Canad. Math. Bull. Vol. 39 (2), 1996 pp. 186–198

BACKWARD CONTINUED FRACTIONS AND THEIR INVARIANT MEASURES

KARLHEINZ GRÖCHENIG AND ANDREW HAAS

ABSTRACT. This paper continues our investigation of backward continued fractions, associated with the generalized Renyi maps $T_u(x) = \langle \frac{1}{u(1-x)} \rangle$ on $[0,1)$. We first show that the dynamics of the shift map on a specific class of shift invariant spaces of nonnegative integer sequences exactly models the maps T_u for $u \in (0,4)$. In the second part we construct a new family of explicit invariant measures for certain values of the parameter u.

1. **Introduction.** The starting point for our considerations is the family of maps $T_u(x) = \langle \frac{1}{u(1-x)} \rangle$ where $u > 0$, $x \in [0,1)$, and $\langle a \rangle$ denotes the fractional part of $a \in \mathbb{R}$. These maps should be viewed as analogues of the classical continued fraction map $S(x) = \langle \frac{1}{x} \rangle$. The case $u = 1$ was studied by Renyi [9] and provides an alternate approach to continued fractions and rational approximation. Varying u in the interval $(0,4)$ results in a one parameter family of continued fractions theories. Termed u-backward continued fractions, they have far more structure than is seen in the general theory of f-transformations [8] and further, possess some attractive properties which are not shared by the classical continued fractions.

Given $u \in (0,4)$ and $x \in [0,1)$ define the u-itinerary of x to be the sequence $(a_i)_{i\in\mathbb{N}}$ of nonnegative integers where we set $x_j = T_u^j(x)$ and $a_j \leq \frac{1}{u(1-x_{j-1})} < a_j + 1$. Then x has the u-backward continued fraction expansion

$$x = [a_1, a_2, \ldots]_u = 1 - \cfrac{1}{un_1 - \cfrac{1}{n_2 - \cfrac{1}{un_3 - \cfrac{1}{n_4 - \cdots}}}} \tag{1}$$

where $n_i = a_i + 1$ and the coefficient of n_i is 1 or u, depending on the parity of i. More precisely, x is the limit of the partial quotients $[a_1, a_2, \ldots, a_n]_u$. This expansion is the unique such for x where all of the partial quotients belong to the unit interval.

In [4] we showed how the dynamics of T_u on $[0,1)$ can be modeled by the shift operator σ acting on the space Σ_u of all itinerary sequences for $x \in [0,1)$ (see also [5, 6]). If we let $\mathbf{z}_u$ denote the itinerary sequence of 0 under T_u, called the *zero sequence*, then Σ_u is characterized as the set of infinite sequences $\mathbf{a}$ of non-negative integers satisfying $\mathbf{z}_u \leq \sigma^k \mathbf{a}$ for all $k \geq 0$, where sequences are ordered lexicographically. In particular, the zero sequence itself must satisfy $\mathbf{z}_u \leq \sigma^k \mathbf{z}_u$ for all $k \geq 0$. Such a sequence is

The first author was partially supported by NSF grant DMS-9306430.
Received by the editors October 24, 1995.
AMS subject classification: 11J70, 58F11, 58F03.
Key words and phrases: Continued fractions, interval maps, invariant measures, symbolic dynamics.

called *admissible*. Our first goal in this paper is to prove a converse to the existence of symbolic representations, which allows us to conclude that the family of dynamical systems $\big(T_u, [0,1)\big)$ for $u \in (0,4)$ is in one-to-one correspondence with the shift spaces (σ, Σ) where $\Sigma = \big\{\mathbf{a} \in (\mathbb{N} \cup \{0\})^{\mathbb{N}} : \mathbf{z} \le \sigma^k \mathbf{a} \text{ for all } k \ge 0\big\}$ for some admissible sequence $\mathbf{z}$. In other words, we show that for any admissible $\mathbf{z}$ there is a number $u \in (0,4)$ so that $\mathbf{z}_u = \mathbf{z}$. It is then easy to deduce the existence of maps T_u which possess the dynamic behavior suited to our needs (Section 3).

In [4] we focused on the basic theory of the u-backward continued fractions and the dynamic properties of the T_u. Of particular interest was the possibility of writing down the explicit form of an absolutely continuous invariant measure similar to the Gauss measure $\frac{1}{1+x}dx$ for $\langle \frac{1}{x} \rangle$ and Renyi's measure $\frac{1}{x}dx$ for T_1. We were successful with two classes of maps: first for the values $u_q = 4\cos^2\frac{\pi}{q}$, by showing that T_u is a factor of the first return map of the geodesic flow on the unit tangent bundle of a $(0; 2, q, \infty)$ hyperbolic surface and then proceeding as in [1, 2, 3, 13]. Secondly, we guessed, by analogy to the u_q cases, the invariant probability measures corresponding to the values $u = \frac{1}{N}$ for positive integers N. The maps $T_{\frac{1}{N}}$ are all combinatorially similar to the map $T_1 = \langle \frac{1}{1-x} \rangle$ studied by Renyi.

The second main result in this paper is a derivation of explicit absolutely continuous invariant probability measures for the family of maps T_u which are combinatorially similar to the maps T_{u_q} for integers $q \ge 3$ (Section 4). These maps all have the property that for some $k \ge 1$ the sequence $T_u^j(0)$, $0 \le j \le k$ is monotone increasing and $T_u^{k+1}(0) = 0$. As in the cases $u = \frac{1}{N}$ considered earlier, each q determines a countable set of values u. Only for the value $u = u_q$ is the measure infinite.

2. **Preliminaries.** In this section we collect the necessary background on backward continued fractions, their symbolic dynamics, and their invariant measures. For more details and proofs the reader is referred to [4].

2.1. *T_u and backward continued fractions.* For fixed $u > 0$, iteration of the map

$$T_u(x) = \left\langle \frac{1}{u(1-x)} \right\rangle$$

is closely connected to the action of the matrices

$$A_u = \begin{pmatrix} \sqrt{u} & -\frac{1}{\sqrt{u}} \\ \sqrt{u} & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

as Möbius transformations. If $x \in [1 - \frac{1}{uk}, 1 - \frac{1}{u(k+1)}) = A_u[k, k+1)$ then

$$T_u(x) = B^{-k} A_u^{-1}(x) \tag{2}$$

It follows by induction that the n^{th} iterate is of the form

$$T_u^n(x) = B^{-a_n} A_u^{-1} B^{-a_{n-1}} A_u^{-1} \cdots B^{-a_1} A_u^{-1}(x) \tag{3}$$

where the a_j's are the unique integers so that $B^{-a_k}A_u^{-1}\cdots B^{-a_1}A_u^{-1}(x) \in [0,1)$ for $k = 1,\ldots,n$.

Let $(\mathbb{N}\cup\{0\})^{\mathbb{N}}$ denote the set of all infinite sequences $\mathbf{a} = (a_1, a_2, \ldots)$ with values in the non-negative integers. We consider $(\mathbb{N}\cup\{0\})^{\mathbb{N}}$ as an ordered topological space. The topology is the product topology, where each factor $\mathbb{N}\cup\{0\}$ is considered discrete. The order is the lexicographical order $\mathbf{a} < \mathbf{b}$, if and only if there exists $k \geq 1$ so that $a_i = b_i$ for $i < k$ and $a_k < b_k$. The shift operator σ on $(\mathbb{N}\cup\{0\})^{\mathbb{N}}$ is defined by $\sigma(a)_i = a_{i+1}$.

Given an infinite sequence $\mathbf{a} \in (\mathbb{N}\cup\{0\})^{\mathbb{N}}$, we define

$$[a_1, a_2, \ldots]_u = \lim_{n\longrightarrow\infty} A_u B^{a_1} A_u B^{a_2} \cdots A_u B^{a_n} A_u(\infty) \tag{4}$$

if it exists, and we write π for the map $\pi\colon (\mathbb{N}\cup\{0\})^{\mathbb{N}} \longrightarrow \mathbb{R}$, $\pi(\mathbf{a}) = [a_1, a_2, \ldots]_u$, whenever it is defined. After setting $n_i = a_i + 1$, it is easy to see that (4) is identical with the informal expansion (1). In general this expansion will not be very nice, unless some restrictions are imposed on the sequences (a_i).

Given $u > 0$ the *zero sequence* $\mathbf{z}_u = (z_1, z_2, \ldots)$ may be defined by

$$\max\Big(0, 1 - \frac{1}{uz_j}\Big) \leq T_u^j(0) < 1 - \frac{1}{u(z_j+1)} \tag{5}$$

in agreement with (2).

DEFINITION 1. The *restricted symbol space* for $u > 0$ is

$$\Sigma_u = \big\{\mathbf{a} \in (\mathbb{N}\cup\{0\})^{\mathbb{N}} : \sigma^k\mathbf{a} \geq \mathbf{z}_u \text{ for all } k \geq 0\big\} \tag{6}$$

The main property of the u-backward continued fractions is given in the following

THEOREM 1. *For any* $0 < u < 4$, π *is well-defined for all* $\mathbf{a} \in \Sigma_u$ *and is a continuous, order-preserving bijection from* Σ_u *onto* $[0,1)$. *Furthermore, the diagram*

$$\begin{array}{ccc} \Sigma_u & \xrightarrow{\sigma} & \Sigma_u \\ \pi\big\downarrow & & \big\downarrow\pi \\ [0,1) & \xrightarrow{T_u} & [0,1) \end{array}$$

commutes.

The following lemma summarizes some important technical properties of the family of matrices A_u, sometimes written $A(u)$. To be consistent with [4], we will also use $\phi_u(x) = A_u^{-1}(x)$. Recall the definition $u_q = 4\cos^2\frac{\pi}{q}$, where q is an integer $q \geq 3$.

LEMMA 1 ([4]). *a) For* $u \in (u_q, u_{q+1})$ *we have*

$$0 < \phi_u(0) < \phi_u^2(0) < \cdots < \phi_u^{q-2}(0) < 1 \quad \textit{and} \quad \phi_u^{q-1}(0) > 1$$

and $\phi_{u_q}^{q-2}(0) = 1$.

https://doi.org/10.4153/CMB-1996-023-8 Published online by Cambridge University Press

b) For $u \in (u_q, u_{q+1})$ *we have*

$$(\phi_u^k)'(0) < 1 \quad \text{for } k = 1, \ldots, q-2 \quad \text{and} \quad (\phi_u^{q-1})'(0) > 1,$$

whereas $(\phi_{u_q}^{q-2})'(0) = 1$.

c) For each $x \in [0,1)$ *and* $k \leq q-1$, $(\phi_u^k)'(x)$ *is a decreasing function of u in* (u_q, u_{q+1}).

d) For $u \in (u_q, u_{q+1})$, *and* $k \leq q-1$, $(\phi_u^k)'(x)$ *is increasing in* $x \in [0,1)$.

e) The zero sequence of u_q *is periodic with period* $q-2$, *and* $\mathbf{z}_{u_q} = \overline{[0, \ldots, 0, 1]}_u$ *with* $q-3$ *consecutive zeros appearing in a period.*

2.2. *Admissible sequences.* Given $\mathbf{z} \in (\mathbb{N} \cup \{0\})^{\mathbb{N}}$ admissible, for $n \geq 1$ define the index function of $\mathbf{z}$ by

$$\kappa(n) = \max\{k : 1 \leq k \leq n, \overline{z_1 \cdots z_k} \text{ is admissible }\} \tag{7}$$

For instance, if $\mathbf{z} = \overline{0\ 1}$, then $\overline{0\ 1\ 0}$ is not admissible and therefore $\kappa(3) = 2$. In general $\kappa(2n+1) = \kappa(2n) = 2n$ for this $\mathbf{z}$.

LEMMA 2. *Let* $\mathbf{z} = (z_1 z_2 \cdots)$ *be an admissible sequence with index function* κ. *Then*

(a) κ *is increasing and either* $\kappa(n+1) = \kappa(n)$ *or* $\kappa(n+1) = n+1$ *for all* $n \geq 1$. *Moreover,* $\overline{z_1 \cdots z_{\kappa(n)}} \leq \mathbf{z}$.

(b) If $\kappa(l) = l$, $z_{l+j} = z_j$ *for* $j = 1, \ldots, n-l-1$ *and* $z_n > z_{n-l}$, *then* $\kappa(n) = n$.

(c) In particular, $z_{\kappa(n)+i} = z_i$ *for* $i = 1, \ldots, n-\kappa(n)$, $\kappa(n) \geq (n+1)/2$, *and* $z_{n+1} \geq z_{n+1-\kappa(n)}$.

(d) If $\kappa(n+1) = n+1$ *and* $z_{n+1} = z_{n+1-\kappa(n)}$, *then there exists an integer* $r \geq 1$ *so that* $\overline{z_1 \cdots z_r} = \overline{z_1 \cdots z_{n+1}}$ *and either* $z_r > z_{r-\kappa(r-1)}$ *or* $r = 1$.

PROOF. (a) If $\kappa(n) < \kappa(n+1) \leq n$, then $\overline{z_1 \cdots z_{\kappa(n+1)}}$ would be admissible in contrast to the definition of $\kappa(n)$. $\overline{z_1 \cdots z_{\kappa(n)}} \leq \mathbf{z}$ follows from $\sigma^{\ell\kappa(n)}\mathbf{z} \geq \mathbf{z}$ for all $\ell \geq 1$.

(b) Consider the sequences $\eta = (\eta_i) = \overline{z_1, \ldots, z_l}$ and $\omega = (\omega_i) = \overline{z_1 \cdots, z_n}$. Observe that $\eta_i = \omega_i$ for $i = 1, \ldots, n-1$ and $\eta_n = z_{n-l} < z_n = \omega_n$. By hypothesis $\sigma^k\eta \geq \eta$ for all $k \geq 0$. Thus for $k < n$ either $\sigma^k\eta = \eta$ or there is a smallest value j so that $\eta_{j+k} > \eta_j$.

If $\sigma^k\eta = \eta$ or if $j > n$, then $\omega_{i+k} = \eta_{i+k} = \eta_i = \omega_i$ for $i = 1, \ldots, n-k-1$, and by the above observation $\sigma^k\omega > \omega$. If $j \leq n$ then it is immediate from the observation that $\sigma^k\omega \geq \omega$. Since ω is n-periodic, we conclude that it is admissible and consequently that $\kappa(n) = n$.

(c) is now an immediate consequence of (b), and $z_{n+1} \geq z_{n+1-\kappa(n)}$ follows from $\sigma^{\kappa(n)}\mathbf{z} \geq \mathbf{z}$.

(d) Set $s = n+1-\kappa(n)$ and $\eta = \overline{z_1 \cdots z_{n+1}} = \overline{z_1 \cdots z_{\kappa(n)} z_1 \cdots z_s}$. By the admissibility of η we obtain $\sigma^{\kappa(n)}\eta = z_1 \cdots z_s \overline{z_1 \cdots z_{n+1}} \geq z_1 \cdots z_s z_{s+1} \cdots z_{n+1} \overline{z_1 \cdots z_{n+1}}$. Omitting the first s terms, this reads as $\eta \geq \sigma^s\eta$ and thus we obtain $\sigma^s\eta = \eta$, *i.e.*, $\overline{z_1 \cdots z_s} = \overline{z_1 \cdots z_{n+1}}$. Let r by the smallest integer so that $\overline{z_1 \cdots z_r} = \overline{z_1 \cdots z_{n+1}}$. Then either $z_r > z_{r-\kappa(r-1)}$ or $r = 1$ by the above argument. ∎

2.3. *The Perron-Frobenius operator.* The density of an absolutely continuous invariant measure for T_u is an eigenfunction of eigenvalue 1 of the Perron-Frobenius operator [10, 7]

$$L_u f(x) = \sum_{\{y \mid T_u(y)=x\}} \frac{1}{|T_u'(y)|} f(y).$$

For $0 < u$ and $N = [\frac{1}{u}]$, L_u is given explicitly by the formula [4]

$$L_u f(x) = \sum_{n=N+1}^{\infty} \frac{1}{u(x+n)^2} f\left(1 - \frac{1}{u(x+n)}\right) + \chi_{[T_u(0),1)}(x) \frac{1}{u(x+N)^2} f\left(1 - \frac{1}{u(x+N)}\right). \tag{8}$$

We shall use this later in Section 4 to verify that a given function is in fact the density for the unique absolutely continuous invariant measure.

2.4. *Ergodicity.* The main facts regarding the ergodic theory of the maps T_u are contained in the following Proposition, which was proved in [4].

PROPOSITION 1. *1. For $u \in (0, 4)$ the maps T_u act ergodically.*

2. If $u = 4\cos^2 \frac{\pi}{q}$ with q an integer, $q \geq 3$, then there exists an infinite T_u-invariant measure on $[0, 1)$, which is, up to a multiplicative constant, the unique invariant measure absolutely continuous with respect to Lebesgue measure.

3. If $u \in (0, 4)$ and $u \neq 4\cos^2 \frac{\pi}{q}$ then there is a unique T_u-invariant probability measure on $[0, 1)$ which is absolutely continuous with respect to Lebesgue measure.

3. Main results.

The main result of this paper is

THEOREM 2. *Given an admissible sequence $\mathbf{z} \in (\mathbb{N} \cup \{0\})^{\mathbb{N}}$, there exists a unique $u \in (0, 4]$ so that $\mathbf{z}$ is the zero sequence of T_u.*

3.1. *Existence of u.* If $u > 0$ is given, then $\mathbf{z}_u = (z_i)$ is defined and

$$B^{-z_n} A_u^{-1} B^{-z_{n-1}} A_u^{-1} \cdots B^{-z_1} A_u^{-1}(0) = T_u^n(0) \in [0, 1) \tag{9}$$

for all $n \geq 1$. Therefore, given an admissible sequence $\mathbf{z} = (z_i)$, we define

$$C_n(u) = B^{-z_n} A_u^{-1} B^{-z_{n-1}} A_u^{-1} \cdots B^{-z_1} A_u^{-1}, \tag{10}$$

$$I_n = \{u > 0 : C_k(u)(0) \in [0, 1) \text{ for } k = 1, \ldots, n\}, \quad \text{and} \tag{11}$$

$$\Psi_n(u) = A_u^{-1} C_n(u)(0), \tag{12}$$

where $C_0(u) = \mathrm{Id}$ and $\Psi_0(u) = A_u^{-1}(0) = 1/u$. Then the intervals I_n are nested and any $u \in \bigcap_{n=1}^{\infty} I_n$ satisfies (9). Consequently if the intersection is nonempty $\mathbf{z}_u = \mathbf{z}$.

The following lemma contains some immediate consequences of these definitions.

LEMMA 3. *If $I_{n+1} \neq \emptyset$, then $\overline{I_{n+1}} \subseteq I_n$ and I_{n+1} is a half-open interval $I_{n+1} = (\alpha_{n+1}, \beta_{n+1}]$. Let $\xi_n = \Psi_n(\beta_n)$, then $\Psi_n : I_n \longrightarrow [\xi_n, \infty)$ is a decreasing C^∞-bijection and*

$$I_{n+1} = \Psi_n^{-1}\big([\xi_n, \infty) \cap [z_{n+1}, z_{n+1} + 1)\big). \tag{13}$$

https://doi.org/10.4153/CMB-1996-023-8 Published online by Cambridge University Press

PROOF BY INDUCTION ON n. The case $n = 0$ is obvious, since $\Psi_0(u) = 1/u$ is decreasing and maps $I_0 = (0, 4]$ onto $[1/4, \infty)$. Then $I_1 = \{0 < u < 4 : B^{-z_1} A_u^{-1}(0) \in [0, 1)\} = \Psi_0^{-1}\big([z_1, z_1 + 1) \cap [\frac{1}{4}, \infty)\big)$.

Assume that the lemma is true for $k < n$. Write $\Psi_n(u) = \frac{1}{u\left(1+z_n-\Psi_{n-1}(u)\right)}$ and differentiate. Then

$$\Psi_n'(u) = \frac{-1}{u^2\big(1 + z_n - \Psi_{n-1}(u)\big)} + \frac{\Psi_{n-1}'(u)}{u\big(1 + z_n - \Psi_{n-1}(u)\big)^2}. \tag{14}$$

Since by induction $\Psi_{n-1}'(u) < 0$ and $\Psi_{n-1}(u) \in [z_n, z_n + 1)$, we have $\Psi_n'(u) < 0$ for $u \in I_n$.

Since $C_{n+1}(u)(0) = B^{-z_{n+1}}\Psi_n(u)$, $u \in I_{n+1}$ if and only if $u \in I_n$ and $\Psi_n(u) \in [z_{n+1}, z_{n+1} + 1)$, from which (13) follows. Consequently, $I_{n+1} = (\alpha_{n+1}, \beta_{n+1}]$ is half-open, and, since $\Psi_n(\alpha_{n+1}) = z_{n+1} + 1$, $\Psi_{n+1}(\alpha_{n+1}) = A(\alpha_{n+1})^{-1}B^{-z_{n+1}}\Psi_n(\alpha_{n+1}) = A(\alpha_{n+1})^{-1}(1) = \infty$. Moreover, $\alpha_{n+1} > \alpha_n$ and thus $\overline{I_{n+1}} \subseteq I_n$. ∎

To prove the existence of u in Theorem 2 it will suffice to show that $I_n \neq \emptyset$ for all $n \geq 1$.

LEMMA 4. *If $\mathbf{z}$ is an admissible sequence, then $I_n \neq \emptyset$ for all $n \geq 1$, and Ψ_n: $(\alpha_n, \beta_n] \longrightarrow [\xi_n, \infty)$ satisfies*

$$\xi_n = \Psi_{n-\kappa(n)}(\beta_{\kappa(n)}), \quad \xi_n \in [z_{n+1-\kappa(n)}, z_{n+1-\kappa(n)} + 1) \tag{15}$$

and $\beta_{n+1} = \beta_n$ if and only if $z_{n+1} = z_{n+1-\kappa(n)}$.

PROOF BY INDUCTION ON n. The case $n = 0$ is obvious. To begin we show that if the lemma holds for n then $I_{n+1} \neq \emptyset$. The relevant observations in this regards are Lemma 2(c) stating that $z_{n+1} \geq z_{n+1-\kappa(n)}$ and the inductive hypothesis $\xi_n \in [z_{n+1-\kappa(n)}, z_{n+1-\kappa(n)} + 1)$. Then $[\xi_n, \infty) \cap [z_{n+1}, z_{n+1} + 1) \neq \emptyset$ and $I_{n+1} = \Psi_n^{-1}\big([\xi_n, \infty) \cap [z_{n+1}, z_{n+1} + 1)\big) \neq \emptyset$.

The assertion regarding the β_n's also follows easily: If $z_{n+1} > z_{n+1-\kappa(n)}$, then $[z_{n+1}, z_{n+1} + 1) \subseteq \big[[\xi_n] + 1, \infty\big)$. Consequently, $\beta_{n+1} = \Psi_n^{-1}(z_{n+1}) > \Psi_n^{-1}(\xi_n) = \beta_n$. On the other hand, as $\Psi_n(\beta_{n+1}) = \max(\xi_n, z_{n+1})$, equality $\beta_{n+1} = \beta_n = \Psi_n^{-1}(\xi_n)$ implies $[\xi_n] = z_{n+1-\kappa(n)} \leq z_{n+1} \leq \xi_n$, *i.e.*, $z_{n+1-\kappa(n)} = z_{n+1}$.

As a consequence of Lemma 2 (c) we have

$$\beta_l = \beta_{\kappa(n)} \quad \text{for } \kappa(n) \leq l \leq n. \tag{16}$$

To prove the claim on ξ_n, we distinguish three cases and use

$$\xi_{n+1} = A(\beta_{n+1})^{-1}B^{-z_{n+1}}\Psi_n(\beta_{n+1})$$

from definition (10) and (12).

CASE 1. Suppose $\kappa(n+1) = \kappa(n) < n+1$. Then by Lemma 2 (c) $z_{n+1} = z_{n+1-\kappa(n)}$ and $\beta_{n+1} = \beta_n = \beta_{\kappa(n)}$ from the previous observation (16). Then using the inductive hypothesis for ξ_n

$$\begin{aligned}\xi_{n+1} &= A(\beta_{\kappa(n)})^{-1}B^{-z_{n+1}}\Psi_n(\beta_n) = A(\beta_{\kappa(n)})^{-1}B^{-z_{n+1}}(\xi_n)\\ &= A(\beta_{\kappa(n)})^{-1}B^{-z_{n+1-\kappa(n)}}\Psi_{n-\kappa(n)}(\beta_{\kappa(n)}) = \Psi_{n+1-\kappa(n)}(\beta_{\kappa(n)})\\ &= \Psi_{n+1-\kappa(n+1)}(\beta_{\kappa(n+1)}) \qquad (17)\end{aligned}$$

Combining Lemma 2(c) and $\kappa(n) = \kappa(n+1)$ gives $\kappa(n) \geq n+2-\kappa(n)$. This then implies that $I_{\kappa(n)} \subseteq I_{n+2-\kappa(n)}$ and that $\xi_{n+1} = \Psi_{n+1-\kappa(n)}(\beta_{\kappa(n)}) \in \Psi_{n+1-\kappa(n)}(I_{n+2-\kappa(n)}) \subseteq [z_{n+2-\kappa(n+1)}, z_{n+2-\kappa(n+1)}+1)$, completing the first case.

CASE 2. If $\kappa(n+1) = n+1$ and $z_{n+1} = z_{n+1-\kappa(n)}$, then by Lemma 2(d) $\overline{z_1\cdots z_{n+1}} = \overline{z_1\cdots z_r}$, where $z_r > z_{r-\kappa(r-1)}$ or $r=1$ and $n+1 = qr$ for some $q \in \mathbb{N}$. Therefore by observation (16), $\beta_j = \beta_{\kappa(j)} = \beta_r$ for all $j \geq r$

Applying the inductive hypothesis to ξ_r yields

$$\begin{aligned}C_r(\beta_r)(0) &= A(\beta_r)\Psi_r(\beta_r) = A(\beta_r)\xi_r\\ &= A(\beta_r)\Psi_{r-\kappa(r)}(\beta_{\kappa(r)}) = A(\beta_r)A(\beta_r)^{-1}(0) = 0.\end{aligned}$$

In this way we obtain

$$\begin{aligned}\xi_{n+1} &= A(\beta_{n+1})^{-1}C_{n+1}(\beta_{n+1})(0) = A(\beta_{\kappa(n+1)})^{-1}C_r(\beta_r)^q(0)\\ &= A(\beta_{\kappa(n+1)})^{-1}(0) = \Psi_{n+1-\kappa(n+1)}(\beta_{\kappa(n+1)})\end{aligned}$$

Writing $\xi_{n+1} = \Psi_0(\beta_{n+1}) \in [z_1, z_1+1) = [z_{n+2-\kappa(n+1)}, z_{n+2-\kappa(n+1)}+1)$ completes this case.

CASE 3. Finally, if $\kappa(n+1) = n+1$, but $z_{n+1} > z_{n+1-\kappa(n)}$, then we must have $\xi_n < z_{n+1}$. Thus $\Psi_n(\beta_{n+1}) = z_{n+1}$ and

$$\begin{aligned}\xi_{n+1} &= \Psi_{n+1}(\beta_{n+1}) = A(\beta_{n+1})^{-1}B^{-z_{n+1}}\Psi_n(\beta_{n+1})\\ &= A(\beta_{n+1})^{-1}B^{-z_{n+1}}z_{n+1} = A(\beta_{n+1})^{-1}(0) = \Psi_0(\beta_{n+1}) \in [z_1, z_1+1).\end{aligned}$$

This completes the proof. ∎

3.2. *Uniqueness of u.* The map T_u is differentiable on $[0,1)$ except at 0 and at the jump points $1 - \frac{1}{nu}$ for integers $n \geq u^{-1}$. For any $s > 0$ and $x \in [0,1)$ we define $(T_u^s)'(x) = \lim_{\epsilon\to 0^+}(T_u^s)'(x+\epsilon)$. This is well-defined and the chain rule holds for any decomposition of T_u^s into a product.

To derive uniqueness in Theorem 2 we will need the following lemma. This result is both a simplification and a strengthening of Proposition 2 in [4].

LEMMA 5. *Let $q \geq 2$ be an integer and suppose that $\alpha, \beta \in \mathbb{R}$ satisfy $u_q < \alpha < \beta < u_{q+1}$. Then for any $\lambda > 0$ there is an integer $s > 0$ so that $(T_u^s)'(x) > \lambda$ for all $u \in [\alpha, \beta]$ and $x \in [0,1)$.*

In the proof, repeated use will be made of Lemma 1.

PROOF. Given $x \in [0,1)$ there is a smallest integer $0 < m < q-1$ for which $\phi_u^m(x) \geq 1$. Observe that m is defined by the inequality $\phi_u^{-1}(1) = 1 - \frac{1}{u} \leq \phi_u^{m-1}(x) < 1$ or $\phi_u^{-m}(1) \leq x < \phi_u^{-m+1}(1)$. We show that there is a number $\eta > 1$ so that $(\phi_u^m)'(x) > \eta$ for any $u \in [\alpha, \beta]$. Since $\phi_u^{q-2}(0) \in (1 - \frac{1}{u}, 1)$, there are two cases to consider. First suppose that $\phi_u^{q-2}(0) \leq \phi_u^{m-1}(x) < 1$, for $0 < m < q-1$. Then the estimate is a consequence of the following chain of inequalities:

$$\begin{aligned} 1 &= (\phi_{u_{q+1}}^{q-1})'(0) < (\phi_\beta^{q-1})'(0) = \eta \leq (\phi_u^{q-1})'(0) \\ &= (\phi_u^m)'\big(\phi_u^{q-m-1}(0)\big)(\phi_u^{q-m-1})'(0) < (\phi_u^m)'\big(\phi_u^{q-m-1}(0)\big) \leq (\phi_u^m)'(x) \end{aligned}$$

for any x with $\phi_u^{q-m-1}(0) \leq x < 1$. We have used Lemma 1(c) in the first and second steps, and (d) in the final step.

The second case is $\phi_u^{-1}(1) \leq \phi_u^{m-1}(x) < \phi_u^{q-2}(0)$, where now $0 < m < q-2$. Again by Lemma 1 we have

$$1 > (\phi_{u_{q+1}}^{-j+q-1})'(0) = (\phi_{u_{q+1}}^{-j})'\big(\phi_{u_{q+1}}^{q-1}(0)\big)(\phi_{u_{q+1}}^{q-1})'(0) = (\phi_{u_{q+1}}^{-j})'(1)$$

for $0 < j < q-1$. Set $\rho = \sup_{0<j<q-1} (\phi_{u_{q+1}}^{-j})'(1)$. Then for $0 < m < q-2$, and $\phi_u^{-m}(1) \leq x < \phi_u^{m+1}(1)$,

$$\begin{gathered} (\phi_u^m)'(x) > (\phi_u^m)'\big(\phi_u^{-m}(1)\big) \quad \text{and} \\ (\phi_u^m)'\big(\phi_u^{-m}(1)\big)(\phi_{u_{q+1}}^{-m})'(1) > (\phi_u^m)'\big(\phi_u^{-m}(1)\big)(\phi_u^{-m})'(1) = 1. \end{gathered}$$

Consequently $(\phi_u^m)'(x) > \frac{1}{\rho} = \eta$. We have used Lemma 1(d) to conclude that $(\phi_{u_{q+1}}^{-m})'(1) > (\phi_u^{-m})'(1)$ for $0 < m < q-2$ and $u_q < u < u_{q+1}$.

Next, by writing $T_u^s(x)$ in a particular product form, which reflects the manner in which successive iterates of x cycle across $[0,1)$, and applying the above, we shall complete the proof.

Given $s > 0$ there are well defined integers $n_1 < \cdots < n_k = s$ so that $T_u^j(x) \in [1 - \frac{1}{u}, 1)$ if and only if $j = n_i - 1$ for some $1 \leq i < k$. Write $T_u^s = T_u^{m_k} \circ \cdots \circ T_u^{m_1}$ where $m_1 = n_1$, $m_0 = 0$ and $m_i = n_i - n_{i-1}$. Applying the chain rule to this composition we have

$$(T_u^s)'(x) = (T_u^{m_k})'\big(T_u^{n_k - 1}(x)\big)(T_u^{m_{k-1}})'\big(T_u^{n_{k-1}-1}(x)\big) \cdots (T_u^{m_1})'(x). \tag{18}$$

With $T_u^{n_j - 1}(x) = x_j$ we see that $T_u^i(x_j) = \phi_u^i(x_j)$ for $i = 1, \ldots, m_j - 1$, $1 - \frac{1}{u} \leq \phi_u^{m_j - 1}(x_j) < 1$, and $\phi_u^{m_j}(x_j) > 1$.

By the initial observation m_j is the unique minimal value m associated with x_j for which $\phi_u^m(x_j) \geq 1$, and so

$$(T_u^{m_j})'\big(T_u^{n_j - 1}(x)\big) = (\phi_u^{m_j})'(x_j) > \eta \quad \text{for } j < k.$$

Applying this to decomposition (18) yields

$$(T_u^s)'(x) > \eta^{k-1}(T_u^{m_k})'(x_k) \geq \eta^{k-1}(T_u^{m_k})'(0).$$

With $(T_u^n)'(0)$ attaining its minimum $C > 0$ for $n = 1 \cdots, q-2$, $u \in [\alpha, \beta]$, and $k \geq \frac{s}{q-1}$ we can finally conclude that

$$(T_u^s)'(x) \geq C\eta^{\frac{s}{q-1}-1}$$

for all $u \in [\alpha, \beta]$ and $x \in [0, 1)$. The lemma follows. ∎

LEMMA 6. *Let $q \geq 2$ be an integer and suppose that $\alpha, \beta \in \mathbb{R}$ satisfy $u_q < \alpha < \beta < u_{q+1}$. Given $M > 0$ there is an $N > 0$ so that $|\Psi_n'(u)| > M$ for all $n \geq N$ and for all $u \in [\alpha, \beta]$.*

PROOF. The modulus of (14) is

$$\begin{aligned} |\Psi_n'(u)| &= \frac{1}{u^2\big(1 + z_n - \Psi_{n-1}(u)\big)} + \frac{|\Psi_{n-1}'(u)|}{u\big(1 + z_n - \Psi_{n-1}(u)\big)^2} \\ &= \frac{1}{u}\Psi_n(u) + u\Psi_n^2(u)|\Psi_{n-1}'(u)| \geq u\Psi_n^2(u)|\Psi_{n-1}'(u)|. \end{aligned} \tag{19}$$

Inducting over n gives

$$|\Psi_n'(u)| \geq |\Psi_0'(u)| \prod_{j=1}^{n} u\Psi_j^2(u).$$

Since $C_j(u)(0) = T_u^j(0)$ we get

$$\Psi_j(u) = A_u^{-1} C_j(u)(0) = \frac{1}{u\big(1 - T_u^j(0)\big)}$$

and so we can write

$$\begin{aligned} |\Psi_n'(u)| &\geq \Big(\prod_{j=1}^{n} u\Psi_j^2(u)\Big)\frac{1}{u^2} = \frac{1}{u}\prod_{j=0}^{n} u\Psi_j^2(u) \\ &= \frac{1}{u}\prod_{j=0}^{n}\frac{1}{u\big(1 - T_u^j(0)\big)^2} = \frac{1}{u}\prod_{j=0}^{n} T_u'\big(T_u^j(0)\big) = \frac{1}{u}(T_u^{n+1})'(0). \end{aligned}$$

Lemma 5 assures us that the last quantity can be made arbitrarily large. ∎

PROOF OF THEOREM 2: UNIQUENESS. We argue by contradiction. Suppose that $\bigcap_{n=1}^{\infty} I_n$ in (13) contains points $0 < u_1 < u_2 < 4$. Then $\mathbf{z}_{u_1} = \mathbf{z}_u$ for all $u \in [u_1, u_2]$. In particular, there is an integer $q \geq 2$ and numbers $\alpha, \beta \in \mathbb{R}$ so that $u_q < \alpha < \beta < u_{q+1}$ and $\mathbf{z}_{u_1} = \mathbf{z}_u$ for all $u \in [\alpha, \beta]$. Ψ_n is a monotone map of $[\alpha, \beta]$ into $[0, 1]$. Choose N in Lemma 6 so that $|\Psi_n'(u)| > \frac{1}{\beta - \alpha}$ for all $n \geq N$ and $u \in [\alpha, \beta]$. Then Ψ_n maps $[\alpha, \beta]$ onto an interval of length greater than one, giving the desired contradiction. ∎

REMARK 1. If $u \geq 4$, then T_u has an attractive fixed point $\xi \in [0, 1 - \frac{1}{u})$, so that $T_u^j(0) \in [0, 1 - \frac{1}{u})$ for all $j \geq 1$ and $\lim_{j\to\infty} T_u^j(0) = \xi$ [4]. Thus for $u \geq 4$ the zero sequence is always $[0, 0, \ldots]_u$.

https://doi.org/10.4153/CMB-1996-023-8 Published online by Cambridge University Press

3.3. *Consequences.* Next we derive some consequences from Theorem 2.

COROLLARY 1. *Suppose that* $\Sigma \subseteq (\mathbb{N} \cup \{0\})^{\mathbb{N}}$ *is a proper shift-invariant subspace of* $(\mathbb{N} \cup \{0\})^{\mathbb{N}}$ *with the property that if* $\mathbf{a} \leq \mathbf{b}$ *and* $\mathbf{a} \in \Sigma$, *then* $\mathbf{b} \in \Sigma$. *Then there exists a continuous order preserving bijection* π *from* Σ *onto either* $I = [0, 1)$ *or* $I = (0, 1)$ *and a unique* $u \in (0, 4]$, *so that the diagram*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ \pi \downarrow & & \downarrow \pi \\ I & \xrightarrow{T_u} & I \end{array}$$

commutes.

PROOF. We define the sequence $\mathbf{z} = (z_n)_{n \in \mathbb{N}}$ by induction. Let $z_1 = \min\{a_1 : \mathbf{a} \in \Sigma\}$. If $z_1, \ldots, z_n$ are already defined, set $z_{n+1} = \min\{a_{n+1} : \mathbf{a} \in \Sigma, a_i = z_i \text{ for } i = 1, \ldots, n\}$. Then it is easy to see that $\{\mathbf{a} \in (\mathbb{N} \cup \{0\})^{\mathbb{N}} : \mathbf{a} > \mathbf{z}\} \subseteq \Sigma \subseteq \{\mathbf{a} \in (\mathbb{N} \cup \{0\})^{\mathbb{N}} : \mathbf{a} \geq \mathbf{z}\}$, and that Σ is one of these sets, depending on whether $\mathbf{z} \in \Sigma$ or not.

By construction $\mathbf{z}$ is an admissible sequence and thus the zero sequence of a unique $u \in (0, 4]$. Therefore either $\Sigma = \Sigma_u$ or $\Sigma = \Sigma_u \setminus \{\mathbf{z}\}$. Since $\Sigma \neq (\mathbb{N} \cup \{0\})^{\mathbb{N}}$, $u < 4$. The diagram now follows from Theorem 1. ∎

COROLLARY 2. *The orbit of* 0 *is (eventually) periodic under* T_u, *if and only if* $\mathbf{z}_u$ *is admissible and (eventually) periodic. Furthermore, if* $\mathbf{z}_u$ *is eventually periodic, then u is an algebraic number.*

PROOF. If $T_u^{jn+r}(0) = T_u^r(0)$ for some $r \geq 0$ and all $j \geq 0$, then by definition $z_{nj+r} = z_r$ and $\mathbf{z}_u$ is eventually periodic.

On the other hand, if $\mathbf{z} \neq \mathbf{o}$ is admissible and eventually periodic, then there exists a $u \in (0, 4)$ with zero sequence $\mathbf{z}_u = \mathbf{z} = z_1 \cdots z_r \overline{z_{r+1} \cdots z_{n+r}}$. Using the diagram of Theorem 1, the periodicity $\sigma^{jn+r}\mathbf{z} = \sigma^r\mathbf{z}$ translates into

$$T_u^{jn+r}(0) = \pi \circ \sigma^{jn+r} \circ \pi^{-1}(0) = \pi \circ \sigma^{jn+r}(\mathbf{z}) = \pi \circ \sigma^r(\mathbf{z}) = T_u^r(0)$$

as desired.

Suppose that $\mathbf{z}_u = z_1 \cdots z_r, \overline{z_{r+1} \cdots z_{n+r}}$. Writing the identity $T_u^{n+r}(0) = T_u^r(0)$ in terms of the matrices A_u and B, as in (9), gives

$$\prod_{j=1}^{r} B^{-z_{r+1-j}} A_u^{-1}(0) = T_u^r(0) = T_u^{n+r}(0) \tag{20}$$

$$= \Big(\prod_{j=1}^{n} B^{-z_{n+r+1-j}} A_u^{-1}\Big)\Big(\prod_{j=1}^{r} B^{-z_{r+1-j}} A_u^{-1}\Big)(0)$$

After defining $V_u = u^{r/2} \prod_{j=1}^{r} B^{-z_{r+1-j}} A_u^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $W_u = u^{n/2} \prod_{j=1}^{n} B^{-z_{n+r+1-j}} A_u^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we obtain from (20) that

$$\frac{\beta}{\delta} = V_u(0) = W_u V_u(0) = \frac{a\beta + b\delta}{c\beta + d\delta}. \tag{21}$$

In (21) the entries of V_u and W_u are all polynomials in u with integer coefficients. We conclude that u is a solution to the equation $\beta^2 c + \beta\delta d - \beta\delta a - b\delta^2 = 0$, which is again a nontrivial polynomial in u with integer coefficients. ∎

3.4. *Examples.* The identity (20) allows us to compute u from a given (eventually) periodic admissible sequence $\mathbf{z}$.

1. If $\mathbf{z} = [N, N, \ldots]$, then $B^{-N}A_u^{-1}(0) = 0$ gives $u = \frac{1}{N}$.
2. If $\mathbf{z} = \overline{N_1 N_2}$, where $N_1 < N_2$, then $B^{-N_2}A_u^{-1}B^{-N_1}A_u^{-1}(0) = 0$ gives $N_2(N_1+1)u^2 - (N_2+1)u = 0$ and thus

$$u = \frac{N_2 + 1}{(N_1 + 1)N_2}. \tag{22}$$

3. If $\mathbf{z} = [N_1\overline{N_2 N_2 \cdots}]$, where $N_1 < N_2$, then $T_u(0)$ is a fixed point and $B^{-N_2}A_u^{-1}B^{-N_1}A_u^{-1}(0) = B^{-N_1}A_u^{-1}(0)$. A little computation reveals that

$$u = \frac{2N_1 - N_2 - \sqrt{(2N_1 - N_2)^2 - 4(N_1+1)(N_1 - N_2)}}{2(N_1+1)(N_1 - N_2)}. \tag{23}$$

In particular, $[0\overline{111\cdots}] = \mathbf{z}_u$ for $u = \frac{1+\sqrt{5}}{2}$.

4. Finally, if 0 has period 3 then $\mathbf{z}_u = \overline{N_1 N_2 N_3}$ with $N_1 \le N_2 < N_3$ or $N_1 < N_2 \le N_3$ and after some computation

$$u = \frac{(N_2+2)N_3 + N_2 + 1 + \sqrt{\{(N_2+2)N_3 + N_2 + 1\}^2 - 4(N_1+1)(N_2+1)N_3}}{2(N_1+1)(N_2+1)N_3}. \tag{24}$$

Observe that in 3 and 4 respectively, the choice of a minus and a plus sign was made in the expression for u to assure that $N_1 \le \frac{1}{u} < N_1 + 1$.

4. **Invariant measures.** Given $u \in (0,4)$, define $\alpha_u = B^{N-1}A_u B$, where $N = [1/u]$. Expanding the notation, we see that

$$\alpha_u(x) = N - \frac{1}{u(x+1)}.$$

THEOREM 3. *Suppose $u \in (0,4)$ has the periodic zero sequence $\mathbf{z}_u = \overline{[N, \ldots, N, N+1]}$, where $N \ge 0$ appears in blocks of length q. Then the function*

$$\rho(x) = \sum_{j=0}^{q} \frac{1}{x + \alpha_u^j(N)} \chi_{[T_u^j(0), T_u^{j+1}(0))}(x) \tag{25}$$

is the invariant density for T_u, where we write $T_u^{q+1}(0) = 1$ for brevity.

PROOF. First we verify that with u defined by its zero sequence, we have

$$\alpha_u^q(N) = \frac{1}{u} - 1 \tag{26}$$

This follows from Corollary 2, since $T_u^{q+1}(0) = B^{-1}(B^{-N}A_u^{-1})^{q+1}(0) = 0$ implies $0 = (AB^N)^{q+1}B(0) = (AB)(B^{N-1}AB)^q B^N(0) = (AB)(\alpha_u^q)(N)$ and thus $\frac{1}{u} - 1 = B^{-1}A^{-1}(0) =$

$\alpha_u^q(N)$. Also, $T_u^j(0) \in [0, 1 - \frac{1}{u(N+1)})$ for $j = 1, \ldots, q-1$ and $T_u^q(0) = 1 - \frac{1}{u(N+1)}$ imply $T_u^j(0) < T_u^{j+1}(0)$ for $j = 0, \ldots, q-1$. The unit interval is therefore the disjoint union of the intervals $\left[T_u^j(0), T_u^{j+1}(0)\right), j = 0, \ldots, q-1$ and $[T_u^q(0), 1) = [1 - \frac{1}{u(N+1)}, 1)$.

The argument is completed by showing that $L_u\rho = \rho$, where L_u is the Perron-Frobenius operator of T_u (8).

For $n > N$ and $x \in [0, 1)$ we have $T_u^q(0) = 1 - \frac{1}{u(N+1)} \leq 1 - \frac{1}{u(x+n)} < 1$ and consequently

$$\rho\left(1 - \frac{1}{u(x+n)}\right) = \frac{1}{x + \alpha_u^q(N)} = \frac{1}{x + \frac{1}{u} - 1}.$$

Suppose $x \in [T_u^j(0), T_u^{j+1}(0)), 1 \leq j \leq q$, then $1 - \frac{1}{u(x+N)} = T_u^{-1}(x+N) \in \left[T_u^{j-1}(0), T_u^j(0)\right)$ and $\rho(1 - \frac{1}{u(x+N)}) = \left(1 - \frac{1}{u(x+N)} + \alpha_u^{j-1}(N)\right)^{-1}$. Thus with $\delta(x) = \chi_{[T_u(0),1)}(x)$ we compute

$$\begin{aligned}
L_u\rho(x) &= \sum_{n=N+1}^{\infty} \frac{1}{u(x+n)^2}\rho\left(1 - \frac{1}{u(x+n)}\right) + \delta(x)\frac{1}{u(x+N)^2}\rho\left(1 - \frac{1}{u(x+N)}\right) \\
&= \sum_{n=N+1}^{\infty} \frac{1}{u(x+n)^2}\frac{1}{\frac{1}{u} - \frac{1}{u(x+n)}} + \delta(x)\frac{1}{u(x+N)^2}\frac{1}{1 - \frac{1}{u(x+N)} + \alpha_u^{j-1}(N)} \\
&= \sum_{n=N+1}^{\infty}\left(\frac{1}{x+n-1} - \frac{1}{x+n}\right) + \delta(x)\left(\frac{1}{(x+N)[u(\alpha_u^{j-1}(N)+1)(x+N)-1]}\right) \qquad (27)\\
&= \frac{1}{x+N} + \delta(x)\left(\frac{1}{x+N-\frac{1}{u(\alpha_u^{j-1}(N)+1)}} - \frac{1}{x+N}\right) \\
&= \left(\frac{1}{x+N} - \delta(x)\frac{1}{x+N}\right) + \delta(x)\frac{1}{x+\alpha_u^j(N)} = \frac{1}{x+\alpha_u^j(N)},
\end{aligned}$$

This is also true for $j = 0$, since $\alpha_u^0(N) = N$. Thus $L_u\rho(x) = \rho(x)$ for $x \in [T_u^j(0), T_u^{j+1}(0))$ and all $j = 0, \ldots, q$. ∎

4.1. *Examples.* If $\mathbf{z} = \overline{[N, N+1]}$, then by (22) $u = \frac{N+2}{(N+1)^2}$, and the invariant density is

$$\rho(x) = \frac{1}{x+N}\chi_{[0,\frac{1}{N+2})}(x) + \frac{1}{x+N-1+\frac{1}{N+2}}\chi_{[\frac{1}{N+2},1)}(x).$$

This density has also been found by Schweiger [12].

If $\mathbf{z} = \overline{[N, N, N+1]}$, then from (24)

$$u = \frac{N^2+3N+3+\sqrt{(N^2+3N+3)^2-4(N+1)^3}}{2(N+1)^3},$$

and $T_u(0) = \frac{1}{u} - N$, $T_u^2(0) = 1 - \frac{1}{u(N+1)}$. According to Theorem 3 the invariant density is

$$\rho(x) = \frac{1}{x+N}\chi_{[0,\frac{1}{u}-N)}(x) + \frac{1}{x+N-\frac{1}{u(N+1)}}\chi_{[\frac{1}{u}-N,1-\frac{1}{u(N+1)})}(x) + \frac{1}{x+\frac{1}{u}-1}\chi_{[1-\frac{1}{u(N+1)},1)}(x).$$

REMARK 2. Fix $q \geq 3$. Given $M, N \geq 0$ let u and v respectively be the unique value given by Theorem 2 with $\mathbf{z}_u = \overline{N \cdots N, N+1}$ and $\mathbf{z}_v = \overline{M \cdots MM+1}$ where M and N appear both $q-2$ times in a period. Define the map $h: [0,1) \longrightarrow [0,1)$, taking $x = [a_1 a_2 \cdots]_u$ to $y = h(x) = [a_1 - N + M, a_2 - N + M, \ldots]_v$. By Theorem 1 and 2 h is a well defined map satisfying

1. h is an increasing homeomorphism, and
2. $h \circ T_u = T_v \circ h$.

Thus T_u and T_v are dynamically indistinguishable. On the other hand, if we let $N = 0$ then the associated $u = 4\cos^2 \frac{\pi}{q}$. For $M > 0$ let μ denote the unique absolutely continuous T_v-invariant probability measure given by Proposition 1. Then the pull back $h_*\mu$ is a T_u-invariant probability measure which by [14] and Proposition 1 must be singular with respect to Lebesgue measure. Thus the ergodic theory of these maps is far from identical.

One would expect that for any $q > 0$ and $N \neq M$ the map h is singular. See [11]. This would follow if one knew that for $N \neq M$, T_u and T_v have different entropies.

REFERENCES

1. R. L. Adler and L. Flatto, *Cross-section maps for geodesic flows*. In: Ergodic Theory and Dynamical Systems, Progress in Math. **2**, (ed. A. Katok), Birkhäuser, Boston, 1980.
2. ———, *The backward continued fraction map and the geodesic flow*, Ergodic Theory Dynamical Systems **4**(1984), 487–492.
3. ———, *Geodesic flows, interval maps, and symbolic dynamics*, Bull. Amer. Math. Soc. **25**(1991), 229–334.
4. K. Gröchenig and A. Haas, *Backward continued fractions, Hecke groups and invariant measures for transformations of the interval*, Ergodic Theory Dynamical Systems, to appear.
5. F. Hofbauer, *On the intrinsic ergodicity of piecewise monotonic transformations with positive entropy*, Israel J. Math. **34**(1979), 213–237.
6. ———, *The structure of piecewise monotonic transformations*, Ergodic Theory Dynamical Systems **1**(1981), 159–178.
7. W. de Melo and S. van Strien, *One-Dimensional Dynamics*, Ergebnisse d. Math. Vol. **25**, Springer, Berlin, Heidelberg, 1993.
8. A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Hungary **8**(1957), 477–493.
9. ———, *Valòs szàmok elöàllitàsàra szölgàlò algoritmusokròl*, M. T. A. Mat. Oszt. Kžl. **7**(1957), 265–293.
10. M. Rychlik, *Bounded variation and invariant measures*, Studia Math. **76**(1983), 69–80.
11. R. Salem, *On some singular monotonic functions which are strictly increasing*, Trans. Amer. Math. Soc. **53**(1943), 427–439.
12. F. Schweiger, *Invariant measures of generalized Renyi maps*, Univ. Salzburg, 1993, preprint.
13. C. Series, *The modular group and continued fractions*, J. London Math. Soc. **31**(1985), 69–80.
14. M. Thaler, *Transformations on* [0, 1] *with infinite invariant measure*, Israel J. Math. **46**(1983), 67–96.

Department of Mathematics
The University of Connecticut
Storrs, Connecticut 06269-3009
U.S.A.
e-mail: GROCH@MATH.UCONN.EDU
e-mail: HAAS@MATH.UCONN.EDU