresponds on the theory of teleparallelism to the field of a constant electric force of magnitude $-a/2\sqrt{\pi}$ on the 1916 theory.

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[1] G. C. McVittie, *Proc. Roy. Soc.* (A), **124** (1929), 366.

§ 6. *Einstein's Approximation.*

The following physical interpretation of the ${}^s h_\mu$ was suggested by Einstein[1]:—Let ${}^s h_\mu = \delta^s_\mu + \bar{h}_{s\mu}$, where $\bar{h}_{s\mu}$ is small. Then equation (2) becomes, to a first approximation,

$$g_{\mu\nu} = \delta^\mu_\nu + e_\mu \bar{h}_{\mu\nu} + e_\nu \bar{h}_{\nu\mu}, \tag{26}$$

and we assume that if $a_{\mu\nu}$ is the electromagnetic force tensor, then to a first approximation

$$a_{\mu\nu} = e_\mu \bar{h}_{\mu\nu} - e_\nu \bar{h}_{\nu\mu}. \tag{27}$$

The approximation (27) gives the correct value for the electric intensity in the case of Einstein and Mayer's solution for the spherically symmetric field of a single particle, but is unsatisfactory in the case of axial symmetry.

Einstein and Mayer's solution[2] for a single charged particle included the following values for the ${}_s h^\mu$,

$${}_s h^4 = 0, \quad {}_4 h^a = \frac{x^a}{\left(1 - \dfrac{e^2}{r^4}\right)^{\frac{1}{2}}} \cdot \frac{e}{r^3}, \tag{28}$$

where s, $a = 1, 2, 3$, and $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$.

The values of the corresponding covariant components, which determine the electric field, are approximately

$${}^1 h_4 = -\frac{ex^1}{r^3}, \quad {}^2 h_4 = -\frac{ex^2}{r^3}, \quad {}^3 h_4 = -\frac{ex^3}{r^3},$$

which gives an electric intensity, by (27)

$$a_{14} = \frac{ex^1}{r^3}, \quad a_{24} = \frac{ex^2}{r^3}, \quad a_{34} = \frac{ex^3}{r^3}.$$

These are the ordinary rectangular components of the electric intensity due to a point charge e according to the inverse square law.

Equation (27) however, is not invariant with respect to a rotation of the fundamental vectors, and we find that in the case of a constant electric force Einstein's approximation is insufficient to determine

[1] A. Einstein, *Berlin Akad. Sitz.*, **1** (1930), 18.

[2] Einstein and Mayer, *Berlin Akad. Sitz.*, **6** (1930), 110.

the intensity. In this case equation (27) always gives a constant intensity, but the value it gives for the intensity corresponding to a particular metric depends of the choice of orientation of the fundamental vectors at the origin.

When the orientation of the ${}^sh_\mu$ at the origin is fixed by the equation ${}^sh_\mu = \delta^s_\mu$, the solution (22) follows, and the components of this solution substituted in equation (27) would give an electric intensity zero everywhere.

We can, however, without altering the metric, give the ${}^sh_\mu$ a rotation

$$\begin{matrix} \sqrt{1+\gamma^2} & 0 & 0 & \gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma & 0 & 0 & \sqrt{1+\gamma^2}, \end{matrix}$$

which gives

$${}^1\bar{h}_4 = \gamma\, {}^4h_4 = \frac{\gamma}{1+\beta x^1}, \quad {}^4\bar{h}_1 = -\gamma\, {}^1h_1 = -\gamma.$$

Substituting these components in equation (27) we have to a first approximation

$$a_{14} = e_1\, {}^1\bar{h}_4 - e_4\, {}^4\bar{h}_1 = -2\gamma.$$

γ is an arbitrary constant, and therefore Einstein's approximation gives a constant intensity of indeterminate magnitude.

If we take $\gamma = a / 4\pi$, equation (27) gives an electric intensity corresponding to the metric (25) which agrees with the 1916 theory.

The indeterminate result given by Einstein's approximation in this case is apparently due to the fact that there is no point at which the electric force is known to be zero. In the Einstein-Mayer solution the orientation of the ${}^sh_\mu$ was fixed by the condition ${}^sh_\mu = \delta^s_\mu$ at infinity, which ensured that all the components of the electromagnetic force tensor would be zero outside the range of influence of the particle.

§ 7. *Conclusion.*

We have shown that Einstein's equations admit a general solution in the case where the field variables depend on one coordinate only, and that the constants in this solution can be chosen so that it agrees to a first approximation with the gravitational field of a constant electric force on the 1916 theory.

Dr McVittie's conclusion that Einstein's approximation gave the electric force zero everywhere is a consequence of the arbitrary restriction ${}_s h^\mu = \delta_s^\mu$ at the origin. We have shown that with a different orientation of the ${}^s h_\mu$ the approximation will give a constant force which is not zero. Since the metric is the same in each case, only one of the infinite number of sets obtained from each other by constant orthogonal rotations will give the correct value for the electric force, and in our case there is no method of selecting this particular set, except by comparison with the 1916 theory.

In conclusion, I wish to express my thanks to Dr McVittie and Dr McCrea, of the Edinburgh Mathematical Society, for their interesting and helpful suggestions for the revision of this paper.