


PAPER

What should a generic object be?

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Abstract

Jacobs has proposed definitions for (weak, strong, split) generic objects for a fibered category; building on his definition of (split) generic objects, Jacobs develops a menagerie of important fibrational structures with applications to categorical logic and computer science, including *higher order fibrations*, *polymorphic fibrations*, *λ 2-fibrations*, *triposes*, and others. We observe that a split generic object need not in particular be a generic object under the given definitions, and that the definitions of polymorphic fibrations, triposes, *etc.* are strict enough to rule out some fundamental examples: for instance, the fibered preorder induced by a partial combinatory algebra in realizability is not a tripos in this sense. We propose a new alignment of terminology that emphasizes the forms of generic object appearing most commonly in nature, *i.e.* in the study of internal categories, triposes, and the denotational semantics of polymorphism. In addition, we propose a new class of *acyclic* generic objects inspired by recent developments in higher category theory and the semantics of homotopy type theory, generalizing the *realignment* property of universes to the setting of an arbitrary fibration.

Keywords: category theory; fibered category theory; fibrations; categorical logic; generic objects; universes; tripos theory; polymorphism

1. Introduction

Since the latter half of the 20th century, *fibered category theory* or the *theory of fibrations* has played an important background role in both the applications and foundations of category theory (Bénabou 1985; Grothendieck 1971). Fibered categories, also known as fibrations, are a formalism for manipulating categories that are defined *relative* to another category, generalizing the way that ordinary categories can be thought of as being defined relative to the category of sets.

The sense in which ordinary category theory is pinned to the category of sets can be illustrated by considering the definition of when a category \mathcal{C} “has products”:

Definition 1. A category \mathcal{C} has products when for any indexed family $\{E_i \in \mathcal{C}\}_{i \in I}$ of objects, there exists an object $\prod_{i \in I} E_i \in \mathcal{C}$ together with a family of morphisms $p_k : \prod_{i \in I} E_i \rightarrow E_k$ such that for any family of morphisms $h_k : H \rightarrow E_k$ there exists a unique morphism $h : H \rightarrow \prod_{i \in I} E_i$ factoring each h_k through p_k .

In the above, the dependency on the category **Set** is clear: the indexing object I is a set. If we had required I to be drawn from a proper subcategory of **Set** (*e.g.* finite sets) or a proper supercategory (*e.g.* classes), the notion of product defined thereby would have been different. The purpose of the

formalism of *fibred categories* is to explicitly control the ambient category that parameterizes all indexed notions, such as products, sums, limits, colimits, *etc.*

Remark 2 (Relevance to computer science). The ability to explicitly control the parameterization of products and sums is very important in theoretical computer science, especially for the denotational semantics of *polymorphic types* of the form $\forall \alpha. \tau[\alpha]$. Such a polymorphic type should be understood as the product of all $\tau[\alpha]$ indexed in the “set” of all types, but a famous result of Freyd (2003) shows that if a category \mathcal{C} has products of this form *parameterized in Set*, then \mathcal{C} must be a preorder. Far from bringing to an early end the study of polymorphic types in computer science, awareness of Freyd’s result sparked and guided the search for ambient categories other than **Set** in which to parameterize these products (Hyland 1988; Hyland et al. 1990; Pitts 1987). Fibred category theory provides the optimal language to understand all such indexing scenarios, and the textbook of Jacobs (1999), discussed at length in the present paper, provides a detailed introduction to the applications of fibred category theory to theoretical computer science.

1.1 Introduction to fibred categories

Before giving a general definition, we will see the way that fibred categorical language indeed makes parameterization explicit by considering the *prototype* of all fibred categories, the category **Fam**(\mathcal{C}) of **Set**-indexed families of objects of a category \mathcal{C} .

Construction 3 (The category of families). We define **Fam**(\mathcal{C}) to be the category of **Set**-indexed families in \mathcal{C} , such that

- (1) an object of **Fam**(\mathcal{C}) is a family $\{E_i \in \mathcal{C}\}_{i \in I}$ where I is a set,
- (2) a morphism $\{E_i\}_{i \in I} \rightarrow \{F_j\}_{j \in J}$ in **Fam**(\mathcal{C}) is given by a function $u : I \rightarrow J$ together with for each $i \in I$ a morphism $\bar{u}_i : E_i \rightarrow F_{u_i}$.

There is an evident functor $p : \mathbf{Fam}(\mathcal{C}) \rightarrow \mathbf{Set}$ taking $(\{E_i\}_{i \in I})$ to I .

Construction 4 (Fiber categories). For each $I \in \mathbf{Set}$, we may define the *fiber* **Fam**(\mathcal{C}) $_I$ of **Fam**(\mathcal{C}) over I to be the category of I -indexed families $\{E_i \in \mathcal{C}\}_{i \in I}$ in \mathcal{C} , with morphisms $\{F_i\}_{i \in I} \rightarrow \{E_i\}_{i \in I}$ given by morphisms $h_i : F_i \rightarrow E_i$ for each $i \in I$.

More abstractly, the fiber category **Fam**(\mathcal{C}) $_I$ is the following pullback:

$$\begin{array}{ccc}
 \mathbf{Fam}(\mathcal{C})_I & \longrightarrow & \mathbf{Fam}(\mathcal{C}) \\
 \downarrow & \lrcorner & \downarrow p \\
 \mathbf{1} & \xrightarrow{I} & \mathbf{Set}
 \end{array}$$

Construction 5 (Reindexing functors). For any function $u : J \rightarrow I$, there is a corresponding reindexing functor $u^* : \mathbf{Fam}(\mathcal{C})_I \rightarrow \mathbf{Fam}(\mathcal{C})_J$ that restricts an I -indexed family into a J -indexed family by precomposition.

With the reindexing functors in hand, can now rephrase the condition that \mathcal{C} has products (Definition 1) in terms of **Fam**(\mathcal{C}).

Proposition 6. A category \mathcal{C} has products if and only if for each product projection function $\pi_{I,J} : I \times J \rightarrow I$, the reindexing functor $\pi_{I,J} : \mathbf{Fam}(\mathcal{C})_I \rightarrow \mathbf{Fam}(\mathcal{C})_{I \times J}$ has a right adjoint $\prod_{(I,J)} : \mathbf{Fam}(\mathcal{C})_{I \times J} \rightarrow \mathbf{Fam}(\mathcal{C})_I$ such that the following Beck–Chevalley condition holds: for any function $u : K \rightarrow I$, the canonical natural transformation $u^* \circ \prod_{(I,J)} \rightarrow \prod_{(K,J)} \circ (u \times \text{id}_J)^*$ is an isomorphism.

The characterization of products in terms of the category of families may seem more complicated, but it has a remarkable advantage: we can replace $\mathbf{Fam}(\mathcal{C}) \rightarrow \mathbf{Set}$ with a different functor satisfying similar properties in order to speak more generally of when one category has “products” that are parameterized in another category. The properties that this functor has to satisfy for the notion to make sense are embodied in the definition of a *fibration* or *fibered category*; a functor $\mathcal{E} \rightarrow \mathcal{B}$ will be called a fibration when it behaves similarly to the functor projecting the parameterizing object from a category of families of objects. We begin with an auxiliary definition of *cartesian morphism*:

Definition 7. Let $p : \mathcal{E} \rightarrow \mathcal{B}$ and let $E \rightarrow F$ be a morphism in \mathcal{E} , which we depict as follows:

$$\begin{array}{ccc} E & \longrightarrow & F \\ \downarrow & & \downarrow \\ pE & \longrightarrow & pF \end{array}$$

In the diagram above, we say that $E \rightarrow F$ lies over $pE \rightarrow pF$. We say that $E \rightarrow F$ is cartesian in p when for any morphism $H \rightarrow F$ in \mathcal{E} and $pH \rightarrow pE$ such that the former lies over the composite $pH \rightarrow pE \rightarrow pF$ in \mathcal{B} , there exists a unique morphism $H \rightarrow E$ lying over $pH \rightarrow pE$ such that $H \rightarrow F$ lies over the composite $pH \rightarrow pE$ as depicted below:

$$\begin{array}{ccccc} & & \text{---} \exists! \text{---} & & \\ & \text{---} & & \text{---} & \\ H & & E & \longrightarrow & F \\ \downarrow & & \downarrow & & \downarrow \\ pH & \longrightarrow & pE & \longrightarrow & pF \end{array}$$

Remark 8 (Explication of cartesian maps). Returning to our example of the category of families $\mathbf{Fam}(\mathcal{C})$ over \mathbf{Set} , we can make sense of the notion of a cartesian map. Given a function $u : J \rightarrow I$ of indexing sets, the reindexing functor u^* takes I -indexed families to J -indexed families. Given an I -indexed family $\{E_i\}_{i \in I}$, we may define a morphism $u^*\{E_i\}_{i \in I} \rightarrow \{E_i\}_{i \in I}$ in \mathcal{C} whose first component is $u : J \rightarrow I$ and whose second component is the identity function $E_{uj} \rightarrow E_{uj}$ at each $j \in J$. The morphism $u^*\{E_i\}_{i \in I} \rightarrow \{E_i\}_{i \in I}$ is then *cartesian* in $p : \mathbf{Fam}(\mathcal{C}) \rightarrow \mathbf{Set}$.

Exercise 9. Verify that the morphism $u^*\{E_i\}_{i \in I} \rightarrow \{E_i\}_{i \in I}$ constructed in Remark 8 is indeed cartesian in $p : \mathbf{Fam}(\mathcal{C}) \rightarrow \mathbf{Set}$.

In fact, we can strengthen Exercise 9 to give an extrinsic characterization of cartesian morphisms in $\mathbf{Fam}(\mathcal{C})$: cartesian morphisms are exactly the “fiberwise isomorphisms”.

Exercise 10. Let $E = \{E_i\}_{i \in I}$ and $F = \{F_j\}_{j \in J}$ be objects of $\mathbf{Fam}(\mathcal{C})$, and let $f : E \rightarrow F$ be a morphism between them. Show that the following are equivalent:

- (1) the morphism $f : E \rightarrow F$ is cartesian;
- (2) for each $i \in I$, the component $f_i : E_i \rightarrow F_{pfi}$ is an isomorphism.

The name “cartesian morphism” is inspired by pullbacks, as we see in Exercise 11 below.

Exercise 11. Let \mathcal{B}^\rightarrow be the arrow category of \mathcal{B} , whose objects are morphisms of \mathcal{B} and whose morphisms are commuting squares between them; let $\text{cod} : \mathcal{B}^\rightarrow \rightarrow \mathcal{B}$ be the codomain functor that projects the codomain of a map $A \rightarrow B$. Show that a morphism $E \rightarrow F \in \mathcal{B}^\rightarrow$ is cartesian if and only if the corresponding square in \mathcal{B} is a pullback square (also called a cartesian square).

The relationship between cartesian morphisms and pullback squares expressed by Exercise 11 suggests a generalization of the conventional “pullback corners” notation to an arbitrary fibration, which we shall use liberally.

Convention 12 (Generalized “pullback corners”). Let $p : \mathcal{E} \rightarrow \mathcal{B}$ be a functor; when we wish to indicate that a morphism $E \rightarrow F$ in \mathcal{E} is cartesian over $pE \rightarrow pF$, we will display it using a “pullback corner” notation as follows:

$$\begin{array}{ccc}
 E & \longrightarrow & F \\
 \downarrow & \lrcorner & \downarrow \\
 pE & \longrightarrow & pF
 \end{array}$$

Finally, we may give the definition of a fibration.

Definition 13. A functor $p : \mathcal{E} \rightarrow \mathcal{B}$ is called a fibration when for any object $E \in \mathcal{E}$ and morphism $B \rightarrow pE \in \mathcal{B}$ there exists a cartesian morphism $H \rightarrow E$ lying over $B \rightarrow pE$. The cartesian morphism is often called the cartesian lift of $B \rightarrow pE$ at E .

Convention 14. We will depict a fibration $p : \mathcal{E} \rightarrow \mathcal{B}$ using triangular arrows. When we wish to leave the functor implicit, we refer to \mathcal{E} as a fibered category over \mathcal{B} . In the same way that one writes $A \times B$ for the apex of a product diagram, we will often write $\bar{u} : u^*E \rightarrow E$ for the cartesian lift of $u : B \rightarrow pE$ at E as depicted below:

$$\begin{array}{ccc}
 u^*E & \xrightarrow{\bar{u}} & E \\
 \downarrow & \lrcorner & \downarrow \\
 B & \xrightarrow{u} & pE
 \end{array}$$

In the case of $\mathbf{Fam}(\mathcal{C})$, the existence of cartesian lifts for each $u : B \rightarrow pE$ corresponds to the reindexing functors $u^* : \mathbf{Fam}(\mathcal{C})_{pE} \rightarrow \mathbf{Fam}(\mathcal{C})_B$.

Exercise 15. Verify that the functor $\mathbf{Fam}(\mathcal{C}) \rightarrow \mathbf{Set}$ is a fibration.

Exercise 16. Conclude from Exercise 11 that the codomain functor $\mathcal{B}^\rightarrow \rightarrow \mathcal{B}$ is a fibration if and only if \mathcal{B} has all pullbacks.

When the codomain functor $\mathcal{B}^\rightarrow \rightarrow \mathcal{B}$ is a fibration, we will refer to it as the *fundamental fibration*, written $\mathbf{P}_{\mathcal{B}}$ following Streicher (2021).

1.1.1 Small categories, internal categories

An ordinary category need not have a set of objects – for instance, the category **Set** of all sets has a *proper class* of objects.¹ Likewise, it is possible to find categories such that between two objects there may be a proper class of morphisms (e.g. the category of sets and isomorphism classes of spans between them). A category that has hom *sets* is called *locally small*, and we will refer to a category that has, up to equivalence, a *set* of objects as *globally small*. A category that has both these properties is equivalent to a *small category* in the ordinary sense. Small categories are very useful: for instance, if \mathbb{C} is a small category then the category of functors $[\mathbb{C}, \mathbf{Set}]$ is a Grothendieck topos. Functor categories of this kind play an important role in theoretical computer science (e.g. Birkedal et al. 2011; Oles 1986; Reynolds 1995).

The idea of a (globally, locally) small category can be relativized from **Set** to another category in two *a priori* different ways that ultimately coincide up to equivalence. The simplest and more naïve way to think of a small category \mathbb{C} in a category \mathcal{B} is as an *algebra* for the sorts and operations of the *theory of a category* internal to \mathcal{B} , which we develop below; the more sophisticated way is to view \mathbb{C} as a fibration over \mathcal{B} satisfying a generalization of the global and local smallness conditions.

Definition 17. Let \mathcal{B} be a category that has pullbacks. An internal category or category object in \mathcal{B} is given by:

- (1) an object $\mathbb{C}_0 \in \mathcal{B}$ of objects,
- (2) and an object $\mathbb{C}_1 \in \mathcal{B}$ of morphisms,
- (3) and source and target maps $s, t : \mathbb{C}_1 \rightarrow \mathbb{C}_0$,
- (4) and a morphism $i : \mathbb{C}_0 \rightarrow \mathbb{C}_1$ choosing the identity maps, such that $s \circ i = \text{id}_{\mathbb{C}_0} = t \circ i$,
- (5) and a morphism $c : \mathbb{C}_1 \times_{\mathbb{C}_0} \mathbb{C}_1 \rightarrow \mathbb{C}_1$ choosing composite maps such that $s \circ c = s \circ \pi_1$ and $t \circ c = t \circ \pi_2$,
- (6) (plus associativity and unit laws for composition and identity)

Observation 18. When $\mathcal{B} = \mathbf{Set}$ we obtain exactly the ordinary notion of a small category, i.e. a small category is the same thing as an *internal category* or *category object* in **Set**.

In the previous section, we have argued that fibrations are a fruitful way to think about categories defined relative to another category. Indeed, we may view an internal category \mathbb{C} as a fibration via a process called *externalization*. This proceeds in two steps; first we construct a *presheaf of categories* on \mathcal{B} , and then we use the *Grothendieck construction* to turn it into a fibration.

Construction 19 (The presheaf of categories associated to an internal category). Let \mathbb{C} be an internal category in \mathcal{B} . We may define a presheaf of categories $\mathbb{C}^\bullet : \mathcal{B}^{\text{op}} \rightarrow \mathbf{Cat}$ like so:

- (1) for $I \in \mathcal{B}$, an object of \mathbb{C}^I is given by a morphism $\alpha : I \rightarrow \mathbb{C}_0$,
- (2) for $I \in \mathcal{B}$, a morphism $\alpha \rightarrow \beta \in \mathbb{C}^I$ is given by a morphism $h : I \rightarrow \mathbb{C}_1$ such that $s \circ h = \alpha$ and $t \circ h = \beta$,
- (3) for $u : J \rightarrow I$ in \mathcal{B} , the reindexing $u^* : \mathbb{C}^I \rightarrow \mathbb{C}^J$ is given on both objects and morphisms by precomposition with u .

Construction 20 (The Grothendieck construction). Let $\mathbb{C}^\bullet : \mathcal{B}^{\text{op}} \rightarrow \mathbf{Cat}$ be a presheaf of categories; we define its *total category* $\int_{\mathcal{B}} \mathbb{C}^\bullet$ as follows:

- (1) an object of $\int_{\mathcal{B}} \mathbb{C}^\bullet$ is given by a pair of an object $I \in \mathcal{B}$ and an object $c \in \mathbb{C}^I$,
- (2) a morphism $(J, c) \rightarrow (I, d)$ is given by a pair of a morphism $u : J \rightarrow I \in \mathcal{B}$ and a morphism $c \rightarrow \mathbb{C}^u d$ in \mathbb{C}^J .

There is an evident functor $p : \int_{\mathcal{B}} \mathbb{C}^\bullet \rightarrow \mathcal{B}$; it is this functor that is referred to as the *Grothendieck construction*.

Exercise 21. Verify that the Grothendieck construction of any presheaf of categories $\mathbb{C}^\bullet : \mathcal{B}^{\text{op}} \rightarrow \mathbf{Cat}$ is a fibration.

Observe that the presheaf of categories associated to an internal category is, in each fiber I , the category object in \mathbf{Set} obtained by restricting along the functor $I : \mathbf{1} \rightarrow \mathcal{B}$.

Definition 22 (Externalization of an internal category). *Let \mathbb{C} be an internal category in \mathcal{B} ; its externalization is defined to be the Grothendieck construction $[\mathbb{C}] := \int_{\mathcal{B}} \mathbb{C}^\bullet \rightarrow \mathcal{B}$ of the associated presheaf of categories (Construction 19).*

Remark 23. When \mathbb{C} is an internal category in \mathbf{Set} , the externalization $[\mathbb{C}]$ is the *family fibration* $\mathbf{Fam}(\mathbb{C})$ described in Construction 3.

As promised we may now isolate the properties of the fibered category $[\mathbb{C}]$ that correspond (up to equivalence) to arising by externalization from an internal category.

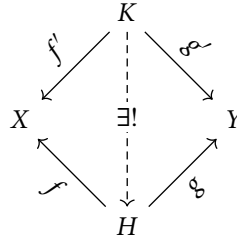
Definition 24. *A fibered category $p : \mathcal{E} \rightarrow \mathcal{B}$ is called globally small if there is an object $T \in \mathcal{E}$ such that for any $X \in \mathcal{E}$ there exists a (not necessarily unique) cartesian map $X \rightarrow T$.*

The property of global smallness described above is often phrased as $p : \mathcal{E} \rightarrow \mathcal{B}$ having a “generic object” $T \in \mathcal{E}$, but the reason for this paper’s existence is that the precise definition of “generic” means in this context is somewhat controversial, and the bulk of the present paper is devoted to justifying the precise meaning for genericity that we have chosen (in agreement with Definition 24). Note that our Definition 24 does not agree with that of Jacobs (1999), who assumes additional (somewhat rare) properties of the object T , namely that for $X \in \mathcal{E}$ there is exactly one morphism $pX \rightarrow pT$ lying underneath a cartesian morphism $X \rightarrow T$. We will show Section 3.1 that ours is the correct definition.

Definition 25 (Bénabou 1975; Streicher 2021). *A fibered category $p : \mathcal{E} \rightarrow \mathcal{B}$ is called locally small when for any $I \in \mathcal{B}$ and $X, Y \in \mathcal{E}_I$ there exists a span $X \xleftarrow{f} H \xrightarrow{g} Y$ with $f : H \rightarrow X$ cartesian over $pg : H \rightarrow Y$ as depicted below,*

$$\begin{array}{ccccc}
 X & \xleftarrow{f} & H & \xrightarrow{g} & Y \\
 \downarrow & & \downarrow & & \downarrow \\
 I & \xleftarrow{pg} & pH & \xrightarrow{pg} & I
 \end{array}$$

such that for any other span $X \xleftarrow{f'} K \xrightarrow{g'} Y$ where $f' : K \rightarrow X$ is cartesian over $pg' : pK \rightarrow pY$, there is a unique map $K \rightarrow H$ making the the following diagram commute:



Exercise 26 (Intermediate). Let \mathcal{C} be an ordinary category show that the fibred category $\mathbf{Fam}(\mathcal{C}) \rightarrow \mathbf{Set}$ is locally small if and only if \mathcal{C} is locally small. Show that \mathcal{C} is equivalent to a small category if and only if $\mathbf{Fam}(\mathcal{C}) \rightarrow \mathbf{Set}$ is both globally small and locally small.

One of the fundamental results of fibred category theory is that, up to equivalence, global and local smallness in the sense of Definitions 24 and 25 suffice to detect internal categories, which we recall from Bénabou (1975):

Proposition 27 (Théorème 2, Bénabou 1975). *A fibration $\mathcal{E} \rightarrow \mathcal{B}$ is equivalent to the externalization of an internal category \mathbb{E} if and only if it is both globally and locally small.*

Although we do not include the (standard) proof of Proposition 27, it is instructive to understand the object $T \in [\mathbb{E}]$ in the externalization of an internal category \mathbb{E} that renders $[\mathbb{E}]$ globally small. Recalling the definition of the externalization via the Grothendieck construction, we define the **weak generic** object T to be the pair $(\mathbb{E}_0, \text{id}_{\mathbb{E}_0} : \mathbb{E}_0 \rightarrow \mathbb{E}_0)$ given by the object of objects and its identity map.

Example 28. When \mathbb{E} is a small category, then the **weak generic** object of $[\mathbb{E}] = \mathbf{Fam}(\mathbb{E})$ can be written as the family $\{E\}_{E \in \mathbb{E}_0}$.

1.1.2 Cleavages and splittings

We briefly recall the definitions of cleavages and splittings for a fibration, as they are relevant to the rest of this paper.

Definition 29. *A cleavage for a fibration $p : \mathcal{E} \rightarrow \mathcal{B}$ is a choice τ of cartesian liftings, assigning to each morphism $u : I \rightarrow pX \in \mathcal{B}$ in the base an object $\tau_u X \in \mathcal{E}$ over I and a cartesian morphism $\overline{\tau}_u : \tau_u X \rightarrow pX$ over u .*

The data of a cleavage τ extends for each $u : I \rightarrow J \in \mathcal{B}$ to a reindexing functor $\tau_u : \mathcal{E}^J \rightarrow \mathcal{E}^I$.

Definition 30. *A cleavage τ for a fibration $p : \mathcal{E} \rightarrow \mathcal{B}$ is called split when the assignment of reindexing functors $u \mapsto \tau_u$ strictly preserves identities and compositions.*

Convention 31. When the cleavage τ is understood, we will often write u^* for τ_u .

A fibration equipped with a cleavage is called a *cloven fibration*; we may use the axiom of choice to equip any fibration with a (non-canonical) cleavage. When the cleavage associated to a cloven fibration is split, we speak of *split fibrations*. Observe that a splitting allows one to view a fibration $\mathcal{E} \rightarrow \mathcal{B}$ as a presheaf of categories $\mathcal{B}^{\text{op}} \rightarrow \mathbf{Cat}$ sending each $I \in \mathcal{B}$ to the fiber \mathcal{E}^I . Recalling Construction 19, we see that the externalization of any internal category is split.

Table 1. A Rosetta stone for the different terminologies for generic objects

Our Proposal	Jacobs (1999)	Phoa (1992)	Hermida (1993)	Streicher (2005)
<u>weak generic</u>	–	–	–	weak generic
<u>generic</u>	weak generic	generic	generic	generic
<u>acyclic generic</u>	–	–	–	–
<u>skeletal generic</u>	generic	–	–	–
<u>gaunt generic</u>	strong generic	skeletal generic	strong generic	classifying

1.2 Goals and structure of this paper

Although the property stated in Definition 24, that there exist cartesian morphisms $X \rightarrow T$ for any X , is the most that can be required for an arbitrary internal category, more restrictive notions of generic object have proved important in practice for different applications. Unfortunately, over the years a number of competing definitions have proliferated throughout the literature – and some of the more established of these definitions lead to false conclusions when taken too literally, as we point out in Section 3.1 in our discussion of Jacobs’ mistaken Corollary 9.5.6.

The goal of this paper, therefore, is to argue for a new alignment of terminology for the different forms of generic object that is both internally consistent *and* reflects the use of generic objects in practice. Because generic objects play an important role in several areas of application (categorical logic, algebraic set theory, homotopy type theory, denotational semantics of polymorphism, *etc.*), we believe that we have sufficient evidence today to correctly draw the map.

- In Section 2, we recall the definitions of several variants of generic object by Jacobs (1999); our main observation is that a **split generic** object in the sense of *op. cit.* need not be a **generic** object in the same sense.
- In Section 3, we analyze the consequences of the definitions discussed in Section 2 for the use of generic objects in internal category theory, tripos theory, denotational semantics of polymorphism, algebraic set theory, and homotopy type theory.
- In Section 4, we propose new unified terminology and definitions for all extant forms of generic object (as well as one *new* one). Our proposal is summarized and compared with the literature in Table 1.

2. Four Kinds of Generic Object

We begin by recalling Definition 5.2.8 of Jacobs (1999), from which we omit some additional characterizations that will not play a role in our analysis.

Consider a fibration $p : \mathcal{E} \rightarrow \mathcal{B}$ and an object T in the total category \mathcal{E} . We call T a

- (i) **weak generic** object if $\forall X \in \mathcal{E}. \exists f : X \rightarrow T.f$ is cartesian.
- (ii) **generic** object if $\forall X \in \mathcal{E}. \exists ! u : pX \rightarrow pT. \exists f : X \rightarrow T.f$ is cartesian over u .
- (iii) **strong generic** object if $\forall X \in \mathcal{E}. \exists ! f : X \rightarrow T.f$ is cartesian.

Jacobs (1999) then defines **split generic** objects for split fibrations in Definition 5.2.1, paraphrased below:

A split fibration $p : \mathcal{E} \rightarrow \mathcal{B}$ has a **split generic** object if there is an object $\Omega \in \mathcal{B}$ together with natural isomorphism $\theta : \mathcal{B}(-, \Omega) \rightarrow \text{ob } \mathcal{E}_\bullet$ in $[\mathcal{B}^{\text{op}}, \mathbf{Set}]$, where the presheaf $\text{ob } \mathcal{E}_\bullet$ is defined using the splitting.

A useful characterization of **split generic** objects is given in Lemma 5.2.2 of *op. cit.*:

*A split fibration $p : \mathcal{E} \rightarrow \mathcal{B}$ has a **split generic** object if and only if there is an object $T \in \mathcal{E}$ with the property that $\forall X \in \mathcal{E}. \exists ! u : pX \rightarrow pT. u^*T = X$. (Jacobs 1999)*

Comment 32. The **weak** and **strong generic** objects of Jacobs (1999) are referred to by Phoa (1992) as *generic objects* and *skeletal generic objects*. Phoa (1992) does not consider the intermediate notion. On the other hand, Phoa (1992) defines *strict generic objects* relative to an arbitrary (non-split) cleavage; a **split generic** object is indeed a *strict generic object* in the sense of Phoa, but even for a split cleavage, a *strict generic object* need not be a **split generic** object. We will discuss Phoa’s terminology more in Comment 58.

2.1 Separating generic objects from strong generic objects

Jacobs (1999) notes that **generic** and **strong generic** objects coincide in fibered preorders, but they may differ otherwise – the difference emanating from the presence of non-trivial vertical automorphisms.

Example 33. Let G be a group containing two distinct elements $u \neq v$, and let \mathbf{BG} be the groupoid with a single object whose hom set is G itself. The family fibration $\mathbf{Fam}(\mathbf{BG}) \rightarrow \mathbf{Set}$ is a fibered category whose objects are just sets, but such that a morphism $f : I \rightarrow J \in \mathbf{Fam}(\mathbf{BG})$ is a pair (f, x) of a function $f : I \rightarrow J \in \mathbf{Set}$ together with a generalized element $x : I \rightarrow G$. Moreover, every morphism in $\mathbf{Fam}(\mathbf{BG})$ is cartesian as \mathbf{BG} is a groupoid. The unique object $T \in \mathbf{Fam}(\mathbf{BG})_{1\mathbf{Set}}$ is clearly **generic**, but not **strong generic**. Indeed, we have two distinct cartesian morphisms $T \rightarrow T$ given by the pairs $(\text{id}_{1\mathbf{Set}}, u) \neq (\text{id}_{1\mathbf{Set}}, v)$.

Example 34. Another class of examples comes from considering skeleta of full subcategories of \mathbf{Set} . For instance, one may take the skeleton of a Grothendieck universe and then externalize to obtain a fibration that has a **generic** object that is not **strong**.

2.2 A split generic object need not be a generic object

Construction 35 (The canonical splitting of the externalization). The externalization of an internal category is split in a canonical way: given $(I, c) \in [\mathbb{C}]$ and $u : J \rightarrow I$, we choose $u^*(I, c) = (J, c \circ u)$. The cartesian morphism $u^*(I, c) \rightarrow (I, c)$ is given by the pair $(u, \text{id} \circ c \circ u)$ where $\text{id} : \mathbb{C}_0 \rightarrow \mathbb{C}_1$ is the generic identity morphism.

Construction 36 (Weak, split generic objects in the externalization). The externalization $p : [\mathbb{C}] \rightarrow \mathcal{B}$ has a **weak generic** object $T = (\mathbb{C}_0, \text{id}_{\mathbb{C}_0})$. Relative to the splitting of $[\mathbb{C}]$ from Construction 35, the **weak generic** object T is also a **split generic** object.

The **weak generic** object of the externalization of an internal category defined in Construction 36 obviously need not be a **strong generic** object, but it may be more surprising to learn that it also need not be a **generic** object at all. This can happen, for instance, when the internal category \mathbb{C} has two distinct isomorphic objects; the following concrete example illustrates the problem.

Example 37 (A **split generic** object that is not a **generic** object). Let U be a set of sets containing two distinct elements A, B with the same cardinality, and $\mathbf{Set}_U \subseteq \mathbf{Set}$ be the full subcategory of \mathbf{Set} spanned by U . Then, the family fibration $\mathbf{Fam}(\mathbf{Set}_U) \rightarrow \mathbf{Set}$ has a **split generic** object T given by

the pair $T = (U, \text{id}_U)$, but T is nonetheless not a **generic** object. Indeed, we have two cartesian morphisms $(1_{\text{Set}}, A) \rightarrow T$ lying over distinct elements $A \neq B : 1_{\text{Set}} \rightarrow U$, respectively.

Corollary 38. *A split generic object is not necessarily a generic object.*

2.3 Skeletal and gaunt small categories

We recall from Example 28 that for a small category \mathbb{C} , the family fibration $\mathbf{Fam}(\mathbb{C})$ has a **weak generic** object $T = \{C\}_{C \in \mathbb{C}_0}$. We will relate properties of the category \mathbb{C} to corresponding properties of the **weak generic** object T .

Definition 39. *A category is called skeletal when any two isomorphic objects are equal.*

Definition 40. *A category is called gaunt when any isomorphism in that category is the identity.*

Comment 41. The term *gaunt* is used by Barwick and Schommer-Pries (2011), nLab (2021), Univalent Foundations Program (2013). In passing, Johnstone (2002) has referred to such categories as *stiff*.

Lemma 42. *The category \mathbb{C} is (respectively skeletal, gaunt) if and only if T is (respectively, **generic**, **strong generic**).*

Proof. (1a) If \mathbb{C} is skeletal, then T is **generic**; fix any family $\{D_i\}_{i \in I}$ and cartesian map $\chi : \{D_i\}_{i \in I} \rightarrow \{C\}_{C \in \mathbb{C}}$. By Exercise 10, $p\chi$ sends each $i \in I$ to an object $p\chi_i \in \mathbb{C}$ that is isomorphic to D_i ; as \mathbb{C} is skeletal, it follows that $p\chi_i = D_i$ and so we conclude that T is **generic**.

(1b) Assume conversely that T is **generic** and fix an isomorphism $f : D_0 \cong D_1$ in \mathbb{C} . We have two cartesian maps $h_0, h_1 : \{D_0\} \rightarrow T$, with one lying over $D_0 : \{*\} \rightarrow \mathbb{C}_0$ via the identity morphism and the other lying over $D_1 : \{*\} \rightarrow \mathbb{C}_0$ via f . Since T is **generic**, these two cartesian maps must lie over the same element of \mathbb{C}_0 , so we have $D_0 = D_1$.

(2a) If \mathbb{C} is gaunt, then T is **strong generic**; fix any two cartesian morphisms $h_0, h_1 : \{D_i\}_{i \in I} \rightarrow \{C\}_{C \in \mathbb{C}}$. Because \mathbb{C} is gaunt and thus skeletal, we know that $ph_0 = ph_1$; thus h_0 assigns to each $i \in I$ an isomorphism $h_{0,i} : D_i \rightarrow ph_{1,i}$ which is (by assumption) necessarily the identity. Thus, both h_0 and h_1 must send every $i \in I$ to the identity map on D_i and are thus equal.

(2b) Conversely, we assume that T is **strong generic** to check that any isomorphism in \mathbb{C} is an identity map; since T is necessarily also **generic**, it follows by the first case of the present lemma that we may consider just the automorphisms in \mathbb{C} , considering Lemma 48. To show that any automorphism $f : D \cong D$ in \mathbb{C} is the identity morphism, we proceed exactly as in the proof of Lemma 49 by observing that the two cartesian maps corresponding to the identity map and the automorphism f respectively are necessarily equal by our assumption that T is **strong generic**. \square

2.4 Generic objects from weak generic objects

Although the externalization of an internal category necessarily has a (split) **weak generic** object, in some cases it may also have a **generic** object T' that embodies the *skeleton* of \mathbb{C} as in Example 34, but T' is usually different from the T .

Construction 43 (Computing the skeleton of a small category). Suppose that $\mathcal{B} = \mathbf{Set}$ and thus \mathbb{C} is an ordinary small category. Then we may consider the quotient \mathbb{C}_0 / \cong of the objects of \mathbb{C} under isomorphism; in other words, this is the set of isomorphism classes of \mathbb{C} -objects. Using the axiom

of choice, we may arbitrarily choose a section $s : \mathbb{C}_0 / \cong \rightarrow \mathbb{C}_0$ to the quotient map; moreover, we may choose a function associating to each $u \in \mathbb{C}_0$ an isomorphism $u \cong s[u]_{/\cong}$.

Lemma 44. *The pair $T' = (\mathbb{C}_0 / \cong, s)$ is a **generic object** for $\mathbf{Fam}(\mathbb{C})$.*

Proof. Fixing $(I, c) \in \mathbf{Fam}(\mathbb{C})$ we must choose a unique $u : I \rightarrow pT'$ such that there exists a cartesian map $(I, c) \rightarrow T$ lying over u . We choose $u(i) = [c(i)]_{/\cong}$, taking each index $i \in I$ to the isomorphism class of $c(i)$.

- (1) First of all, it is clear that there exists a cartesian map lying over u in the correct configuration.
- (2) Fixing $v : I \rightarrow pT'$ such that there exists a cartesian map $(I, c) \rightarrow T$ lying over v , it remains to show that $v = u$. This follows because such a cartesian map ensures that v and u are the same family of isomorphism classes of objects. □

Lemma 45. *If the T' defined above is a **split generic object**, then \mathbb{C} is skeletal.*

Proof. We have already seen that T is a **split generic object**, hence if T' is also **split generic** we have isomorphisms $pT \cong \mathbb{C}_0 \cong pT'$ and so we have $\mathbb{C}_0 \cong \mathbb{C}_0 / \cong$. □

We finally observe in Lemma 49 that if T' is **strong generic**, then \mathbb{C} is *essentially gaunt* in the sense defined below.

Definition 46. *A category is called essentially gaunt when it is equivalent to a gaunt category.*

Simon Henry has made the following observation:²

Observation 47. A category is essentially gaunt if and only if any *automorphism* in that category is the identity.

Lemma 48. *A category is gaunt if and only if it is skeletal and essentially gaunt.*

Proof. If \mathbb{C} be gaunt, it is obviously both essentially gaunt and skeletal. Conversely, if \mathbb{C} is essentially gaunt and skeletal, given any isomorphism $f : D \cong C$ we have $D = C$ and this f is an automorphism, which (by gauntness) is the identity. □

Lemma 49. *If T' as defined above is a **strong generic object**, then \mathbb{C} is essentially gaunt in the sense of Definition 46.*

Proof. Let $f : c \rightarrow c$ be an automorphism in \mathbb{C} , i.e. a vertical isomorphism in $\mathbf{Fam}(\mathbb{C})_{1_{\mathbf{Set}}}$. Since T' is **strong generic**, there exists a unique cartesian morphism $(1_{\mathbf{Set}}, c) \rightarrow T'$; this means that there is a unique element $[c] \in \mathbb{C}_0 / \cong$ and a unique isomorphism $h : c \rightarrow s[c]$. Writing $\phi_c : c \cong s[c]$ for the (globally) chosen isomorphism, we have $f; \phi_c = h = \phi_c$ and hence $f = \text{id}_c$, so \mathbb{C} is essentially gaunt. □

Later on in Section 4, we will see that the original **weak generic** object T being **strong generic** corresponds to \mathbb{C} being gaunt.

Thus, we conclude that although the family fibration $\mathbf{Fam}(\mathbb{C})$ over \mathbf{Set} of a small category \mathbb{C} does have a **generic object**, this generic object cannot be either a **split generic object** or a **strong generic object** except in somewhat contrived scenarios.

2.5 Weak generic objects are the correct generalization of split generic objects

It is clear that any split generic object is in particular a weak generic object; but the converse *also* holds in a certain sense that we make precise below.

Construction 50 (Presheaf of categories). Let $p : \mathcal{E} \rightarrow \mathcal{B}$ be a fibered category equipped with a cleavage τ , and let $T \in \mathcal{E}$ be a **weak generic** object for T . We may construct a presheaf of categories $\mathcal{E}^\bullet : \mathcal{B}^{\text{op}} \rightarrow \mathbf{Cat}$ like so:

- (1) an object of \mathcal{E}^I is a morphism $\alpha : I \rightarrow pT$,
- (2) a morphism $\alpha \rightarrow \beta$ in \mathcal{E}^I is a vertical map $h : \tau_\alpha T \rightarrow \tau_\beta T$ over I .

Construction 51. Let $\mathbb{E} : \mathcal{B}^{\text{op}} \rightarrow \mathbf{Cat}$ be a presheaf of categories; then the Grothendieck construction $q : \int \mathbb{E} \rightarrow \mathcal{B}$ has a splitting. Given $(J, \alpha) \in \int \mathcal{E}$ and $u : I \rightarrow J$, the object $u^*(J, \alpha)$ is chosen to be $(I, \alpha \circ u)$ and the cartesian morphism $u^*(J, \alpha) \rightarrow (J, \alpha)$ over u in $\int \mathcal{E}^\bullet$ is defined to be the pair $(u, \text{id}_{u^*\alpha})$.

Lemma 52. *There is a fibered equivalence from $\int \mathcal{E}^\bullet$ to \mathcal{E} over \mathcal{B} .*

Lemma 53. *The pair $T' = (T, \text{id}_T)$ is a **split generic** object in $\int \mathcal{E}^\bullet \rightarrow \mathcal{B}$.*

Proof. Fixing $X \in \int \mathcal{E}^\bullet$, we must check that there exists a unique $u : pX \rightarrow pT'$ such that $u^*T' = X$. Unfolding definitions, we fix $I \in \mathcal{B}$ and $\alpha : I \rightarrow T$ to check that there is a unique $u : I \rightarrow T$ such that $(I, u) = (I, \alpha)$. Of course, this is true with $u = \alpha$. □

Thus, a **weak generic** object for a fibration $\mathcal{E} \rightarrow \mathcal{B}$ generates in a canonical way a new *equivalent* split fibration $\mathcal{E}' \rightarrow \mathcal{B}$ that has a **split generic** object. This is the correspondence between **weak generic** and **split generic** objects.

3. Consequences for Internal Categories, Tripos Theory, Polymorphism, etc.

We recall several definitions from Jacobs (1999) below in order to illustrate a pattern.

- (1) A **higher order fibration** is a first order fibration³ with a **generic** object and a cartesian closed base category. Such a higher order fibration will be called **split** if the fibration is split and all of its fibered structure (including the **generic** object) is **split**. (Jacobs 1999, Definition 5.3.1)
- (2) A **tripos** is a **higher order fibration** $\mathcal{E} \rightarrow \mathbf{Set}$ over \mathbf{Set} for which the induced products \prod_u and coproducts \coprod_u along an arbitrary function u satisfy the Beck–Chevalley condition. (Jacobs 1999, Definition 5.3.3)
- (3) A **polymorphic fibration** is a fibration with a **generic** object, with fibered finite products and with finite products in its base category. It will be called **split** whenever all this structure is **split**. (Jacobs 1999, Definition 8.4.1).
- (4) A **polymorphic fibration** with Ω in the base as a generic object will be called
 - (a) a **$\lambda \rightarrow$ -fibration** if it has fibered exponents;
 - (b) a **$\lambda 2$ -fibration** if it has fibered exponents and also simple Ω -products and coproducts;
 - (c) a **$\lambda \omega$ -fibration** if it has fibered exponents, simple products, and coproducts, and exponents in its base category. (Jacobs 1999, Definition 8.4.3)
- (5) Let \diamond be $\rightarrow, 2, \omega$. A $\lambda \diamond$ -fibration will be called **split** if all of its structure is **split**. In particular, its underlying **polymorphic fibration** is **split**. (Jacobs 1999, Definition 8.4.4)

A **split generic** object need not be a **generic** object as we have seen in Section 2.2, and indeed, is only quite rarely one. Consequently, a **split polymorphic fibration** is not the same thing as a

split fibration with a **split generic** object and split fibered finite products. This disorder suggests that a change of definitions is in order, which we propose in Section 4.

3.1 Consequences for internal category theory

The somewhat chaotic status of **generic** objects *vis-à-vis* **split generic** objects has led to an erroneous claim by Jacobs (1999) that the externalization of an internal category has a **generic** object. What is actually proved by *op. cit.* in Lemma 7.3.2 is that the externalization of an internal category has a **split generic** object, but this is later claimed (erroneously) to give rise to a **generic** object in the proof of Corollary 9.5.6. Thus, the claimed result of Corollary 9.5.6, that a fibration is small (equivalent to the externalization of an internal category) if and only if it is globally small (has a **generic** object) and locally small, is mistaken. To give a correct definition in the terminology of *op. cit.*, one would define globally small categories to assume a **weak generic** object, as we have done in Definition 24.

3.2 Consequences for tripos theory

Jacobs (1999) sketches in Example 5.3.4 the standard construction of a tripos from a partial combinatory algebra, referring to Hyland et al. (1980) for several parts of the construction (including the **generic** object); the standard definition of a tripos involves a **weak generic** object only (Hyland et al. 1980; Pitts 1981, 2002), a discrepancy that has already been noted by Birkedal (2000, p. 110).

This difference is quite destructive, as the **weak generic** object of a realizability tripos will not generally be **generic**, as pointed out by Streicher (2017, Theorem 4.6). Thus, we conclude that definition of tripos proposed by Jacobs (1999) rules out the main examples of naturally occurring triposes.

Remark 54. A final subtlety: over the category of assemblies, the fibration of *regular* subobjects does indeed have a **strong generic** object, but this is not part of the structure of the tripos.

3.3 Consequences for polymorphism

The reception in the community studying polymorphism has been to either avoid or tacitly correct the definition of **generic** object. For instance, Hermida (1993) speaks of **weak generic** objects and **strong generic** objects and gives the correct definition of $\lambda \rightarrow, \lambda 2, \lambda \omega$ fibration in terms of **weak generic** objects. On the other hand Birkedal et al. (2005), Johann and Sojakova (2017), Sojakova and Johann (2018), Ghani et al. (2019) deal mainly with the **split generic** objects, and thus do not seem to run into problems. It can be seen, however, that the examples of (split) $\lambda \diamond$ -fibrations in the cited works are *not* in fact (**split**) $\lambda \diamond$ -fibrations in the sense of Jacobs (1999), because they do not have **generic** objects. Of course, the problem lies with the definitions rather than the examples.

3.4 Consequences for type theory and algebraic set theory

The idea of a *universe* in a category has been abstracted from Grothendieck's universes (Artin et al. 1972) by way of the contributions of a number of authors including Bénabou (1973), Martin-Löf (1975), Street (1980), Joyal and Moerdijk (1995), Hofmann and Streicher (1997), Streicher (2005). In fibered categorical language, a universe usually is a *full subfibration* that has a **weak generic** object – potentially equipped with additional structure.

In certain cases, the **weak generic** object of a universe is also a **generic** object that is nonetheless not strong; one example is Voevodsky's construction of a universe of well-ordered simplicial sets in the context of the simplicial model of homotopy type theory, reported by Kapulkin and

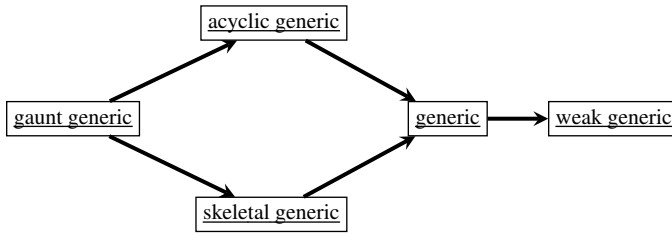


Figure 1. An analysis of the comparability of different notions of generic object, where the rightward direction represents (strictly) decreasing strength.

Lumsdaine (2021): it is **strong generic** for the fibered category of well-ordered families of simplicial sets and order-preserving morphisms, but only **generic** when considering morphisms that need not preserve the well-orderings. On the other hand, universes of propositions, such as the subobject classifier of a topos or the regular subobject classifier of a quasitopos, are the main source of **strong generic** objects in nature.

The theory of universes as applied to *type theory* on the one hand and *algebraic set theory* on the other hand motivates two additional variants of generic object:

- (1) In applications to type theory, it has been increasingly important for universes to have a **weak generic** object T that satisfies an additional *realignment* property relative to a class of monomorphisms \mathcal{M} ; in particular, given a span of cartesian maps $U \leftarrow X \rightarrow T$ where $p(U \leftarrow X) \in \mathcal{M}$, we need an extension $U \rightarrow T$ factoring $X \rightarrow T$ through $U \rightarrow X$. This realignment property has proved essential for the semantics of univalent universes in homotopy type theory (Gratzer et al. 2022; Kapulkin and Lumsdaine 2021; Shulman 2015, 2019) as well as Sterling’s *synthetic Tait computability* (Sterling 2021), an abstraction of Artin gluing and logical relations. In Section 4.3, we will discuss the generalization of the realignment property to an arbitrary fibered category as the acyclic generic object, which lies strictly between **weak generic** and **generic** objects.
- (2) In algebraic set theory (Joyal and Moerdijk 1995), universes are considered that enjoy a form of generic object that is even weaker than **weak generic** object. For each $E \in \mathcal{E}$ one only “locally” has a morphism into T ; the “very weak” generic objects of algebraic set theory seem to relate to **weak generic** objects in the same way that weak completeness relates to strong completeness in the context of stack completions (Hyland 1988; Hyland et al. 1990), with important applications to the theory of polymorphism. As these very weak generic objects seem to play a fundamental role, we discuss them in more detail in Section 4.2 as part of our proposal for a new alignment of terminology.

4. A Proposal for a New Alignment of Terminology

Based on the data and experience of the applications of fibered categories in the theory of triposes, polymorphism, and universes in type theory and algebraic set theory, we may now proceed with some confidence to propose a new alignment of terminology for generic objects that better reflects fibered categorical practice. In this section, we distinguish our proposed usage from that of other authors by underlining.

We will define several notions of generic object in conceptual order rather than in order of strength; in Table 1, we summarize our terminology and compare it to several representatives from the literature. In Figure 1, we compare the strength of the different kinds of generic object.

4.1 Generic objects: Ordinary, skeletal, and gaunt

Let $p: \mathcal{E} \rightarrow \mathcal{B}$ be a fibered category. Our most basic definition below corresponds to the **weak generic** objects of Jacobs (1999).

Definition 55. A *generic* object for $p: \mathcal{E} \rightarrow \mathcal{B}$ is defined to be an object $T \in \mathcal{E}$ such that for any $X \in \mathcal{E}$ there exists a cartesian morphism $X \rightarrow T$.

Definition 56. A *generic* object $T \in \mathcal{E}$ is called *skeletal* when for any $X \in \mathcal{E}$, there exists a unique $pX \rightarrow pT$ such that there exists a cartesian map $X \rightarrow T$ lying over $pX \rightarrow pT$.

Definition 57. A *generic* object $T \in \mathcal{E}$ is called *gaunt* when for any $X \in \mathcal{E}$, any two cartesian morphisms $X \rightarrow T$ are equal.

Comment 58. Our *skeletal* and *gaunt* generic objects are the same as Jacobs' **generic** objects and **strong generic** objects, respectively. Our terminology is inspired by a comparison between the properties of an internal category \mathbb{C} and its externalization: in particular, the generic object over \mathbb{C}_0 is *skeletal* when \mathbb{C} is a skeletal category and it is *gaunt* when \mathbb{C} is a gaunt category. Unfortunately, Phoa (1992) has used the word "skeletal" to describe what we call *gaunt* generic objects; but this usage accords with *op. cit.*'s unconventional definition of a *skeletal category*: usually a skeletal category is one in which any two isomorphic objects are equal, but Phoa defines it to be one in which the only isomorphisms are identity maps. Thus, Phoa's skeletal categories would ordinarily be referred to as *gaunt* categories.

To illustrate the comparison between (skeletal, gaunt) categories and (*skeletal*, *gaunt*) generic objects, respectively, we recall Lemma 42, which is restated in the new terminology as follows:

Corollary 59. Let \mathbb{C} be a small category; then \mathbb{C} is (respectively skeletal, gaunt) if and only if the *generic* object $T = \{C\}_{C \in \mathbb{C}_0}$ for $\mathbf{Fam}(\mathbb{C})$ is (respectively *skeletal*, *gaunt*).

4.2 Weak generic objects and stack completions

In this section, we are concerned with *stacks* – a 2-dimensional variation on the concept of a sheaf.

Definition 60. Suppose that a pretopos \mathcal{B} is equipped with the structure of a site. Then a fibration $\mathcal{E} \rightarrow \mathcal{B}$ is a *stack* when for any object $X \in \mathcal{B}$ and covering sieve $S \subseteq \mathcal{B} \downarrow X$, the canonical restriction functor $\mathcal{E}_X \rightarrow \text{hom}_{\mathbf{Fib}_{\mathcal{B}}}(S, \mathcal{E})$ is an equivalence.⁴

Our examples will mainly concern stacks on a pretopos \mathcal{B} equipped with its regular topology.

Definition 61 (Frey 2013, Definition 2.3.5). A cartesian morphism $X \rightarrow Y$ in a fibered category $p: \mathcal{E} \rightarrow \mathcal{B}$ is called *cover-cartesian* when $pX \twoheadrightarrow pY$ is a regular epimorphism.

Definition 62. A *weak generic* object for $p: \mathcal{E} \rightarrow \mathcal{B}$ is defined to be an object $T \in \mathcal{E}$ such that for all $X \in \mathcal{E}$ there exists a span of cartesian maps $X \leftarrow \tilde{X} \rightarrow T$ where $\tilde{X} \rightarrow X$ is cover-cartesian.

Comment 63. Our definition and terminology agrees with that of Streicher (2005, (5.2)) and Battenfeld (2008, Definition 6.3.7), differing only in that both of the cited works specialize the definition for Bénabou-definable full subfibrations of the fundamental fibration of a topos, *i.e.* full subfibrations that are stacks. Our definition is also identical to the *representability axiom* (S2) for classes of small maps in algebraic set theory (Joyal and Moerdijk 1995, Definition 1.1).

Remark 64. The notion of weak generic object defined above should be thought of as an *internal* version of the property of being a generic object – internal to a fibration that is a stack for the regular topology. Indeed, our explicit definition ought to be the translation of ordinary genericity through a generalization of Shulman’s stack semantics (Shulman 2010) that is stated for stacks other than the fundamental fibration $\mathbf{P}_{\mathcal{B}} = \mathcal{B}^{\rightarrow} \rightarrow \mathcal{B}$. The same (informal) translation is used by Hyland et al. (1990) to correctly define *weak equivalences* and *weak completeness* for categories fibered over a regular category.

Weak generic objects in the sense of Definition 62 arise very naturally.

Example 65. Let $\pi : E \rightarrow U$ be a morphism in a topos \mathcal{B} ; then the class of maps arising as pullbacks of $\pi : E \rightarrow U$ determines a full subfibration $[\pi] \subseteq \mathbf{P}_{\mathcal{B}}$ of the fundamental fibration $\mathbf{P}_{\mathcal{B}} = \mathcal{B}^{\rightarrow} \rightarrow \mathcal{B}$ for which $\pi : E \rightarrow U$ is a generic object. The *stack completion* of $[\pi]$ is a weakly (but not necessarily strongly) equivalent full subfibration $\{\pi\} \subseteq \mathbf{P}_{\mathcal{B}}$, and it can be computed like so: an object of $\{\pi\}_I$ is a morphism $X \rightarrow I$ such that there exists a regular epimorphism $\tilde{I} \twoheadrightarrow I$ such that the pullback $X \times_I \tilde{I} \rightarrow \tilde{I}$ lies in $[\pi]_{\tilde{I}}$. In other words, we have the pullback squares in the following configuration:

$$\begin{array}{ccccc}
 X & \longleftarrow & X \times_I \tilde{I} & \longrightarrow & E \\
 \downarrow & & \lrcorner \downarrow \lrcorner & & \downarrow \\
 I & \longleftarrow & \tilde{I} & \longrightarrow & U
 \end{array}$$

Unless $[\pi]$ was already a stack, it is not necessarily the case that $E \rightarrow U$ is a generic object for the stack completion $\{\pi\}$. But Streicher (2005) points out that $E \rightarrow U$ is nevertheless a weak generic object for $\{\pi\}$, essentially by definition.

Scenarios of the form described in Example 65 are easy to come by; a canonical example is furnished by the modest objects of a realizability topos as described by Hyland et al. (1990), with critical implications for the denotational semantics of polymorphism.

Example 66 (Hyland et al. 1990). Let \mathbf{Eff} be the *effective topos* (Hyland 1982), and let \mathbb{N} be its object of realizers, i.e. the partitioned assembly given by \mathbb{N} such that $n \Vdash m \in \mathbb{N} \Leftrightarrow m = n$. Write $\mathbf{P} \subseteq \Omega^{\mathbb{N} \times \mathbb{N}}$ for the object of $\neg\neg$ -closed partial equivalence relations on \mathbb{N} and $\mathbf{P}' \in \mathbf{Eff} \downarrow \mathbf{P}$ for the *generic $\neg\neg$ -closed subquotient* of \mathbb{N} ; internally, this is the subquotient $R : \mathbf{P} \vdash \{x : \mathbb{N} \mid x R x\} / R$.

The family $\pi : \mathbf{P}' \rightarrow \mathbf{P}$ then induces a full subfibration $[\pi] \subseteq \mathbf{P}_{\mathbf{Eff}}$ spanned by morphisms that arise by pullback from π ; an element $[\pi]_I$ is an object at stage I that is the subquotient of \mathbb{N} by some partial equivalence relation defined at stage I . The fibration $[\pi] \rightarrow \mathbf{Eff}$ is small with generic object π , but it is *not* complete. Although for every \mathbf{Eff} -indexed diagram $\mathbb{C} \rightarrow [\pi]$ there “exists” in the internal sense a limit, we cannot globally choose this limit. This situation is referred to by Hyland et al. (1990) as *weak completeness* as opposed to *strong completeness*.

In contrast, we may consider the stack completion $\{\pi\}$ of $[\pi]$. An element of $\{\pi\}_I$ is given by an object E at stage I such that there “exists” (in the internal sense) a partial equivalence relation that it is the quotient of – externally, this means that there is a regular epimorphism $\tilde{I} \twoheadrightarrow I$ such that $\tilde{I}^* E$ is the subquotient of \mathbb{N} by some partial equivalence relation defined at stage \tilde{I} . The stack $\{\pi\}$ is weakly but not strongly equivalent to $[\pi]$; on the other hand, $\{\pi\}$ is complete in the strong sense. Finally, we observe that $\pi : \mathbf{P}' \rightarrow \mathbf{P}$ is a weak generic object for the stack completion $\{\pi\}$.

If we pull back $[\pi]$ to along the inclusion $i : \mathbf{Asm} \hookrightarrow \mathbf{Eff}$ of assemblies / $\neg\neg$ -separated objects in \mathbf{Eff} , then we have a *complete* fibered category $i^*[\pi]$ over \mathbf{Asm} which turns out to be (strongly) equivalent to the familiar fibration $\mathbf{UFam}(\mathbf{PER}) \rightarrow \mathbf{Asm}$ of uniform families of PERs over

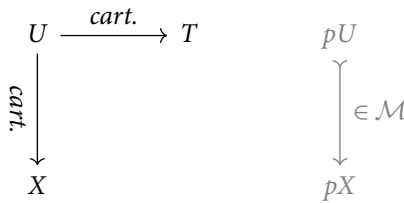
assemblies. In fact, $i^*[\pi]$ is strongly equivalent to $i^*\{\pi\}$ – in other words, over assemblies there is no difference between an object that is locally isomorphic to a subquotient of N and an actual subquotient of N .

There is another way to think of the situation described at the end of Example 66, using the observations of Streicher (2005) on the relationship between Bénabou’s notion of *definable class* and stackhood: the class of families of subquotients of N is not definable in **Eff**, but it is definable in **Asm**.

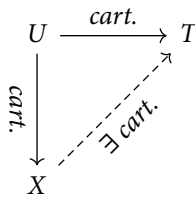
4.3 A new class: acyclic generic objects

Inspired by the crucial *realignment* property of type theoretic universes (Gratzer et al. 2022) that was discussed in Section 3.4(1), we define a new kind of generic object for an arbitrary fibration that restricts in the case of a full subfibration to a universe satisfying realignment. We refer to Gratzer et al. (2022) for an explanation of the applications of realignment. In this section, let $p : \mathcal{E} \rightarrow \mathcal{B}$ be a fibration and let \mathcal{M} be a class of monomorphisms in \mathcal{B} .

Definition 67. A *generic object* T for $p : \mathcal{E} \rightarrow \mathcal{B}$ is called *\mathcal{M} -acyclic* when for any span of cartesian maps in \mathcal{E} as depicted below in which $pU \rightarrow pX$ lies in \mathcal{M} ,



there exists a cartesian map $X \rightarrow T$ making the following diagram commute in \mathcal{E} :



Convention 68. When \mathcal{M} is understood to be the class of *all* monomorphisms in \mathcal{B} , we will simply speak of *acyclic generic* objects.

All of the examples of *acyclic generic* objects that we are aware of so far have arisen in the context of the full subfibration spanned by a *universe* in the sense of Streicher (2005), where *acyclicity* reduces to the *realignment* property studied in detail by Gratzer et al. (2022).

Remark 69. It is reasonable to ask whether there is a “weak” version of *acyclicity* that pertains to stack completions in the same way that *weak generic* objects relate to *generic* objects, e.g. by asking for a cover-cartesian morphism $\tilde{X} \twoheadrightarrow X$ and an extension of $U \times_X \tilde{X} \rightarrow U \rightarrow T$ along $U \times_X \tilde{X} \rightarrow \tilde{X}$. It remains somewhat unclear to this author whether this notion is useful, so we do not include it in our proposal.

Comment 70. In the context of universes, the *realignment* or *acyclicity* condition has been referred to as “Axiom (2’)” by Shulman (2015), as “strictification” by Orton and Pitts (2016),

as “stratification” by Stenzel (2019), as “alignment” by Awodey (2021), and as “strict gluing” by Sterling and Angiuli (2021). See also Riehl (2022) for further discussion.

Origins of the terminology The origin of the term “acyclicity” is explained by Shulman (2019) and Riehl (2022) in essentially the following way. Let $\mathcal{E} \subseteq \mathbf{P}_{\mathcal{B}}$ be a full subfibration equipped with a generic object T ; we will write $\mathbf{Core}(\mathcal{E})$ for the *groupoid core* of \mathcal{E} , and for any $I \in \mathcal{B}$ we will write $\mathbf{y}_{\mathcal{B}}I \rightarrow \mathcal{B}$ for the discrete fibration whose fiber at $J \in \mathcal{B}$ is the set of morphisms $J \rightarrow I$. Then, we have a canonical morphism of fibered categories $\mathbf{y}_{\mathcal{B}}pT \rightarrow \mathbf{Core}(\mathcal{E})$ corresponding under the fibered Yoneda lemma (Streicher 2021, Section 3) to T itself. Realignment for \mathcal{E} is then the property that $\mathbf{y}_{\mathcal{B}}pT \rightarrow \mathbf{Core}(\mathcal{E})$ has the following (2-categorical) extension property with respect to any monomorphism $m : J \rightarrow I \in \mathcal{M}$, in which we write $[X] : \mathbf{y}_{pX} \rightarrow \mathbf{Core}(\mathcal{E})$ for the morphism that we identify with $X \in \mathcal{E}$ under the fibered Yoneda lemma:

$$\begin{array}{ccc}
 \mathbf{y}_{\mathcal{B}}J & \xrightarrow{\mathbf{y}_{\mathcal{B}}\alpha} & \mathbf{y}_{\mathcal{B}}pT \\
 \mathbf{y}_{\mathcal{B}}m \downarrow & \dashrightarrow \beta & \downarrow [T] \\
 \mathbf{y}_{\mathcal{B}}I & \xrightarrow{[B]} & \mathbf{Core}(\mathcal{E})
 \end{array}$$

More formally, for any square as above commuting up to an isomorphism $\phi : [B] \circ \mathbf{y}_{\mathcal{B}}m \cong [T] \circ \mathbf{y}_{\mathcal{B}}\alpha$, the extension property states that there exists a (not necessarily unique) morphism $\beta : I \rightarrow pT$ and isomorphisms $\phi_0 : \mathbf{y}_{\mathcal{B}}\beta \circ \mathbf{y}_{\mathcal{B}}m \cong \mathbf{y}_{\mathcal{B}}\alpha$ and $\phi_1 : [T] \circ \mathbf{y}_{\mathcal{B}}\beta \cong [B]$ such that ϕ factors as the pasting of ϕ_0 and ϕ_1 . Moreover, as the boundary of ϕ_0 is discrete, it must be an identity and thus $\beta : I \rightarrow pT$ extends $\alpha : J \rightarrow pT$ along $J \rightarrow I$ in the (strict) 1-categorical sense. Under the fibered Yoneda lemma, the isomorphism $\phi : [B] \circ \mathbf{y}_{\mathcal{B}}m \cong [T] \circ \mathbf{y}_{\mathcal{B}}\alpha$ corresponds to an isomorphism $[\phi] : m^*B \cong \alpha^*T$ and the isomorphism $\phi_1 : [T] \circ \mathbf{y}_{\mathcal{B}}\beta \cong [B]$ corresponds to an isomorphism $[\phi_1] : \beta^*T \cong B$ that extends $[\phi]$ along $m : J \rightarrow I$.

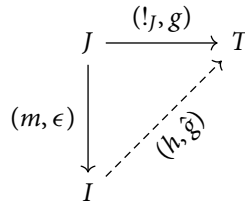
In a model categorical setting where \mathcal{M} is understood to be the class of cofibrations, the extension property above expresses that $\mathbf{y}_{\mathcal{B}}pT \rightarrow \mathbf{Core}(\mathcal{E})$ is an *acyclic fibration*, whence the terminology.

4.3.1 Incomparability of acyclic and skeletal generic objects

It is not the case that a skeletal generic object need be acyclic. We may consider the following counterexample, which was kindly suggested by one of the anonymous referees of this paper; in what follows, assume that \mathcal{B} is a category with finite limits and that G is a group object in \mathcal{B} . As in Example 33, we may regard G as an internal groupoid \mathbf{BG} in \mathcal{B} with a single object $(\mathbf{BG})_0 = 1_{\mathcal{B}}$, such that the hom object $(\mathbf{BG})_1$ is exactly G . The externalization $[\mathbf{BG}]$ can be seen to have the same objects as \mathcal{B} ; a morphism $I \rightarrow J$ in $[\mathbf{BG}]$ is given by a pair of a morphism $f : I \rightarrow J$ together with a generalized element $u : I \rightarrow G$. As in any fibered groupoid, every morphism of $[\mathbf{BG}]$ is cartesian. Observe that the object $T = 1_{\mathcal{B}}$ is a skeletal generic object in $[\mathbf{BG}]$; below, we show that T need not be acyclic.

Lemma 71. *The skeletal generic object $T \in [\mathbf{BG}]$ is acyclic if and only if $G \in \mathcal{B}$ has the right lifting property with respect to any element of \mathcal{M} .*

Proof. Suppose that T is an acyclic generic object, and fix a monomorphism $m : J \rightarrow I \in \mathcal{M}$ together with a generalized element $g : J \rightarrow G$ in \mathcal{B} . We may exhibit a span of cartesian maps in $[\mathbf{BG}]$ as follows, which has a lift by acyclicity:



Above we have exactly a morphism $\hat{g} : I \rightarrow G$ extending $g : J \rightarrow G$ along $m : J \rightarrow I$. The other direction is analogous. \square

It is clear that an acyclic generic object need not be skeletal; the above discussion confirms that acyclic and skeletal generic objects are in fact incomparable.

4.4 Splittings and generic objects

We have shown in Section 2.5 that the correct generalization to non-split fibrations of a **split generic** object in the sense of Jacobs (1999) is what we have proposed to call a generic object, *i.e.* the **weak generic** object of *op. cit.* Thus we conclude that the correct relationship can be established between *split* \blacksquare -fibrations and \blacksquare -fibrations when re-expressed using our definitions, where \blacksquare ranges over the different kinds of fibered structures (*e.g.* λ 2-fibration, polymorphic fibration, *etc.*). For example, the following definition expresses the correct relationship between λ 2-fibrations and split λ 2-fibrations:

Definition 72. A λ 2-fibration is a fibration with a generic object T , fibered finite products, and simple pT -products and coproducts. A λ 2-fibration will be called split if all its structure is split.

4.5 In univalent mathematics

The theory of fibered (1-)categories carries over *mutatis mutandis* to the univalent foundations of mathematics, as shown by Ahrens and Lumsdaine (2019). The pertinent notions of generic object are, however, a bit different in a univalent setting. In particular, gaunt generic objects play a more fundamental role in univalent mathematics than in non-univalent mathematics, because the uniqueness of the classifying squares becomes a statement of contractibility rather than strict uniqueness. In non-univalent mathematics, uncontrived instances of gaunt generic objects tend to follow the pattern of subobject classifiers in representing a fibered poset; in univalent mathematics, we also have *object* classifiers (univalent universes) which are gaunt generic objects for groupoid cores of the full subfibrations they induce.

Just as in non-univalent mathematics, skeletal generic objects are not expected to play a significant structural role although they may appear when considering the deloopings of groups, as in our 1-categorical examples. The most interesting case is that of acyclic generic objects: the acyclicity property is the correct way to present in non-univalent mathematics a generic object that is gaunt in the homotopical sense, as Shulman has recently argued (Shulman 2022). Thus, in a homotopical/univalent setting, one expects to work directly with gaunt generic objects rather than acyclic ones.

5. Concluding Remarks

For a number of years, the disorder in the variants of generic object has led to a proliferation of subtle differences in terminology between different papers applying fibered categories to categorical logic, type theory, and the denotational semantics of polymorphic types. Based on the kinds

of generic object that occur most naturally or have the most utility, we have proposed a unified terminological scheme for generic objects that we believe will meet the needs of scientists working in these areas.

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Notes

1 In this paper, we are somewhat agnostic about set theoretic foundations. Our discussion is compatible with the viewpoint of ZFC, in which classes are taken to be formulas at the meta-level; our discussion is, however, also compatible with other accounts of the “set–class” distinction, such as NBG set theory, MK set theory, or the universe-based approaches of Grothendieck (Artin et al. 1972) and Mac Lane (1998).

2 Comment on the MathOverflow thread entitled *Name for ‘Category without nontrivial automorphisms?’*, <https://mathoverflow.net/q/370764>.

3 We do not recall the definition of first order fibrations here, as the specifics are not important for the present paper.

4 For more on stacks, see The Stacks Project (Stacks Project Authors 2017) or the tutorial of Vistoli (2004).

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