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Perturbations of Von Neumann Subalgebras With Finite Index

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Abstract. In this paper, we study uniform perturbations of von Neumann subalgebras of a von Neumann algebra. Let M and N be von Neumann subalgebras of a von Neumann algebra with finite probabilistic index in the sense of Pimsner and Popa. If M and N are sufficiently close, then M and N are unitarily equivalent. The implementing unitary can be chosen as being close to the identity.

1 Introduction

In 1972, the uniform perturbation theory of operator algebras was initiated by Kadison and Kastler [15]. They defined a metric on the set of operator algebras on a fixed Hilbert space by the Hausdorff distance between their unit balls. We get basic examples of close operator algebras by small unitary perturbations. Namely, given an operator algebra $N \subset \mathbb{B}(H)$ and a unitary operator $u \in \mathbb{B}(H)$, if u is close to the identity operator, then uNu^* is close to N. Conversely, Kadison and Kastler suggested that suitably close operator algebras must be unitarily equivalent. This conjecture was solved positively for injective von Neumann algebras in [5, 12, 24] with earlier special cases [4, 18]. Cameron et al. [2] and Chan [3] gave classes of non-injective von Neumann algebras for which this conjecture was valid. In [6], for von Neumann subalgebras in a finite von Neumann algebra, Kadison-Kastler conjecture was solved positively. However, for general von Neumann algebras, this conjecture is still open.

Examples of non-separable C^*-algebras that are arbitrary close but non-isomorphic were found in [1]. However, for general separable C^*-algebras, Kadison–Kastler conjecture is still open. In [9], the conjecture was solved positively for separable nuclear C^*-algebras. Earlier special cases of [9] were studied in [7, 16, 19, 20]. The author and Watatani showed that for an inclusion of simple C^*-algebras with finite index, sufficiently close intermediate C^*-subalgebras are unitarily equivalent in [11]. Although our constants depend on inclusions, Dickson obtained universal constants independent of inclusions in [10].

In this paper, we study uniform perturbations of von Neumann subalgebras of a von Neumann algebra with finite index. Let M and N be von Neumann subalgebras of a von Neumann algebra L with conditional expectations $E_M\colon L \to M$ and $E_N\colon L \to N$ of finite probabilistic index in the sense of Pimsner–Popa [21]. If M is sufficiently close to N, then M and N are unitarily equivalent. Moreover, the implementing unitary can be chosen as being close to the identity. In general, there exist examples of arbitrarily

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close unitarily conjugate C^*-algebras where the implementing unitaries could not be chosen to be close to the identity in [13]. Compared with the author and Watatani's C^*-algebraic case [11], we do not assume that M and N have a common subalgebra with finite index.

2 Distance and the Relative Dixmier Property

In this paper, all von Neumann algebras are countably decomposable; that is, they have faithful normal states.

We recall the distance defined by Kadison and Kastler in [15] and near inclusions defined by Christensen in [7]. For a von Neumann algebra N, we denote by N_1 and N^u the unit ball of N and the unitaries in N, respectively.

Definition 2.1 Let M and N be von Neumann algebras in $\mathbb{B}(H)$. Then the distance between M and N is defined by

$$d(M,N) := \max\Big\{ \sup_{n\in N_1} \inf_{m\in M_1} \|n-m\|, \sup_{m\in M_1} \inf_{n\in N_1} \|m-n\| \Big\}.$$

Let $\gamma > 0$. We say that N is γ contained in M and write $N \subseteq_\gamma M$ if for any $n \in N_1$, there exists $m \in M$ such that $\|n-m\| \le \gamma$.

If $d(M,N) < \gamma$, then for any x in either M_1 or N_1, there exists y in the other unit ball such that $\|x-y\| \le \gamma$.

The following well-known fact is needed to show that maps are onto in Proposition 3.1.

Lemma 2.2 *Let M and N be von Neumann algebras in $\mathbb{B}(H)$. If $N \subset M$ and $d(M,N) < 1$, then $M = N$.*

The next lemma records some standard estimates.

Lemma 2.3 *Let A be a unital C^*-algebra.*

(i) *Let $x \in A$ satisfy that $\|x-I\| < 1$ and let $u \in A$ be the unitary factor in the polar decomposition $x = u|x|$. Then*
$$\|u-I\| \le \sqrt{2}\|x-I\|.$$
(ii) *Let p and q be projections in A with $\|p-q\| < 1$. Then there exists a unitary $w \in A$ such that*
$$wpw^* = q \quad \text{and} \quad \|w-I\| \le \sqrt{2}\|p-q\|.$$

Jones introduced an index for inclusions of type II_1 factors in [14]. For arbitrary factors, Kosaki extended Jones' notion of the index in [17]. The following definition was introduced by Pimsner and Popa in [21].

Definition 2.4 Let $N \subset M$ be an inclusion of von Neumann algebras and let $E\colon M \to N$ be a conditional expectation. Then we call E is *of finite probabilistic index* if there exists $c \ge 0$ such that $E(x^*x) \ge cx^*x$ for all $x \in M$. When E is of finite probabilistic index, we define the probabilistic index of E by $(\sup\{c \geq 0 : E(x^*x) \geq cx^*x \text{ for } x \in M\})^{-1}$.

We recall the basic construction (see [22]). Let $N \subset M$ be an inclusion of von Neumann algebras with a faithful normal conditional expectation $E_N: M \to N$ and let ψ be a faithful normal state on N. Put $\phi := \psi \circ E_N$. Then ϕ is a faithful normal state on M. Let (H, π, ξ) be the GNS triplet associated with ϕ. Then we get the Jones projection $e_N \in \mathbb{B}(H)$ satisfying

$$\Im(e_N) = [N\xi] \quad \text{and} \quad e_N(x\xi) = E_N(x)\xi, \quad x \in M.$$

The basic construction $\langle M, e_N \rangle$ is the von Neumann algebra in $\mathbb{B}(H)$ generated by M and e_N. If E_N is of finite probabilistic index, then there exists a conditional expectation $E_M: \langle M, e_N \rangle \to M$ of finite probabilistic index by [22].

Let $N \subset M$ be an inclusion of von Neumann algebras. For any $x \in M$, we will denote by $C_N(x)$ the norm closure of the convex hull of

$$\{uxu^* : u \text{ is unitary element in } N\}.$$

We recall the relative Dixmier property for inclusions of von Neumann algebras after Popa [23].

Definition 2.5 Let $N \subset M$ be an inclusion of von Neumann algebras. Then we say that $N \subset M$ has *the relative Dixmier property* if for any $x \in M$, $C_N(x) \cap N' \cap M \neq \emptyset$.

In [23], Popa proved the following theorem.

Theorem 2.6 (Popa [23]) *Let $N \subset M$ be an inclusion of von Neumann algebras with a conditional expectation $E: M \to N$ of finite probabilistic index. Then $N \subset M$ has the relative Dixmier property.*

We shall establish relations between the relative Dixmier property and the distance.

Let $N \subset M$ be an inclusion of von Neumann algebras. For any $x \in M$, the map $\mathrm{ad}(x): N \to M$ is defined by $(\mathrm{ad}(x))(y) = yx - xy$.

The proof of the next proposition follows from [8, Proposition 2.5].

Proposition 2.7 *Let M and N be von Neumann subalgebras of a von Neumann algebra L with $N \subseteq_\gamma M$. If $N \subset L$ has the relative Dixmier property, then*

$$M' \cap L \subseteq_{2\gamma} N' \cap L.$$

Proof For any $x \in M' \cap L_1$, there exists $y \in C_N(x) \cap N' \cap L$. Since for any unitary $u \in N$,

$$\|uxu^* - x\| = \|ux - xu\| = \|(\mathrm{ad}(x))(u)\| \leq \|\mathrm{ad}(x)\|,$$

we have $\|y - x\| \leq \|\mathrm{ad}(x)\|$. On the other hand, for any $n \in N_1$, there exists $m \in M$ such that $\|n - m\| \leq \gamma$. Thus,

$$\begin{aligned}\|(\mathrm{ad}(x))(n)\| &= \|nx - xn\| = \|nx - mx + xm - xn\| \\ &\leq \|n - m\| + \|m - n\| \leq 2\gamma.\end{aligned}$$

Namely, $\|x - y\| \leq \|\mathrm{ad}(x)\| \leq 2\gamma$. ∎

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3 Perturbations

In the following proposition, we construct a surjective $*$-isomorphism between von Neumann subalgebras of a von Neumann algebra with finite probabilistic index. The argument is originated in early work of Christensen [5, 6].

Proposition 3.1 *Let M and N be von Neumann subalgebras of a von Neumann algebra L with conditional expectations $E_M\colon L \to M$, $E_N\colon L \to N$ of finite probabilistic index. If $d(M,N) < 1/15$, then there exists a normal surjective $*$-isomorphism $\Phi\colon N \to M$ such that $\|\Phi - \mathrm{id}_N\| < 14d(M,N)$.*

Proof Put $\gamma := (1.01)d(M,N)$. Let $\langle L, e_M\rangle$ be the basic construction by using $E_M\colon L \to M$. Then there exists a conditional expectation $E_L\colon \langle L, e_M\rangle \to L$ of finite probabilistic index. Since $E_N \circ E_L\colon \langle L, e_M\rangle \to N$ is of finite probabilistic index, $N \subset \langle L, e_M\rangle$ has the relative Dixmier property by Theorem 2.6. Therefore,

$$M' \cap \langle L, e_M\rangle \subseteq_{2\gamma} N' \cap \langle L, e_M\rangle$$

by Proposition 2.7. Thus, there exists $t \in N' \cap \langle L, e_M\rangle$ such that $\|t - e_M\| \le 2\gamma < 1/2$. Put $p := \chi_{[1-2\gamma, 1+2\gamma]}((t+t^*)/2)$. Since we have $\|p - e_M\| \le \|p - t\| + \|t - e_M\| \le 4\gamma < 1$, there exists a unitary $w \in \langle L, e_M\rangle$ such that

$$we_M w^* = p \quad \text{and} \quad \|w - I\| \le 4\sqrt{2}\gamma$$

by Lemma 2.3. For any $x \in N$, we define $\widetilde{\Phi}(x) := e_M w^* x w e_M = w^* p x p w$. Then $\widetilde{\Phi}\colon N \to e_M\langle L, e_M\rangle e_M$ is a normal $*$-homomorphism, because $p \in N'$. Now, there exists a surjective $*$-isomorphism $\iota\colon e_M\langle L, e_M\rangle e_M \to M$. Hence, we can define a normal $*$-homomorphism $\Phi := \iota \circ \widetilde{\Phi}\colon N \to M$. For any $x \in N_1$,

$$\begin{aligned}\|\Phi(x) - E_M(x)\| &= \|e_M(\Phi(x) - E_M(x))e_M\| = \|e_M w^* x w e_M - e_M x e_M\| \\ &\le 2\|w - I\| \le 8\sqrt{2}\gamma.\end{aligned}$$

Therefore, by [11, Lemma 3.2],

$$\|\Phi - \mathrm{id}_N\| \le \|\Phi - E_M|_N\| + \|E_M|_N - \mathrm{id}_N\| \le (8\sqrt{2} + 2)\gamma < 14d(N,M) < 1.$$

This gives that Φ is a $*$-isomorphism.

Moreover, for any $x \in M_1$, there exists $y \in N_1$ such that $\|x - y\| \le \gamma$. Then

$$\|x - \Phi(y)\| \le \|x - y\| + \|y - \Phi(y)\| \le \gamma + (8\sqrt{2} + 2)\gamma < 15d(N,M) < 1.$$

Since this gives that $d(M, \Phi(N)) < 1$, $\Phi(N) = M$ by Lemma 2.2. ∎

The following is our main theorem in this paper. It is based on Christensen's work [5, Proposition 4.2] and [6, Proposition 3.2].

Theorem 3.2 *Let M and N be von Neumann subalgebras of a von Neumann algebra L with conditional expectations $E_M\colon L \to M$, $E_N\colon L \to N$ of finite probabilistic index. If $d(M,N) < 1/15$, then there exists a unitary $u \in L$ such that $uMu^* = N$ and $\|u - I\| < 20d(M,N)$.*

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Proof By Proposition 3.1, there exists a normal surjective $*$-isomorphism $\Phi: N \to M$ such that $\|\Phi - \mathrm{id}_N\| < 14d(M, N)$. Put

$$K := \left\{ \begin{pmatrix} x & 0 \\ 0 & \Phi(x) \end{pmatrix} : x \in N \right\}.$$

Then we can define a conditional expectation $E_K: \mathbb{M}_2(L) \to K$ of finite probabilistic index by

$$E_K\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \frac{E_N(a)+\Phi^{-1}(E_M(d))}{2} & 0 \\ 0 & \frac{\Phi(E_N(a))+E_M(d)}{2} \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2(L).$$

Therefore, $K \subset \mathbb{M}_2(L)$ has the relative Dixmier property by Theorem 2.6. Applying the relative Dixmier property for $\left(\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right) \in \mathbb{M}_2(L)$, we obtain x in $C_K\left(\left(\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right)\right) \cap K' \cap \mathbb{M}_2(L)$. Then there exists $y \in L$ such that $x = \left(\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}\right)$, because for any unitary $u \in N$,

$$\begin{pmatrix} u & 0 \\ 0 & \Phi(u) \end{pmatrix}\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u^* & 0 \\ 0 & \Phi(u^*) \end{pmatrix} = \begin{pmatrix} 0 & u\Phi(u^*) \\ 0 & 0 \end{pmatrix}.$$

Furthermore,

$$\|y - I\| \le \sup_{u \in N^u} \|u\Phi(u^*) - I\| = \sup_{u \in N^u} \|\Phi(u^*) - u^*\| \le \|\Phi - \mathrm{id}_N\| < 1.$$

By Lemma 2.3, the unitary $u \in L$ in the polar decomposition $y = u|y|$ satisfies

$$\|u - I\| \le \sqrt{2}\|\Phi - \mathrm{id}_N\| < 20d(N, M).$$

Since $x = \left(\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}\right) \in K'$, for any $n \in N$,

$$\begin{pmatrix} 0 & y\Phi(n) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}\begin{pmatrix} n & 0 \\ 0 & \Phi(n) \end{pmatrix} = \begin{pmatrix} n & 0 \\ 0 & \Phi(n) \end{pmatrix}\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ny \\ 0 & 0 \end{pmatrix}.$$

By taking adjoints, we have $\Phi(n)y^* = y^*n$ for any $n \in N$. Therefore,

$$y^*y\Phi(n) = y^*ny = \Phi(n)y^*y, \quad n \in N.$$

This gives $|y|\Phi(n) = \Phi(n)|y|$. Therefore,

$$u\Phi(n) = y|y|^{-1}\Phi(n) = y\Phi(n)|y|^{-1} = ny|y|^{-1} = nu, \quad n \in N.$$

Hence, $uMu^* = u\Phi(N)u^* = N$. ∎

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