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Theories of convection and the spectrum of turbulence in the solar photosphere

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Abstract. Classical theories of turbulence do not describe accurately inertial range scaling laws in turbulent convection and notably fail to model the shape of the turbulent spectrum of solar photospheric convection. To understand these discrepancies, a detailed study of scale-by-scale budgets in turbulent Rayleigh-Bénard convection is presented, with particular emphasis placed on anisotropy and inhomogeneity. A generalized Kolmogorov equation applying to convection is derived and its various terms are computed using numerical simulations of turbulent Boussinesq convection. The analysis of the isotropic part of the equation shows that the third-order velocity structure function is significantly affected by buoyancy forcing and large-scale inhomogeneities. Anisotropic contributions to this equation are also shown to be comparable to their isotropic counterpart at moderate to large scales. Implications of these results for convection in the solar photosphere, mesogranulation and supergranulation are discussed.

Keywords. Convection, turbulence, hydrodynamics, Sun: photosphere, Sun: granulation

1. Introduction

Fundamental information about the physical processes at work in turbulent flows can be obtained by studying their energy spectrum and the associated spectral scaling laws. Various studies of the solar surface have focused on understanding the spectrum of turbulent convection in the photosphere (Zahn 1987; Espagnet *et al.* 1993; Hathaway *et al.* 2000). In spite of these efforts, a globally satisfying theoretical picture of the distribution of energy at various scales (like granulation, mesogranulation, supergranulation) is still lacking (Rincon, Lignières & Rieutord 2005): solar convection does not seem to follow the simple predictions of linear stability theories or those of classical theories of isotropic and homogeneous turbulence.

There are several possible explanations for this disagreement between classical theories and observations (see for instance Petrovay 2001). On the one hand, solar convection is not laminar, therefore linear theories will fail unless mean-field effects (like turbulent viscosity) can be parametrized accurately. On the other hand, turbulence in the solar photosphere is neither isotropic (because it is driven by the unstable stratification in the vertical direction) nor homogeneous (because the observed flow occurs in the vicinity of the $\tau = 1$ optical depth surface, which acts as a natural boundary). Buoyancy effects are also very likely present at most observable scales, which are thus not strictly part of an "inertial" range in the Kolmogorov sense. The focus of this essay is on these last three points. Results regarding inertial range scalings in numerical simulations of turbulent convection, which have been obtained using modern tools of turbulence theory, like the SO(3) decomposition of structure functions to quantify anisotropic effects and a description of statistical inhomogeneities, are used to illustrate how convection spectra are affected simultaneously by forcing, anisotropic and inhomogeneous effects.

In § 2, a quick summary of the current status of theories of turbulent convection is given. § 3 is devoted to the analysis of turbulent statistics and scaling laws in a numerical simulation of turbulent Boussinesq convection. A discussion of these results in the context of the solar photosphere is presented in § 4 and a short conclusion follows.

2. Theories of turbulent convection

The aim of this section is to present a quick summary of currently available theories to describe turbulent convection. Emphasis is placed on a description in physical space using structure and correlation functions rather than on turbulent spectra, because scaling predictions result from exact laws (Kolmogorov and Yaglom equations, see Monin & Yaglom 1975) which are valid in physical space only. Intermittency effects are not considered. Moments of increments of various quantities such as velocity $\vec{u}$ or temperature θ fluctuations between two points separated by a correlation vector $\vec{r}$ are used, and increments of any quantity f are denoted by δf. Simple forms for these exact laws are usually obtained assuming homogeneity (turbulence looks statistically the same everywhere) and isotropy (turbulence looks statistically the same for any orientation of $\vec{r}$).

2.1. *Kolmogorov theory*

Kolmogorov's 1941 theory (hereinafter K41) predicts $\delta u \sim r^{1/3}$ and $\delta\theta \sim r^{1/3}$. The first prediction can be obtained using dimensional analysis or one of the only exact results known for turbulence, namely the Kolmogorov equation, also called the 4/5 law:

$$\left\langle (\delta u_r)^3 \right\rangle = -\frac{4}{5} \left\langle \varepsilon \right\rangle r + 6\nu \frac{\partial}{\partial r} \left\langle (\delta u_r)^2 \right\rangle , \tag{2.1}$$

where ν is the kinematic viscosity and $\langle \varepsilon \rangle = \nu \left\langle (\partial_i u_j)^2 \right\rangle$. This law states that the third-order structure function for longitudinal velocity increments $\left\langle (\delta u_r)^3 \right\rangle$ is proportional to r in the inertial range of turbulence, where both viscosity and forcing are negligible. The prediction for temperature increments (at scales larger than the thermal diffusion scale) results from the extra assumption that temperature behaves as a passive scalar, which is strictly speaking only valid for a neutrally stratified flow. The corresponding spectra are both proportional to $k^{-5/3}$ in the framework of this theory.

2.2. *Bolgiano-Obukhov theory*

Atmospheric science research has led to the development of Bolgiano-Obukhov (Bolgiano 1959 ; Obukhov 1959, hereinafter BO59) theory for stably stratified turbulence. This theory is equally valid for an unstably stratified medium (L'vov 1991). Unlike K41, it does not require that the forcing acts on large scales only and relies on a dominant balance in a "generalized" 4/5 law between $\langle (\delta u_r)^3 \rangle$ and a forcing/damping term due to unstable/stable stratification (Yakhot 1992), whereas in the classical 4/5 law, the balance is between $\langle (\delta u_r)^3 \rangle$ and the 4/5 term. Using the equivalent of the 4/5 law for temperature, called Yaglom equation or 4/3 law, one can infer the following scalings using dimensional analysis: $\delta u \sim r^{3/5}$ and $\delta\theta \sim r^{1/5}$, leading respectively to $k^{-11/5}$ and $k^{-7/5}$ spectra for the velocity and temperature fluctuations. The crossover scale between K41 and BO59 is called the Bolgiano length (Chilla *et al.* 1993; Benzi *et al.* 1998)

$$L_B = \frac{Nu^{1/2} d}{(RaPr)^{1/4}}, \tag{2.2}$$

where Nu is the Nusselt number, d is the typical scale-height of the layer, Ra is the Rayleigh number and Pr is the Prandtl number. At this scale, the 4/5 term equals the

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forcing/damping term in the Kolmogorov equation. BO59 should hold for $r > L_B$, while K41 predictions should be observed for $r < L_B$, leading to *a priori* observable kinks in both velocity and temperature spectra. Note that equation (2.2) is only a dimensional estimate which, from previous studies (Calzavarini *et al.* 2002, Rincon 2006), appears to underestimate the actual L_B by a factor ∼10.

2.3. *Dealing with anisotropy and inhomogeneity*

As mentioned earlier, important differences between theories and observations or experimental convection (*e. g.* Verzicco & Camussi 2003) may be related to anisotropic or inhomegeneous effects (Hill 1997). A brief description of modern techniques used to quantify these effects is given here. Many details on these issues are given by Biferale & Procaccia (2005), Danaila *et al.* (2001), and their applications to turbulent convection can be found in the work of Biferale *et al.* (2003) and Rincon (2006). The essential point is to project the full $(\vec{x},\vec{r})$ dependent (as opposed to a $|\vec{r}|$ only dependence, with $\vec{x}$ being the measurement position) Kolmogorov and Yaglom equations onto a spherical harmonics basis. Projections on the $\ell = 0$ degree lead to the usual isotropic Yaglom and Kolmogorov equations. Inhomogeneous effects are included by keeping all terms involving variations of statistical quantities with respect to the location at which they are computed (*e. g.* at the centre). The isotropic part of the generalized Kolmogorov equation, used in § 3, reads

$$-\langle U_R\rangle_0^0 + \frac{2\alpha g}{r^2}\int_0^r y^2\,\langle\delta u_z\delta\theta\rangle_0^0 \mathrm{d}y + 2\nu\frac{\partial}{\partial r}\left\langle(\delta u_i)^2\right\rangle_0^0 + \langle NH\rangle_0^0 = \frac{2}{r^2}\int_0^r y^2[\langle\varepsilon\rangle + \langle\varepsilon'\rangle]_0^0 \mathrm{d}y, \quad (2.3)$$

where the $^0_0$ notation indicates projection onto the Y_0^0 spherical harmonic (equivalently average over a $\vec{r}$-sphere), $\langle U_R\rangle_0^0 = \langle(\delta u_i)^2\delta u_r\rangle_0^0$ is a generalized third-order velocity structure function, $\langle\varepsilon'\rangle = \langle\varepsilon\rangle(\vec{x}+\vec{r})$, the associated $\langle\varepsilon\rangle + \langle\varepsilon'\rangle$ term is a generalized form of the 4/5 term in equation (2.1), the αg term is the contribution of buoyancy forcing in the generalized Kolmogorov equation, and $\langle NH\rangle$ stands for all inhomogeneous terms. All these statistical quantities depend on the vertical (gravity) coordinate. This equation is *exact*: only statistical stationarity has to be assumed to obtain it. Evaluating its various terms to determine the dominant balances and thus potential inertial range scalings for $\langle U_R\rangle_0^0$ and $\langle\Theta_R\rangle_0^0$ is the aim of the next section.

3. Numerical results

Three-dimensional direct numerical simulations of turbulent Boussinesq convection at $Ra = 10^6$ and $Pr = 1$ with aspect ratio $A = 5$ have been performed in order to compute the scale-by-scale budgets of equation (2.3) in the middle of the convection box.

3.1. *Isotropic component*

Figure 1 shows that the buoyancy term in Kolmogorov equation is smaller than the 4/5 term, so that the conditions for BO59 are not met. The two curves seem to intersect at scales comparable to the box depth ($r/\eta = 60$, where η is the Kolmogorov dissipation scale) rather than $0.1\,d$, predicted by equation (2.2) for this simulation. However, K41 is not verified either. The reasons are that inhomogeneous and buoyancy terms interfere in the scale-by-scale budget to prevent the K41 balance between $\langle U_R\rangle$ and the 4/5 term. A detailed investigation (Rincon 2006) shows that $Ra = 10^9$ is a mimimum requirement for buoyancy to be negligible for some $r < L_B$ and to observe one K41 decade. The reason why scaling laws are not observed on velocity spectra or structure functions is therefore very different from the usual argument that viscous dissipation (plotted in figure 1) is too

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large. In fact, scaling laws are observed in this simulation for mixed velocity-temperature third-order structure functions appearing in Yaglom equation.

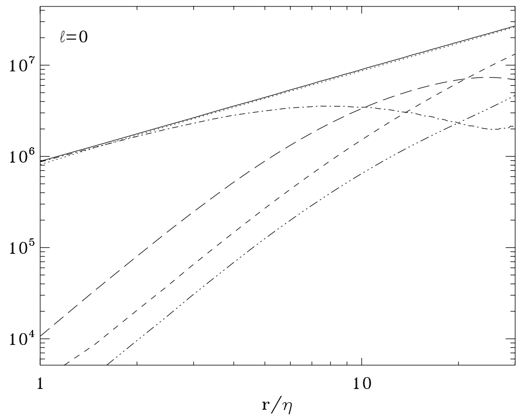

Figure 1. Scale-by-scale budget for the isotropic part of Kolmogorov equation: log-log plots of $-\left\langle U_R\right\rangle_0^0$ (— —), $2\alpha g/r^2 \int_0^r y^2 \left\langle \delta u_z \delta\theta\right\rangle_0^0 \mathrm{d}y$ (- - -), $2\nu\partial_r \left\langle(\delta u_i)^2\right\rangle_0^0$ (- · -), $\left\langle NH\right\rangle_0^0$ (— · · · —). The sum of the l.h.s. terms of equation (2.3) (——) matches the r.h.s. "4/5 term" (· · ·).

3.2. *Anisotropy and projection effects*

The other effect which may prevent observations of scaling laws is the anisotropic driving mechanism, which generates $\ell \neq 0$ components of structure functions comparable to the isotropic component at moderate to large scales (figure 2 depicts the spherical harmonics spectrum of $\langle U_R\rangle$ for even ℓ and various r). This is important because "reduced" structure functions based on correlation vectors lying in horizontal planes are linear combinations of these anisotropic components, which may follow different r-power laws according to modern turbulent theories (Biferale & Procaccia 2005):

$$\left\langle U_R\right\rangle(r,\pi/2) = \sum_\ell \left\langle U_R\right\rangle_0^\ell(r) Y_\ell^0(\pi/2). \tag{3.1}$$

Thus, unless turbulence is really isotropic, these reduced quantities behave differently from isotropic structure functions, and may not exhibit any scaling behaviour. The local scaling exponents of $\left\langle U_R\right\rangle(r,\pi/2)$ and of the r.h.s. of equation (3.1) reconstructed up to $\ell = 6$ are plotted in figure 2 using the associated ℓ-spectrum. It is straightforward to see that the $\ell = 2$ and $\ell = 4$ degrees strongly affect the local slope of the "reduced" third-order structure function (slope 1 corresponds to the isotropic K41 scaling law).

4. Understanding the spectrum of photospheric convection

This section discusses the theoretical arguments exposed in § 2 in the context of the solar photosphere, in the light of the numerical results presented in § 3.

4.1. *Inertial and injection ranges for the photosphere*

Recent studies indicate that $Nu \sim Ra^{1/2}Pr^{1/2}$ holds for $Ra > 10^{11}$ (Grossmann & Lohse 2000; Calzavarini *et al.* 2005), so that L_B as given by equation (2.2) is *independent of any nondimensional parameter* in the hard turbulence regime and is *of order a few d for large Ra* (taking into account the prefactor). Close to $\tau = 1$, identifying d with the local density scale height H_ρ leads to an actual L_B of a few thousands kilometers. Thus, surface

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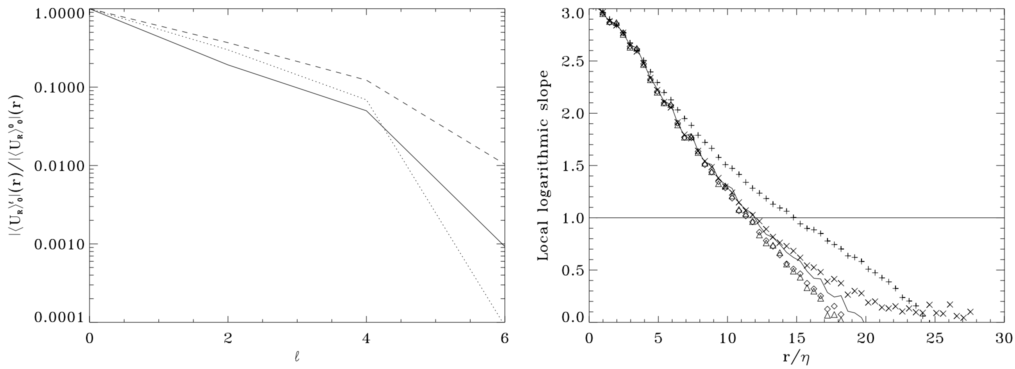

Figure 2. Left: spherical harmonics spectrum of $\langle U_R \rangle$ at the centre of the box for $r/\eta = 7.9$ (——), $r/\eta = 15.3$ (· · ·) and $r/\eta = 22.6$ (- - -). Right: local log-slope of $\langle U_r \rangle\,(r, \pi/2)$ (——) and using spherical harmonics up to $\ell = 0$ (+), $\ell = 2$ (×), $\ell = 4$ (⋄), $\ell = 6$ (△) (see equation (3.1)).

turbulence from granular to supergranular scales is strongly affected by buoyancy forcing, and such scales are part of the injection range, as suggested earlier by Zahn (1987) and Petrovay (2001). This is supported by spectral-space budgets computed for large aspect ratio simulations by Rincon *et al.* (2005), in which most energy is pumped predominantly at mesogranular scales and cascaded down to the dissipation scales.

4.2. *The effects of inhomogeneity and anisotropy on data analysis*

According to this estimate for L_B, K41 should be observed at subgranular scales, while BO59 is expected at larger scales. In practice, there is no clear evidence for this. A reasonable explanation for this disagreement on large scales is that inhomogeneities dominate the scale-by-scale budget at scales larger than the typical scale-height and prevent the BO59 balance (see Danaila *et al.* 2001 for large-scale inhomogeneous effects in a channel flow). As granular scales are close to L_B, there is no dominant balance in the Kolmogorov equation for this range of scales and therefore no definite inertial range scaling exponent. Finally, photospheric convection spectra presented so far in literature have been obtained using either *line-of-sight* measurements or by analysing local *horizontal* motions, *i.e.* along specific directions only. As shown previously, these quantities involve a mixture of different power laws and attempts to extract (isotropic) exponents from them may prove impossible. To disentangle anisotropic effects would notably require the knowledge of the vertical velocity field at various depths to measure correlations along the vertical. Only very high-resolution local helioseismology may provide this information in the future.

5. Conclusion

This study is only a first step towards a fully consistent description of the spectrum of photospheric convection, but shows that considering generalized Kolmogorov and Yaglom equations is crucial to understand *anisotropic and inhomogeneous turbulence forced in a full range of scales.* An interesting result is the independence of the estimated L_B on both Pr and Ra at $Ra > 10^{11}$, which makes it directly applicable to the Sun. More work is needed to understand scaling laws in the low Pr regime typical of the Sun.

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Discussion

J. P. Zahn: Is your statement that energy is injected on mesogranular scales based on numerical simulations?

F. Rincon: Not only. This is supported by theory (eq. (2.2) with $Nu \sim (RaPr)^{1/2}$).

F. H. Busse: What boundary conditions did you use in your Boussinesq simulations?

F. Rincon: Fixed temperature and stress-free boundaries are used for both plates.

H. G. Ludwig: Can a stationary theory (like Kolmogorov's) describe the properties of a transient flow like photospheric convection?

F. Rincon: Yes. Statistical stationarity does not prevent time-dependence in the flow.

H. G. Ludwig: Do you associate the observable granular cells with turbulent eddies?

F. Rincon: Granules are undoubtedly turbulent. But they are very much influenced by anisotropy, inhomogeneity and injection, unlike classical inertial eddies.

R. F. Stein: On the Sun, large temperature fluctuations are at granular scales, which is where there are the large entropy fluctuations that give rise to the buoyancy driving.

F. Rincon: True. Granular scales are very much influenced by buoyancy at $\tau = 1$. Deeper and deeper, driving occurs on larger and larger scales, as $L_B \sim$ a few H_ρ shows.

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