Canad. Math. Bull. Vol. **43** (1), 2000 pp. 87–89

# Lomonosov's Techniques and Burnside's Theorem

Mikael Lindström and Georg Schlüchtermann

*Abstract.* In this note we give a proof of Lomonosov's extension of Burnside's theorem to infinite dimensional Banach spaces.

## 1 Introduction

In [L2] Lomonosov proved an extension of Burnside's theorem to infinite dimensional Banach spaces. The proof is based on the argument of de Branges' proof of the Stone-Weierstrass theorem. In the Hilbert space case, Brown [B] proved the extended Burnside theorem for commutative subalgebras using Lomonosov's celebrated invariant subspace techniques for compact operators [L1]. On Hilbert spaces, Simonic [S] recently obtained an extended version of Burnside's theorem for weakly closed subalgebras. In this note we also make use of Lomonosov's invariant subspace techniques for compact operators to obtain a proof of the extended Burnside theorem for infinite dimensional Banach spaces. We are grateful to the referee for informing us that a similar project has independently been carried out by Chevreau, Li and Pearcy in [CLP].

All Banach spaces $E$ will be complex. By an operator we mean a bounded linear map. We denote by $\mathcal{L}(E)$ the algebra of all operators on the Banach space $E$. If $\mathcal{A}$ is subalgebra of $\mathcal{L}(E)$, then we denote by $\mathcal{A}^*$ the subalgebra $\{T^* : T \in \mathcal{A}\}$ of $\mathcal{L}(E^*)$. The uniform operator topology on $\mathcal{L}(E)$ is the topology of norm convergence and the weak operator topology on $\mathcal{L}(E)$ is the topology of pointwise weak convergence. That the subalgebra $\mathcal{A} \subset \mathcal{L}(E)$ has a non-trivial invariant subspace means that there exists a non-trivial closed subspace of $E$ that is invariant under all operators in $\mathcal{A}$.

Let $\|T\|_e$ denote the essential norm of $T \in \mathcal{L}(E)$, *i.e.*, the distance from the compact operators,

$$\|T\|_e = \inf\{\|T - K\| : K \in \mathcal{K}(E)\}.$$

Recall that $\lambda \in \mathbb{C}$ is called an *eigenvalue of finite algebraic multiplicity*, if $\lambda$ is an isolated point of $\sigma(T)$ and is a pole of the resolvent $(\lambda I - T)^{-1}$ of $T$, with spectral projection $P$ (*cf.* [K], III-6.5); $n = \dim R(P)$ is the *algebraic multiplicity* of $\lambda$. If $T \in \mathcal{L}(E)$, then

$$r_e(T) = \sup\{|\lambda| : \lambda \in \sigma(T) \text{ but not an eigenvalue of finite algebraic multiplicity}\},$$

is the *essential spectral radius* and $r_e(T) = \lim_n \|T^n\|_e^{\frac{1}{n}} \leq \|T\|_e$. For more details, see the works of Lebow-Schechter [LS], Voigt [V] and Weis [W].

Received by the editors February 4, 1998; revised October 22, 1998.
AMS subject classification: 47A15.

## 2 Results

Since the map $T \mapsto T^*$ is isometric, we get the following simple result.

***Lemma 2.1*** *If $\mathcal{A}$ is a uniformly closed subalgebra of $\mathcal{L}(E)$, then $\mathcal{A}^*$ is also a uniformly closed subalgebra of $\mathcal{L}(E^*)$.*

The proofs of Lemma 2.2 and Theorem 2.3 are based on ideas from [B] and [L1].

***Lemma 2.2*** *Let $\mathcal{A} \subset \mathcal{L}(E)$ be a subalgebra. Suppose that for every non-zero $l \in E^*$ the set $\{T^*(l) : T \in \mathcal{S}\}$ is dense in $E^*$, where $\mathcal{S} := \{T \in \mathcal{A} : \|T\|_e \le \frac{1}{24}\}$. Then there is an $R \in \mathcal{S}$ such that $1$ is an eigenvalue of finite algebraic multiplicity of $R^*$.*

**Proof** Let $l_0 \in E^*$ with $\|l_0\| = 2$. Consider $D = \{l \in E^* : \|l - l_0\| \le 1\}$. By assumption, for $l \in D$ there is a $T_l \in \mathcal{S}$, such that $\|T_l^*(l) - l_0\| \le \frac{1}{4}$. Since $\|T_l\|_e \le \frac{1}{24}$, there is $K_l \in \mathcal{K}(E)$ such that $\|T_l - K_l\| \le \frac{1}{23}$. Since every $K_l^* \colon (D, w^*) \to (E^*, \|\cdot\|)$ is continuous, we obtain that for each $l \in D$, there is a relatively weak*-open neighbourhood $V_l \subset D$ of $l$ with

$$\|K_l^*(m) - K_l^*(l)\| < \frac{1}{8} \quad \text{for all } m \in V_l.$$

For all $m \in V_l$,

$$\|T_l^*(m) - T_l^*(l)\| \le \|T_l - K_l\| \cdot \|m\| + \frac{1}{8} + \|T_l - K_l\| \cdot \|l\| < \frac{1}{2}.$$

Hence $\|T_l^*(m) - l_0\| \le \|T_l^*(m) - T_l^*(l)\| + \|T_l^*(l) - l_0\| < \frac{3}{4}$ for all $m \in V_l$. Thus

$$T_l^*(V_l) \subset D \quad \text{for each } l \in D.$$

By Alaoglu's theorem $D$ is weak*-compact, so there is a finite number $l_1, \dots, l_p \in D$ with $D = \bigcup_{i=1}^p V_{l_i}$. Using the partition of unity, there are continuous functions $f_i \colon (D, w^*) \to [0,1]$, $i = 1, \dots, p$, such that $\sum_{i=1}^p f_i(l) = 1$ for all $l \in D$ and $f_i(l) = 0$ for $l \in D \setminus V_{l_i}$ and $1 \le i \le p$. Define $g \colon (D, w^*) \to (D, w^*)$ by $g(l) := \sum_{i=1}^p f_i(l) T_{l_i}^*(l)$. Then $g$ is well-defined and continuous. Since $D$ is convex and $w^*$-compact, Schauder's fixed point theorem implies that there exists a non-zero $m \in D$, such that $g(m) = m$. Put $R := \sum_{i=1}^p f_i(m) T_{l_i}$. Then $R \in \mathcal{L}(E)$ and $R^*(m) = m$, so 1 is an eigenvalue of $R^*$. Further, $\|R\|_e \le \frac{1}{24}$ and consequently $\|R^*\|_e \le \frac{1}{24}$. Hence $r_e(R^*) \le \frac{1}{24}$, so 1 is an eigenvalue of finite algebraic multiplicity of $R^*$. ■

***Theorem 2.3*** **(Lomonosov, 1991)** *Let $E$ be an infinite dimensional Banach space and let $\mathcal{A}$ be a weakly closed subalgebra of $\mathcal{L}(E)$ with $\mathcal{A} \ne \mathcal{L}(E)$. Then there exist nonzero $u \in E^{**}$, $l \in E^*$ such that*

$$|u(T^*(l))| \le \|T\|_e \quad \textit{for all } T \in \mathcal{A}.$$

**Proof** Suppose not; then by the Hahn-Banach theorem $\mathcal{A}^*$ does not have a non-trivial invariant subspace. Furthermore we may assume that there is *no* non-zero $l \in E^*$, such that $\{T^*(l) : T \in \mathcal{S}\}$ is not dense in $E^*$, where $\mathcal{S} := \{T \in \mathcal{A} : \|T\|_e \le \frac{1}{24}\}$. Otherwise, by the Hahn-Banach theorem there is a non-zero $u \in E^{**}$ such that $|u(T^*(l))| \le 1$ for every

https://doi.org/10.4153/CMB-2000-013-8 Published online by Cambridge University Press

$T \in \mathcal{S}$. Thus there is a non-zero $v \in E^{**}$ with $|v(T^*(l))| \leq \|T\|_e$ for all $T \in \mathcal{A}$. Hence, the assumption of Lemma 2.2 is fulfilled, and we find an $R \in \mathcal{S}$, such that 1 is an eigenvalue of finite algebraic multiplicity of $R^*$. Thus $P\colon E^* \to E^*$ has finite rank, where

$$P = \frac{1}{2\pi i} \int_\Gamma (\lambda I - R^*)^{-1}\, d\lambda$$

and $\Gamma$ is a closed curve enclosing 1 and no other point of $\sigma(R^*)$ lies within or on $\Gamma$. Since $\|R^*\|_e \leq \frac{1}{24}$, it follows that every $|\lambda| > \frac{1}{24}$, $\lambda \in \sigma(R^*)$, is an eigenvalue of finite algebraic multiplicity. Therefore all $\lambda \in \Gamma$ belongs to the unbounded component of the resolvent set $\rho(R^*)$. Since $(\lambda I - R^*)^{-1}$ exists in $\mathcal{L}(E^*)$ for every $\lambda \in \Gamma$, it thus follows from Theorem 10.18 in [R] and Lemma 2.1 that $(\lambda I - R^*)^{-1} \in \mathcal{A}^*$ for all $\lambda \in \Gamma$ (note that we may assume that I$\in \mathcal{A}$, since the algebra generated by $\mathcal{A}$ and $I$ satisfies the assumption of the theorem). Hence we get by Lemma 2.1 that $\int_\Gamma (\lambda I - R^*)^{-1}\, d\lambda \in \mathcal{A}^*$, so $P \in \mathcal{A}^*$. Since $\mathcal{A}^*$ has no non-trivial invariant subspace and $\mathcal{A}^*$ contains a finite rank operator, Theorem 8.2 in [RR] (*cf.* Lemma 10 in [L2]) implies that $\mathcal{A}^* \subset \mathcal{L}(E^*)$ is weakly dense. But then also $\mathcal{A} \subset \mathcal{L}(E)$ is weakly dense, so $\mathcal{A} = \mathcal{L}(E)$ contradicting the assumption $\mathcal{A} \neq \mathcal{L}(E)$. ■

## References


[B] S. W. Brown, *Lomonosov's theorem and essentially normal operators*. New Zealand J. Math. **23**(1994), 11–18.

[CLP] B. Chevreau, W. S. Li and C. Pearcy, *A new Lomonosov lemma*. J. Operator Theory, (to appear).

[K] T. Kato, *Perturbation theory for linear operators*. Springer-Verlag, Berlin, Heidelberg, New York, 1966.

[LS] A. Lebow and M. Schechter, *Semigroups of operators and measures of non-compactness*. J. Funct. Anal. **7**(1971), 1–26.

[L1] V. Lomonosov, *Invariant subspaces for operators commuting with compact operators*. Functional Anal. Appl. **7**(1973), 213–214.

[L2] ———, *An extension of Burnside's theorem to infinite dimensional spaces*. Israel J. Math. **75**(1991), 329–339.

[RR] H. Radjavi and P. Rosenthal, *Invariant Subspaces*. Springer-Verlag, New York, 1973.

[R] W. Rudin, *Functional Analysis*. McGraw-Hill, New York, 1973.

[S] A. Simonic, *An extension of Lomonosov's techniques to non-compact operators*. Trans. Amer. Math. Soc. **348**(1996), 975–995.

[V] J. Voigt, *A perturbation theorem for the essential spectral radius of the strongly continuous semigroups*. Monatsh. Math. **90**(1980), 153–161.

[W] L. Weis, *A generalization of the Vidav-Jörgens perturbation theorem for semigroups and its application to transport theory*. J. Math. Anal. Appl. **129**(1988), 6–23.



*Department of Mathematics*
*Åbo Akademi University*
*FIN-20500 Åbo*
*Finland*
*email: mikael.lindstrom@abo.fi*

*Mathematisches Institut der Universität München*
*Theresienstr. 39*
*D-80333 München*
*Germany*
*email: schluech@rz.mathematik.uni-muenchen.de*