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Ample Vector Bundles of Curve Genus One

Antonio Lanteri and Hidetoshi Maeda

Abstract. We investigate the pairs $(X, \mathcal{E})$ consisting of a smooth complex projective variety X of dimension n and an ample vector bundle $\mathcal{E}$ of rank $n-1$ on X such that $\mathcal{E}$ has a section whose zero locus is a smooth elliptic curve.

0 Introduction

Let $\mathcal{E}$ be an ample vector bundle of rank $n-1$ on a complex projective manifold X of dimension n. Following [LMS], we define the *curve genus* $g(X, \mathcal{E})$ of $(X, \mathcal{E})$ by setting

$$2g(X, \mathcal{E}) - 2 = \big(K_X + c_1(\mathcal{E})\big)c_{n-1}(\mathcal{E}),$$

where K_X is the canonical bundle of X. This number was first introduced by Ballico [B], who called it the sectional genus. Needless to say, $g(X, \mathcal{E})$ is nothing but the usual sectional genus in case $n = 2$.

In this paper we consider the following set-up.

0.1

$\mathcal{E}$ is an ample vector bundle of rank $n-1$ on a complex projective manifold X of dimension $n \geq 3$ and there exists a section $s \in \Gamma(\mathcal{E})$ whose zero locus $Z = (s)_0$ is a smooth curve in X.

Note that if $\mathcal{E}$ is ample and spanned, then the last condition in (0.1) is satisfied with any general section $s \in \Gamma(\mathcal{E})$. Under the assumption (0.1) the smooth curve Z represents the top Chern class $c_{n-1}(\mathcal{E})$, and then the adjunction formula combined with the fact that $N_{Z/X} \cong \mathcal{E}_Z$ shows that $g(X, \mathcal{E})$ coincides with $g(Z)$, the genus of the smooth curve Z. In particular this implies that $g(X, \mathcal{E}) \geq 0$. The classification of pairs $(X, \mathcal{E})$ as in (0.1) with $g(X, \mathcal{E}) = 0$ follows from more general results proven in [LM] and is given by the following

Theorem 0 *Let $(X, \mathcal{E})$ be as in (0.1) with $g(X, \mathcal{E}) = 0$. Then $(X, \mathcal{E})$ is one of the following:*

1. $\big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)}\big)$,
2. $\big(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}\big)$,
3. $\big(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}}(1)^{\oplus(n-1)}\big)$, *where $\mathbb{Q}^n$ is a smooth hyperquadric in $\mathbb{P}^{n+1}$,*
4. *X is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ and $\mathcal{E}_F = \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)}$ for every fibre F of the bundle projection.*

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The aim of this paper is to provide the classification of pairs $(X, \mathcal{E})$ as in (0.1) with $g(X, \mathcal{E}) = 1$. Our result is the following

Theorem 1 *Let $(X, \mathcal{E})$ be as in (0.1) with $g(X, \mathcal{E}) = 1$. Then $(X, \mathcal{E})$ is one of the following:*

1. *X is a $\mathbb{P}^{n-1}$-bundle over an elliptic curve (isomorphic to Z) and $\mathcal{E}_F = \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)}$ for every fibre F of the bundle projection,*
2. $\bigl(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-3)}\bigr)$,
3. $\bigl(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(3) \oplus \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-2)}\bigr)$,
4. $\bigl(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}}(2) \oplus \mathcal{O}_{\mathbb{Q}}(1)^{\oplus(n-2)}\bigr)$,
5. *X is a Del Pezzo manifold with $-K_X = (n-1)\mathcal{H}$ and $\mathcal{E} = \mathcal{H}^{\oplus(n-1)}$,*
6. *$\bigl(\mathbb{P}^3, N(2)\bigr)$, where N is the null correlation bundle on $\mathbb{P}^3$ (see [OSS, p. 76]),*
7. *$\bigl(\mathbb{Q}^3, S(2)\bigr)$, where S is the spinor bundle on $\mathbb{Q}^3$ (see [Ot, Definition 1.3]),*
8. *$\bigl(\mathbb{P}^2 \times \mathbb{P}^1, p^*T_{\mathbb{P}^2} \otimes \mathcal{O}(0,1)\bigr)$, where p denotes the first projection,*
9. $\bigl(\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{O}(2,1) \oplus \mathcal{O}(1,1)\bigr)$,
10. *$\bigl(\mathbb{Q}^4, S(2) \oplus \mathcal{O}_{\mathbb{Q}}(1)\bigr)$, where S is a spinor bundle on $\mathbb{Q}^4$,*
11. *$n = 4$ and $\mathbb{P}(\mathcal{E})$ has two projective $\mathbb{P}^2$-bundle structures over smooth Fano 4-folds of index 1 with $b_2 = 1$ and pseudoindex ≥ 3.*

Remark (i) Any of the pairs listed above except (1) and (11) satisfies the condition (0.1). Indeed, if $(X, \mathcal{E})$ is one of those listed as (2), (3), (4), (6), (7), (8), (9) or (10), then $\mathcal{E}$ is ample and spanned on X, hence (0.1) automatically holds. Now let $(X, \mathcal{E})$ be as in case (5). If $\mathcal{H}^n \geq 2$, then $\mathcal{H}$ is ample and spanned on X by [F1, I, (1.5)]. Thus so is $\mathcal{E}$. If $\mathcal{H}^n = 1$, then it follows from [F1, I, (1.4)] that $(X, \mathcal{H})$ has a smooth ladder. Therefore (0.1) holds for $\mathcal{E} = \mathcal{H}^{\oplus(n-1)}$.

(ii) There exists a pair $(X, \mathcal{E})$ of type (1) satisfying the condition (0.1). To see this, let X be a $\mathbb{P}^{n-1}$-bundle over a smooth projective (not necessarily elliptic) curve C. Then we can write $X = \mathbb{P}_C(\mathcal{F})$ for some vector bundle $\mathcal{F}$ of rank n on C. Take an arbitrary ample line bundle L on C. Then $\mathcal{F} \otimes tL$ is ample and spanned for $t \gg 0$. Let H be the tautological line bundle on X and let π be the projection $X \to C$. Then $H + \pi^*(tL)$ is ample and spanned on X. If we set $\mathcal{E} = \bigl(H + \pi^*(tL)\bigr)^{\oplus(n-1)}$, then $\mathcal{E}$ satisfies (0.1) and $\mathcal{E}_F = \mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)}$ for every fibre F of π.

(iii) We do not know whether a pair $(X, \mathcal{E})$ of type (11), which comes from [PSW, (7.4)], exists or not.

The classification of pairs $(X, \mathcal{H})$ as in (5) is due to Fujita [F1] (see also [F2, Ch. I, Section 8]). For $n = 3$ the same situation as in Theorem 1 was considered by Ballico [B, Theorem 0] on the assumption that $\mathcal{E}$ is spanned in addition. However, (8) and (9) are missing in his list.

Here is the idea of the proof. Since Z is smooth of the expected dimension, we can identify $\mathcal{E}_Z$ with the normal bundle to Z in X; by the adjunction formula, we get $(K_X + \det \mathcal{E})_Z = K_Z$, which is trivial, Z being an elliptic curve. Therefore $K_X + \det \mathcal{E}$ cannot be ample. This allows us to use results of Ye and Zhang [YZ] and of Andreatta, Ballico and Wiśniewski [ABW] providing a list of possibilities for our pairs $(X, \mathcal{E})$, which we have to examine through a case-by-case analysis.

In Section 1 we present a slightly improved version of the result in [ABW], including some descriptions of $\mathcal{E}$, which we need for our purpose. Section 2 is devoted to the proof of Theorem 1.

1 Background Material

In this paper varieties are always assumed to be defined over the complex number field $\mathbf{C}$. We use the standard notation from algebraic geometry. The words "vector bundles" and "locally free sheaves" are used interchangeably. The tensor products of line bundles are denoted additively. The pullback $i^*\mathcal{E}$ of a vector bundle $\mathcal{E}$ on X by an embedding $i\colon Y \hookrightarrow X$ is denoted by $\mathcal{E}_Y$. The canonical bundle of a smooth variety X is denoted by K_X.

Now we state the basic result which we will use to prove our Theorem 1.

Theorem 1.1 *Let $\mathcal{E}$ be an ample vector bundle of rank $n-1$ on a projective manifold X of dimension $n \geq 3$. Then the adjoint bundle $A := K_X + \det \mathcal{E}$ has the following properties:*

A) *([YZ, Theorem 3]) A is nef unless $(X, \mathcal{E})$ is one of the following pairs:*
   - *(a)* $\left(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}}(1)^{\oplus(n-1)}\right)$,
   - *(b)* $\left(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}}(2) \oplus \mathcal{O}_{\mathbf{P}}(1)^{\oplus(n-2)}\right)$,
   - *(c)* $\left(\mathbf{Q}^n, \mathcal{O}_{\mathbf{Q}}(1)^{\oplus(n-1)}\right)$,
   - *(d) X is a $\mathbf{P}^{n-1}$-bundle over a smooth curve and $\mathcal{E}_F = \mathcal{O}_{\mathbf{P}}(1)^{\oplus(n-1)}$ for every fibre F of the bundle projection.*

B) *([ABW, Theorem, B] for $n \geq 4$) Assume that A is nef. Let φ be the morphism defined by tA for $t \gg 0$ and let $\pi\colon X \to W$ be the fibration obtained through its Stein factorization. Then A is big except the following cases:*
   - *(a) X is a Fano manifold and $\det \mathcal{E} = -K_X$,*
   - *(b) π gives X the structure of a $\mathbf{P}^{n-1}$-bundle over a smooth curve W and either $\mathcal{E}_F = \mathcal{O}_{\mathbf{P}}(2) \oplus \mathcal{O}_{\mathbf{P}}(1)^{\oplus(n-2)}$ or $\mathcal{E}_F = T_{\mathbf{P}}$ for every fibre F of π,*
   - *(c) π gives X the structure of a hyperquadric fibration over a smooth curve W and $\mathcal{E}_F = \mathcal{O}_{\mathbf{Q}}(1)^{\oplus(n-1)}$ for a general fibre $F(\cong \mathbf{Q}^{n-1})$ of π,*
   - *(d) π gives X the structure of a $\mathbf{P}^{n-2}$-fibration over a smooth surface W, locally trivial in the complex topology and $\mathcal{E}_F = \mathcal{O}_{\mathbf{P}}(1)^{\oplus(n-1)}$ for every fibre F ($\cong \mathbf{P}^{n-2}$) of π.*

C) *([BeS, Theorem 1.3] for $n = 3$ and [ABW, Theorem, C] for $n \geq 4$) Assume that A is nef and big but not ample. Then there exist a birational morphism $f\colon X \to X'$ expressing X as a projective manifold X' blown up at a finite set B and an ample vector bundle $\mathcal{E}'$ of rank $n-1$ on X' such that $\mathcal{E} = f^*\mathcal{E}' \otimes [-f^{-1}(B)]$ and that $K_{X'} + \det \mathcal{E}'$ is ample.* ■

Remark 1.2 (1) For $n = 3$, case B is not covered by [ABW]. However, the same result as above follows from the argument in [BeS, Proof of Theorem 3.1, starting from line 12 on p. 67].

(2) As to cases (b), (c) and (d) of B, we included in the statement the descriptions of $\mathcal{E}_F$, which do not appear in [ABW, Theorem]. So we give here a proof for the convenience of the reader.

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Proof Let F be an arbitrary fibre of π in cases (b) and (d), and a general fibre of π in case (c). Then its canonical bundle is the restriction of K_X. Since $K_X + \det \mathcal{E} = \pi^* H$ for some ample line bundle H on W by virtue of the fibration theorem (see for example [F3, (1.1)]), we have

$$\mathcal{O}_F = (K_X + \det \mathcal{E})_F = K_F + \det \mathcal{E}_F,$$

i.e., $K_F + \det \mathcal{E}_F$ is trivial. Since $\operatorname{rank} \mathcal{E}_F \geq \dim F$, [F3, Main Theorem] tells us that the descriptions of $\mathcal{E}_F$ are as follows according to cases:

(b) $\mathcal{E}_F$ is either $\mathcal{O}_{\mathbf{P}}(2) \oplus \mathcal{O}_{\mathbf{P}}(1)^{\oplus(n-2)}$ or $T_{\mathbf{P}}$.
(c) $\mathcal{E}_F = \mathcal{O}_{\mathbf{Q}}(1)^{\oplus(n-1)}$.
(d) $\mathcal{E}_F = \mathcal{O}_{\mathbf{P}}(1)^{\oplus(n-1)}$. ■

As mentioned above, $K_X + \det \mathcal{E}$ is the pullback of an ample line bundle on W. This trivially implies the following

Lemma 1.3 *With things as in (b), (c), or (d) in case* B, *let C be a curve on X such that $(K_X + \det \mathcal{E})C = 0$. Then $\pi(C)$ is a point on W.* ■

2 Curve Genus One

Let us prove Theorem 1. Let $(X, \mathcal{E})$ be as in (0.1) with $g(X, \mathcal{E}) = 1$. Then, as explained in the Introduction, the adjoint bundle

(2.0) $$K_X + \det \mathcal{E} \quad \text{is not ample,}$$

so that Theorem 1.1 applies.

2.1

First, let $(X, \mathcal{E})$ be as in (1.1, A). Then cases (a), (b) and (c) cannot occur, since a simple computation shows that $g(X, \mathcal{E}) = 0$ in each case. In case (d), since the restriction s_F of our section s to F is an element of $\Gamma\big(\mathcal{O}_{\mathbf{P}}(1)^{\oplus(n-1)}\big)$ for every fibre F of the bundle projection and Z is irreducible, $Z \cap F = (s_F)_0$ is a single point. Then the $\mathbf{P}$-bundle projection restricted to Z is an isomorphism. This gives case (1) of Theorem 1.

2.2

Now let $(X, \mathcal{E})$ be as in (1.1, B). In case (a), applying [PSW, (0.3), (0.4) and (7.4)] to $(X, \mathcal{E})$, we get cases (2) through (11).

Claim *Cases (b), (c) and (d) do not occur.*

Proof Suppose to the contrary that they occur. We begin with (b) and (c). With the same notation as in (1.1, B), for a general fibre F of π, $\mathcal{E}_F$ is either $\mathcal{O}_{\mathbf{P}}(2) \oplus \mathcal{O}_{\mathbf{P}}(1)^{\oplus(n-2)}$ or $T_{\mathbf{P}}$ in case (b), while $\mathcal{E}_F = \mathcal{O}_{\mathbf{Q}}(1)^{\oplus(n-1)}$ in case (c). Hence in either event $Z \cap F = (s_F)_0 \neq \varnothing$ for a general fibre F. This contradicts Lemma 1.3.

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Now we consider case (d). Since $s_F \in \Gamma\big(\mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)}\big)$ for any fibre F of π, $Z \cap F = (s_F)_0$ is either empty or a single point. Thus $\pi_{|Z}\colon Z \to W$ is an injection, which is also contrary to Lemma 1.3. ■

2.3

Finally, let $(X, \mathcal{E})$ be as in (1.1, C). Then for every exceptional divisor E $(\cong \mathbb{P}^{n-1}) \subset X$ of f we have $s_E \in \Gamma\big(\mathcal{O}_{\mathbb{P}}(1)^{\oplus(n-1)}\big)$. Since Z cannot contain linear spaces of positive dimension, we thus conclude that $Z \cap E = (s_E)_0$ is a single point. Let $Z' := f(Z)$; then Z' is a smooth elliptic curve on X', which is the zero locus of the section $s' \in \Gamma(\mathcal{E}')$ corresponding to s. So $(X', \mathcal{E}')$ also satisfies the assumption in Theorem 1, but this contradicts (2.0), since $K_{X'} + \det \mathcal{E}'$ is ample. We have completed the proof of Theorem 1.

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Dipartimento di Matematica "F. Enriques"
Università
Via C. Saldini, 50
I-20133 Milano
Italy
email: lanteri@vmimat.mat.unimi.it

Department of Mathematical Sciences
School of Science and Engineering
Waseda University
3-4-1 Ohkubo, Shinjuku
Tokyo 169-8555
Japan
email: hmaeda@mse.waseda.ac.jp

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