Forum of Mathematics, Sigma (2024), **Vol. 12**:e117 1–24
doi:10.1017/fms.2024.126

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Eichler–Selberg relations for singular moduli

Yuqi Deng[1], Toshiki Matsusaka[2] and Ken Ono[3]

[1]Graduate School of Mathematics, Kyushu University, Motooka 744, Nishi-ku, Fukuoka, 819-0395, Japan;
E-mail: deng.yuqi.608@s.kyushu-u.ac.jp.
[2]Faculty of Mathematics, Kyushu University, Motooka 744, Nishi-ku, Fukuoka, 819-0395, Japan;
E-mail: matsusaka@math.kyushu-u.ac.jp.
[3]Department of Mathematics, University of Virginia, Charlottesville, VA, 22904, USA;
E-mail: ko5wk@virginia.edu (corresponding author).

Received: 20 June 2024; **Revised:** 2 September 2024; **Accepted:** 17 October 2024

2020 Mathematics Subject Classification: *Primary* – 11F37; *Secondary* – 11F50, 11F67

Abstract

The Eichler–Selberg trace formula expresses the trace of Hecke operators on spaces of cusp forms as weighted sums of Hurwitz–Kronecker class numbers. We extend this formula to a natural class of relations for traces of singular moduli, where one views class numbers as traces of the constant function $j_0(\tau) = 1$. More generally, we consider the singular moduli for the Hecke system of modular functions

$$j_m(\tau) := mT_m(j(\tau) - 744).$$

For each $\nu \geq 0$ and $m \geq 1$, we obtain an *Eichler–Selberg relation*. For $\nu = 0$ and $m \in \{1, 2\}$, these relations are Kaneko's celebrated singular moduli formulas for the coefficients of $j(\tau)$. For each $\nu \geq 1$ and $m \geq 1$, we obtain a new Eichler–Selberg trace formula for the Hecke action on the space of weight $2\nu + 2$ cusp forms, where the traces of $j_m(\tau)$ singular moduli replace Hurwitz–Kronecker class numbers. These formulas involve a new term that is assembled from values of symmetrized shifted convolution L-functions.

Contents

1. Introduction and statement of results

Let $j(\tau)$ be the usual modular function for $\mathrm{SL}_2(\mathbb{Z})$ with Fourier expansion

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots,$$

where $q := e^{2\pi i\tau}$. Its values at imaginary quadratic arguments in the upper-half of the complex plane are examples of *singular moduli* [34]. They are algebraic integers that generate Hilbert class fields of imaginary quadratic fields, in addition to serving as isomorphism class invariants of elliptic curves with complex multiplication. Well-known examples of these values include

$$j\left(\frac{1+\sqrt{-3}}{2}\right) = 0, \quad j(i) = 1728, \quad \text{and} \quad j\left(\frac{1+\sqrt{-15}}{2}\right) = \frac{-191025 - 85995\sqrt{5}}{2}.$$

We consider the sequence of modular functions $j_0(\tau) := 1, j_1(\tau) := j(\tau) - 744, \ldots$ that satisfy

$$j_m(\tau) = q^{-m} + O(q).$$

Each $j_m(\tau)$ is a monic degree m polynomial in $\mathbb{Z}[j(\tau)]$, and the set $\{j_m(\tau) : m \geq 0\}$ is a basis of $M_0^!$, the space of weakly holomorphic modular functions on $\mathrm{SL}_2(\mathbb{Z})$. The first examples are $j_0(\tau) = 1$ and

$$\begin{aligned}
j_1(\tau) &= j(\tau) - 744 = q^{-1} + 196884q + \cdots,\\
j_2(\tau) &= j(\tau)^2 - 1488j(\tau) + 159768 = q^{-2} + 42987520q + \cdots,\\
j_3(\tau) &= j(\tau)^3 - 2232j(\tau)^2 + 1069956j(\tau) - 36866976 = q^{-3} + 2592899910q + \cdots.
\end{aligned}$$

In terms of the Hecke operators T_m (see [28, Ch. VII] and [34]), for positive integers m, we have

$$j_m(\tau) = q^{-m} + \sum_{n=1}^{\infty} c_m(n)q^n = mT_m(j(\tau) - 744). \tag{1.1}$$

We shall derive infinitely many relations for the singular moduli of these functions. To make this precise, for positive integers d with $-d \equiv 0, 1 \pmod 4$, we let $\mathcal{Q}_d$ be the set of integral positive definite binary quadratic forms $Q(X, Y) = [A, B, C] := AX^2 + BXY + CY^2$ with discriminant $-d = B^2 - 4AC$. The group $\Gamma := \mathrm{PSL}_2(\mathbb{Z})$ acts on $\mathcal{Q}_d$ by

$$\left(Q \circ \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(X, Y) := Q(aX + bY, cX + dY)$$

and does so with finitely many orbits, the number of which is the discriminant $-d$ *class number*. For each $Q \in \mathcal{Q}_d$, we let $\alpha_Q \in \mathbb{H}$ be a root of $Q(\tau, 1) = 0$. The numbers $j_m(\alpha_Q)$ are its *singular moduli*.

We study the weighted traces of these values which are defined as follows. If we let Γ_Q be the stabilizer of Q in Γ, then it is well known that

$$\#\Gamma_Q = \begin{cases} 3 & \text{if } Q \sim a(X^2 + XY + Y^2),\\ 2 & \text{if } Q \sim a(X^2 + Y^2),\\ 1 & \text{if otherwise.} \end{cases}$$

https://doi.org/10.1017/fms.2024.126 Published online by Cambridge University Press

Following Zagier [34], the *trace functions* we consider are

$$\mathbf{t}_m(d) := \sum_{Q \in \mathcal{Q}_d/\Gamma} \frac{j_m(\alpha_Q)}{\#\Gamma_Q}. \tag{1.2}$$

For $m = 0$, where $j_0(\tau) = 1$, we obtain the *Hurwitz–Kronecker* class numbers $H(d) := \mathbf{t}_0(d)$. These numbers are prominent in the Eichler–Selberg trace formula (for example, see [32]) for the trace $\mathrm{Tr}(n; 2k)$ of the action of the Hecke operators T_n on S_{2k}, the complex vector space of weight $2k$ cusp forms on $\mathrm{SL}_2(\mathbb{Z})$.

Theorem (The Eichler–Selberg trace formula). For integers $k \geq 2$, we have

$$\mathrm{Tr}(n; 2k) = -\frac{1}{2} \sum_{r \in \mathbb{Z}} p_{2k}(r, n) \mathbf{t}_0(4n - r^2) - \lambda_{2k-1}(n), \tag{1.3}$$

where $\lambda_k(n) := \frac{1}{2} \sum_{d|n} \min(d, n/d)^k$ and

$$p_k(r, n) = \sum_{0 \leq j \leq \frac{k}{2}-1} (-1)^j \binom{k-2-j}{j} n^j r^{k-2-2j} = \mathrm{Coeff}_{X^{k-2}}\left(\frac{1}{1 - rX + nX^2}\right). \tag{1.4}$$

We generalize these formulas to traces of singular moduli, where (1.3) are the $m = 0$ cases of a doubly infinite suite of formulas in $m \geq 0$ and $\nu \geq 0$. The general formulas involve the trace functions $\mathbf{t}_m(4n - r^2)$. To make this precise, for every $\nu \geq 0$ and $m \geq 0$, we define the generating function

$$\mathcal{G}_{m,\nu}(\tau) := -\frac{1}{2} \sum_{n \gg -\infty} \sum_{r \in \mathbb{Z}} p_{2\nu+2}(r, n) \mathbf{t}_m(4n - r^2) q^n, \tag{1.5}$$

where for $d \leq 0$, we let

$$\mathbf{t}_m(d) := \begin{cases} 2\sigma_1(m) & \text{if } d = 0, \\ -\kappa & \text{if } d = -\kappa^2 \text{ and } \kappa \mid m, \\ 0 & \text{if otherwise.} \end{cases} \tag{1.6}$$

By (1.3), each $\mathrm{Tr}(n; 2k)$ is essentially the nth coefficient of $\mathcal{G}_{0,k-1}(\tau)$. Therefore, we refer to any explicit formula for $\mathcal{G}_{m,\nu}(\tau)$ as an Eichler–Selberg relation for m and ν.

Our first result establishes that these generating functions are *weakly holomorphic modular forms*, meromorphic modular forms whose poles (if any) are supported at cusps. For convenience, we let $M_k^!$ denote the space of such weight k forms on $\mathrm{SL}_2(\mathbb{Z})$.

Theorem 1.1. *If* $\nu \geq 0$ *and* $m \geq 1$, *then we have that* $\mathcal{G}_{m,\nu}(\tau) \in M_{2\nu+2}^!$.

The $\nu = 0$ Eichler–Selberg relations only involve derivatives of the $j_m(\tau)$, as they generate $M_2^!$ due to the absence of holomorphic modular forms. For convenience, we let $D := \frac{1}{2\pi i} \frac{\mathrm{d}}{\mathrm{d}\tau} = q \frac{\mathrm{d}}{\mathrm{d}q}$.

Theorem 1.2. *For positive integers* m, *the following are true.*

1) *We have*

$$\mathcal{G}_{m,0}(\tau) = -\frac{1}{2} \sum_{\kappa | m} \sum_{0 < r < \kappa} \frac{\kappa}{r(\kappa - r)} \cdot D j_{r(\kappa - r)}(\tau).$$

https://doi.org/10.1017/fms.2024.126 Published online by Cambridge University Press

2) *If n is a positive integer, then we have*

$$\sum_{r\in\mathbb{Z}} \mathbf{t}_m(4n-r^2) = n\sum_{\kappa|m}\sum_{0<r<\kappa} \frac{\kappa}{r(\kappa-r)} c_{r(\kappa-r)}(n).$$

Example. Theorem 1.2, with $m \in \{1,2\}$, gives Kaneko's identities [16]

$$\sum_{r\in\mathbb{Z}} \mathbf{t}_1(4n-r^2) = 0 \quad \text{and} \quad \sum_{r\in\mathbb{Z}} \mathbf{t}_2(4n-r^2) = 2nc_1(n),$$

which he used to derive his well-known singular moduli formula for the coefficients of $j(\tau)$

$$c_1(n) = \frac{1}{n}\left\{\sum_{r\in\mathbb{Z}} \mathbf{t}_1(n-r^2) + \sum_{r\geq 1\ odd}\Big((-1)^n\mathbf{t}_1(4n-r^2) - \mathbf{t}_1(16n-r^2)\Big)\right\}.$$

Such formulas have been extended to higher levels N in subsequent works [18, 19, 25]. Finally, as a different kind of generalization, Theorem 1.2 (2) shows how to express the coefficients of each $j_m(\tau)$ in terms of traces of singular moduli.

For $\nu > 0$, there are holomorphic modular forms, and so the relations have richer structure. To make this precise, we recall the weight $2k$ modular Poincaré series [4, Ch. 6.3]

$$P_{2k,h}(\tau) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} q^h|_{2k}\gamma, \tag{1.7}$$

where $|_{2k}$ is the slash operator, $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, and Γ_∞ is the stabilizer for the cusp infinity. The usual Eisenstein series is $P_{2k,0}(\tau) = E_{2k}(\tau)$, and for negative integers $-h$, we have the weakly holomorphic

$$P_{2k,-h}(\tau) = q^{-h} + \sum_{n=1}^{\infty} c_{2k,-h}(n)q^n. \tag{1.8}$$

For small ν, when there are no cusp forms, we obtain the following Eichler–Selberg relations.

Theorem 1.3. *If* $\nu \in \{1,2,3,4,6\}$, *then for every positive integer m, the following are true.*
1) *We have that*

$$\mathcal{G}_{m,\nu}(\tau) = \sum_{\kappa|m}\sum_{0<r\leq\kappa} r^{2\nu+1} P_{2\nu+2,-r(\kappa-r)}(\tau).$$

2) *If n is a positive integer, then we have*

$$\sum_{r\in\mathbb{Z}} p_{2\nu+2}(r,n)\mathbf{t}_m(4n-r^2) = -2\sum_{\kappa|m}\sum_{0<r\leq\kappa} r^{2\nu+1} c_{2\nu+2,-r(\kappa-r)}(n).$$

Remark. The Poincaré series in Theorem 1.3 are easily described in terms of the Eisenstein series

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n \quad \text{and} \quad E_6(\tau) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n.$$

https://doi.org/10.1017/fms.2024.126 Published online by Cambridge University Press

For $k \in \{4, 6, 8, 10, 14\}$, we have

$$P_{k,-1}(\tau) = \begin{cases} E_4(\tau) \cdot (j(\tau) - 984) & \text{if } k = 4, \\ E_6(\tau) \cdot (j(\tau) - 240) & \text{if } k = 6, \\ E_4^2(\tau) \cdot (j(\tau) - 1224) & \text{if } k = 8, \\ E_4(\tau)E_6(\tau) \cdot (j(\tau) - 480) & \text{if } k = 10, \\ E_4^2(\tau)E_6(\tau) \cdot (j(\tau) - 720) & \text{if } k = 14. \end{cases}$$

Generalizing (1.1), for $m > 1$, we have the Hecke formula

$$P_{k,-m}(\tau) = m^{-k+1} \cdot T_m P_{k,-1}(\tau).$$

Example. For positive integers n, Theorem 1.3 with $\nu = 1$ and $m = 1$ implies that

$$\sum_{r \in \mathbb{Z}} r^2 \mathbf{t}_1(4n - r^2) = -480\sigma_3(n).$$

Cusp forms arise in the general case. Special values of symmetrized shifted convolution L-functions, and Petersson norms control these cusp forms in these Eichler–Selberg relations. Throughout, we let d_{2k} denote the dimension of S_{2k}, the space of weight $2k$ cusp forms on $\mathrm{SL}_2(\mathbb{Z})$.

Theorem 1.4. *If $\nu \geq 1$ and $m \geq 1$, then we have*

$$\mathcal{G}_{m,\nu}(\tau) = \sum_{\kappa | m} \sum_{0 < r \leq \kappa} r^{2\nu+1} P_{2\nu+2, -r(\kappa - r)}(\tau) - \sum_{j=1}^{d_{2\nu+2}} \left(24\sigma_1(m) - \frac{\Gamma(2\nu+1)}{(4\pi)^{2\nu+1}} \frac{\widehat{L}(f_j, m; 2\nu+1)}{\|f_j\|^2} \right) f_j,$$

where the f_j's are normalized Hecke eigenforms of $S_{2\nu+2}$ and

$$\widehat{L}(f, m; s) := \sum_{n=1}^{\infty} \frac{c_f(n)c_f(n+m)}{n^s} - \sum_{n=1}^{\infty} \frac{c_f(n)c_f(n-m)}{n^s}.$$

Example. Example 1 of [22] gives $\widehat{L}(\Delta, 1; 11) = -33.383\ldots$ and $\widehat{L}(\Delta, 2; 11) = 266.439\ldots$, which arise in Theorem 1.4 when $\nu = 5$ and $m \in \{1, 2\}$. By brute force computation, we have

$$\mathcal{G}_{1,5}(\tau) = E_{12}(\tau) - \frac{82104}{691}\Delta(\tau),$$
$$\mathcal{G}_{2,5}(\tau) = P_{12,-1}(\tau) + 2049E_{12}(\tau) - \left(\alpha - \frac{1746612}{691}\right)\Delta(\tau),$$

where

$$P_{12,-1}(\tau) = \Delta(\tau)(j_2(\tau) + 24j_1(\tau) + 324 + \alpha) = q^{-1} + \alpha q + \cdots,$$

with $\alpha = 1842.894\ldots$. Using $\|\Delta\|^2 = \langle \Delta, \Delta \rangle = 0.0000010353\ldots$, these numerics illustrate Theorem 1.4

$$\frac{82104}{691} = 24 + \frac{65520}{691} = 24 - \frac{\Gamma(11)}{(4\pi)^{11}} \frac{(-33.383\ldots)}{\|\Delta\|^2},$$
$$\alpha - \frac{1746612}{691} = 24 \cdot 3 - \frac{\Gamma(11)}{(4\pi)^{11}} \frac{(266.439\ldots)}{\|\Delta\|^2}.$$

Theorem 1.4 gives a doubly infinite family of modified Eichler–Selberg trace formulas, where Hecke eigenvalues are weighted by shifted convolution L-values and where traces of singular moduli $\mathbf{t}_m(4n - r^2)$

replace the Hurwitz–Kronecker class numbers $\mathbf{t}_0(4n-r^2) = H(4n-r^2)$. To make this precise, we let

$$\mathrm{Tr}_m(n;2k) := \frac{\Gamma(2k-1)}{(4\pi)^{2k-1}} \sum_{j=1}^{d_{2k}} \frac{\widehat{L}(f_j, m; 2k-1)}{\|f_j\|^2} \cdot c_{f_j}(n), \tag{1.9}$$

where, as above, $c_{f_j}(n)$ is the eigenvalue of T_n for the Hecke eigenform $f_j \in S_{2k}$.

Corollary 1.5. *If* $2k \in 2\mathbb{Z}^+ \setminus \{2,4,6,8,10,14\}$ *and m is a positive integer, then we have*

$$\mathrm{Tr}(n;2k) = \frac{1}{24\sigma_1(m)} \cdot \left(\mathrm{Tr}_m(n;2k) + \frac{1}{2}\sum_{r\in\mathbb{Z}} p_{2k}(r,n)\mathbf{t}_m(4n-r^2) + \sum_{\kappa \mid m} \sum_{0<r\le\kappa} r^{2k-1} c_{2k,-r(\kappa-r)}(n) \right).$$

To obtain these results, we adapt Zagier's novel (unpublished) proof [32] of the Eichler–Selberg trace formula. In Section 2, we recall his proof and his work on traces of singular moduli, and we prove Theorems 1.1–1.3. The proof of Theorem 1.4 is more involved, as we make use of the theory of vector-valued Poincaré series, the arithmetic of half-integral weight Kloosterman sums, Rankin–Cohen bracket operators and symmetrized shifted convolution L-functions. In Section 3, we recall important formalities regarding vector-valued modular forms that transform according to the Weil representation. In Section 4, we relate the Fourier coefficients of half-integral weight Maass–Poincaré series to traces of singular moduli, and finally, in Section 5, we assemble these facts to prove Theorem 1.4.

2. Zagier's work and the proofs of Theorems 1.1–1.3

In unpublished notes [32], Zagier gave a novel proof of the Eichler–Selberg trace formula using harmonic Maass forms (see [7] or [4] for background on harmonic Maass forms). Saad and the third author [26] obtained further such formulas by modifying his argument. We adapt his argument in a different aspect.

2.1. Zagier's Proof

We begin by sketching his proof, which relies on the following theorem.

Theorem (Zagier [33]). We have that

$$\mathcal{H}(\tau) := -\frac{1}{12} + \sum_{\substack{d>0 \\ d\equiv 0,3 \pmod 4}} H(d)q^d + \frac{1}{8\pi\sqrt{v}} + \frac{1}{4\sqrt{\pi}} \sum_{n=1}^{\infty} n\Gamma\left(-\frac{1}{2}; 4\pi n^2 v\right) q^{-n^2}$$

is a harmonic Maass form of weight 3/2 on $\Gamma_0(4)$, where $\tau = u + iv$ and $\Gamma(s;x)$ is the incomplete Gamma function. Its *holomorphic part* is the Fourier series

$$\mathcal{H}^+(\tau) := -\frac{1}{12} + \sum_{\substack{d>0 \\ d\equiv 0,3 \pmod 4}} H(d)q^d.$$

Zagier uses a sequence of modular forms he constructs from $\mathcal{H}(\tau)$ and Jacobi's weight 1/2 theta function

$$\theta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + \cdots. \tag{2.1}$$

https://doi.org/10.1017/fms.2024.126 Published online by Cambridge University Press

To define these modular forms, he requires Atkin's U-operator defined by

$$(f|U_m)(\tau) := \frac{1}{m}\sum_{j=0}^{m-1} f\left(\frac{\tau+j}{m}\right) \tag{2.2}$$

and the Rankin–Cohen bracket differential operators. For modular forms f and g (possibly non-holomorphic), with weights k and l, respectively, these operators are defined by

$$[f,g]_\nu := \sum_{\substack{r,s\geq 0\\ r+s=\nu}} (-1)^r \frac{\Gamma(k+\nu)\Gamma(l+\nu)}{s!\Gamma(k+r)r!\Gamma(l+s)} D^r(f)D^s(g), \tag{2.3}$$

where $D = \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\tau} = q\frac{\mathrm{d}}{\mathrm{d}q} = \frac{1}{2\pi i}\frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)$. These functions are weight $2\nu + k + l$ (possibly non-holomorphic) modular forms, which one can project to obtain a holomorphic modular form via an integral map π_{hol}.

Zagier studies the resulting sequence of modular forms $\pi_{\mathrm{hol}}([\mathcal{H},\theta]_\nu|U_4)$, where $\nu \geq 1$. He computes them in two ways. The first method is combinatorial, and it uses the identity (for example, see [20, 21])

$$\pi_{\mathrm{hol}}([\mathcal{H},\theta]_\nu|U_4) = [\mathcal{H}^+,\theta]_\nu|U_4 + 2\binom{2\nu}{\nu}\sum_{n=1}^{\infty}\lambda_{2\nu+1}(n)q^n.$$

A straightforward brute force calculation with (2.3) gives

$$[\mathcal{H}^+,\theta]_\nu|U_4 = \binom{2\nu}{\nu}\sum_{n=0}^{\infty}\left(\sum_{r\in\mathbb{Z}} p_{2\nu+2}(r,n)H(4n-r^2)\right)q^n. \tag{2.4}$$

Therefore, the nth coefficient of $\pi_{\mathrm{hol}}([\mathcal{H},\theta]_\nu|U_4)$ is

$$\binom{2\nu}{\nu}\left(\sum_{r\in\mathbb{Z}} p_{2\nu+2}(r,n)H(4n-r^2) + 2\lambda_{2\nu+1}(n)\right). \tag{2.5}$$

As an alternate calculation, Zagier combines (for example, see [13, Theorem 5.5]) the Rankin–Cohen bracket operators with Hecke–Petersson theory. As each $\pi_{\mathrm{hol}}([\mathcal{H},\theta]_\nu|U_4)$ is a cusp form, we have

$$\pi_{\mathrm{hol}}([\mathcal{H},\theta]_\nu|U_4) = \sum_{j=1}^{d_{2\nu+2}} a_j f_j,$$

where the f_j's form a basis of Hecke eigenforms for $S_{2\nu+2}$. In particular, we have $T_n f_j = c_{f_j}(n) f_j$, where

$$f_j(\tau) = q + \sum_{n\geq 2} c_{f_j}(n)q^n.$$

To compute the a_j, he expresses $\mathcal{H}(\tau)$ in terms of Eisenstein series (see [26, Section 2.2] or [14, Ch. 2]), which allows him to use the method of unfolding and the Rankin–Selberg method to derive the Petersson inner product identity (for example, see [4, Ch. 6.3])

$$a_j\langle f_j, f_j\rangle = \langle \pi_{\mathrm{hol}}([\mathcal{H},\theta]_\nu|U_4), f_j\rangle = -2\binom{2\nu}{\nu}\langle f_j, f_j\rangle.$$

For each j, this gives $a_j = -2\binom{2\nu}{\nu}$. Therefore, the nth coefficient of $\pi_{\mathrm{hol}}([\mathcal{H},\theta]_\nu|U_4)$ is $-2\binom{2\nu}{\nu}\cdot \mathrm{Tr}(n; 2\nu+2)$, which when equated with (2.5) gives the Eichler–Selberg trace formula.

https://doi.org/10.1017/fms.2024.126 Published online by Cambridge University Press

2.2. Proofs of Theorems 1.1–1.3

Zagier's proof begins with the fact that $\mathcal{H}^+(\tau)$ is the holomorphic part of a weight 3/2 harmonic Maass form. In 2002, Zagier [34] greatly generalized this fact.

Theorem 5 of [34]. *For positive integers m, we have that*

$$g_m(\tau) := -\sum_{\kappa|m} \kappa q^{-\kappa^2} + 2\sigma_1(m) + \sum_{\substack{d>0 \\ d\equiv 0,3 \pmod 4}} \mathbf{t}_m(d) q^d \tag{2.6}$$

is a weakly holomorphic modular form of weight $3/2$ *on* $\Gamma_0(4)$.

Proof of Theorem 1.1. Emulating Zagier's proof of the Eichler–Selberg trace formula, we replace $\mathcal{H}^+(\tau)$ in (2.4) with the $g_m(\tau)$. Namely, we define

$$\mathcal{G}_{m,\nu}(\tau) := -\frac{1}{2\binom{2\nu}{\nu}} \cdot [g_m, \theta]_\nu | U_4.$$

By the combinatorial calculation that gave (2.4), we obtain the earlier definition (1.5)

$$\mathcal{G}_{m,\nu}(\tau) = -\frac{1}{2} \sum_{n \gg -\infty} \sum_{r\in\mathbb{Z}} p_{2\nu+2}(r,n) \mathbf{t}_m(4n-r^2) q^n.$$

Furthermore, the theory of Rankin–Cohen brackets in this setting (see [13, Theorem 5.5]) implies that $\mathcal{G}_{m,\nu}(\tau)$ is a weakly holomorphic modular form in $M^!_{2\nu+2}$. □

Proof of Theorem 1.2. The space of weight 2 holomorphic modular forms is $M_2 = \{0\}$ and

$$Dj_{-n}(\tau) = nq^n + O(q) \in M_2^!.$$

Therefore, we have

$$\mathcal{G}_{m,0}(\tau) + \frac{1}{2} \sum_{-\frac{m^2}{4} \le n < 0} \frac{1}{n} \left(\sum_{r\in\mathbb{Z}} \mathbf{t}_m(4n-r^2) \right) Dj_{-n}(\tau) = 0.$$

The first claim follows from (1.6). By comparing the nth coefficients, the second claim is obtained. □

Proof of Theorem 1.3. For $\nu > 0$, we note that

$$\mathcal{G}_{m,\nu}(\tau) + \frac{1}{2} \sum_{-\frac{m^2}{4} \le n \le 0} \sum_{r\in\mathbb{Z}} p_{2\nu+2}(r,n) \mathbf{t}_m(4n-r^2) P_{2\nu+2,n}(\tau) \tag{2.7}$$

is a cusp form. We are merely cancelling the poles at infinity with Poincaré series that satisfy (1.8), and we capture the constant term with Eisenstein series $P_{2\nu+2,0}(\tau) = E_{2\nu+2}(\tau) = 1 + \cdots$. For $\nu \in \{1,2,3,4,6\}$, the space of cusp forms $S_{2\nu+2} = \{0\}$ is trivial. Therefore, the theorem follows from the identity

$$p_{2\nu+2}\left(r, \frac{r^2-\kappa^2}{4}\right) = \frac{(\kappa-r)^{2\nu+1} + (\kappa+r)^{2\nu+1}}{2^{2\nu+1}\kappa}. \tag{2.8}$$

□

https://doi.org/10.1017/fms.2024.126 Published online by Cambridge University Press

3. Vector-valued modular forms

The proof of Theorem 1.4 is much more involved than the proofs of Theorems 1.2 and 1.3. Nevertheless, its proof is still based on Theorem 1.1, and the aim is to understand the Fourier expansion of $\mathcal{G}_{m,\nu}(\tau)$ arithmetically in terms of traces of Hecke operators and shifted convolution L-functions. These calculations shall depend on the arithmetic of half-integral weight vector-valued modular forms that transform with respect to the Weil representation. To this end, here we recall essential preliminaries.

3.1. The Weil representation

Let $\mathcal{O}(\mathbb{H})$ be the set of all holomorphic functions $\phi : \mathbb{H} \to \mathbb{C}$. For $z \in \mathbb{C} \setminus \{0\}$, we take the principal branch of $z^{1/2}$ as $\arg(z^{1/2}) \in (-\pi/2, \pi/2]$. For an integer $k \in \mathbb{Z}$, we put $z^{k/2} = (z^{1/2})^k$. For $n \in \mathbb{Z}_{\geq 0}$, we put $x^{\overline{n}} := \Gamma(x+n)/\Gamma(x) = x(x+1)\cdots(x+n-1)$, and $x^{\underline{n}} := \Gamma(x+1)/\Gamma(x-n+1) = x(x-1)\cdots(x-n+1)$.

The *metaplectic group* $\mathrm{Mp}_2(\mathbb{R})$ is a group defined by

$$\mathrm{Mp}_2(\mathbb{R}) := \left\{(\gamma, \phi(\tau)) : \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), \phi \in \mathcal{O}(\mathbb{H}) \text{ satisfying } \phi(\tau)^2 = c\tau + d\right\},$$

where the group operation is $(\gamma_1, \phi_1(\tau)) \cdot (\gamma_2, \phi_2(\tau)) := (\gamma_1\gamma_2, \phi_1(\gamma_2\tau)\phi_2(\tau))$.

As usual, we have $\gamma\tau := \frac{a\tau+b}{c\tau+d}$, and for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, we define $j(\gamma, \tau) = c\tau + d$ and $\widetilde{\gamma} = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), j(\gamma, \tau)^{1/2}\right) \in \mathrm{Mp}_2(\mathbb{R})$. Let $\mathrm{Mp}_2(\mathbb{Z})$ be the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the projection $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$. As usual, we let $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. It is well known that $\mathrm{Mp}_2(\mathbb{Z})$ is generated by $\widetilde{T}$ and $\widetilde{S}$, (see [6, p.16]) and its center is generated by

$$\widetilde{-I} = \widetilde{S}^2 = (\widetilde{S}\widetilde{T})^3 = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i\right).$$

Moreover, we let $\widetilde{\Gamma}_\infty := \langle \widetilde{T} \rangle \times \langle \widetilde{-I} \rangle$, representing the metaplectic stabilizer for the cusp at infinity.

We recall the *Weil representation*,[1] the unitary representation $\rho : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{C})$ defined by

$$\rho(\widetilde{T}) := \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad \text{and} \quad \rho(\widetilde{S}) := \frac{1}{\sqrt{2i}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{3.1}$$

We note that $\rho(\widetilde{-I}) = \rho(\widetilde{S}^2) = -iI$. We let $\rho^* : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{C})$ be the dual representation of ρ

$$\rho^*((\gamma, \phi)) := {}^t\rho((\gamma, \phi))^{-1} = \overline{\rho((\gamma, \phi))}.$$

We recall an explicit formula for $\rho(\widetilde{\gamma})$, which is easily derived from work of both Shintani [29, Proposition 1.6] and Bruinier [6, Proposition 1.1], where for odd integers d, we let

$$\epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases} \tag{3.2}$$

Proposition 3.1. *For $c \geq 0$, we have*

$$\rho(\widetilde{\gamma}) = \begin{cases} \dfrac{\epsilon_c}{1+i}\left(\dfrac{a}{c}\right)\begin{pmatrix} 1 & i^{cd} \\ i^{ac} & -i^{(a+d)c} \end{pmatrix} & \textit{if } c \equiv 1 \pmod 2, \\ \epsilon_a^{-1}\left(\dfrac{c}{a}\right)\begin{pmatrix} 0 & i^{ab} \\ 1 & 0 \end{pmatrix} & \textit{if } c \equiv 2 \pmod 4, \\ \epsilon_a^{-1}\left(\dfrac{c}{a}\right)\begin{pmatrix} 1 & 0 \\ 0 & i^{ab} \end{pmatrix} & \textit{if } c \equiv 0 \pmod 4. \end{cases}$$

[1]For more general settings, see Bruinier [6, Ch. 1] and Borcherds [3]. As mentioned in the Bruinier's book, this representation is essentially the Weil representation associated with the discriminant group $L'/L \cong \mathbb{Z}/2\mathbb{Z}$, where L is a certain lattice with a quadratic form. For specific settings, refer to [9, Section 2].

https://doi.org/10.1017/fms.2024.126 Published online by Cambridge University Press

We now give the definition of a vector-valued modular form that transforms under the Weil representation. If $k \in \frac{1}{2}\mathbb{Z}$ and $f : \mathbb{H} \to \mathbb{C}^2$. For $(\gamma, \phi(\tau)) \in \mathrm{Mp}_2(\mathbb{Z})$, then we define the *slash operator*

$$(f|_{k,\rho}(\gamma, \phi))(\tau) := \phi(\tau)^{-2k}\rho((\gamma, \phi))^{-1} f(\gamma\tau).$$

We say that $f : \mathbb{H} \to \mathbb{C}^2$ is a *weight k (vector-valued) modular form with respect to* ρ if

$$f|_{k,\rho}(\gamma, \phi) = f$$

for every $(\gamma, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$. We define them for ρ^* in a similar manner.

3.2. Jacobi's theta functions

For later use, we recall the Jacobi theta functions (for example, see [13, Section 5]) in this context. If we set $\zeta := \mathbf{e}(z)$, where $\mathbf{e}(z) := e^{2\pi i z}$, we have

$$\theta_0(\tau, z) := \sum_{\substack{r \in \mathbb{Z} \\ r \equiv 0\ (2)}} q^{r^2/4}\zeta^r \quad \textit{and} \quad \theta_1(\tau, z) := \sum_{\substack{r \in \mathbb{Z} \\ r \equiv 1\ (2)}} q^{r^2/4}\zeta^r \tag{3.3}$$

and $\Theta(\tau, z) := \begin{pmatrix} \theta_0(\tau,z) \\ \theta_1(\tau,z) \end{pmatrix}$. The specialization $\Theta(\tau, 0)$ is a weight 1/2 vector-valued modular form with respect to ρ, and in general is a (vector-valued) Jacobi form, which for $(\gamma, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$, in this case, means that

$$(\Theta|_{1/2,1,\rho}(\gamma, \phi))(\tau, z) := \phi(\tau)^{-1}\mathbf{e}\left(\frac{-cz^2}{c\tau + d}\right)\rho((\gamma, \phi))^{-1}\Theta\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = \Theta(\tau, z). \tag{3.4}$$

4. Maass–Poincaré series and traces of singular moduli

The proof of Theorem 1.4 relies on Maass–Poincaré series that transform with respect to the Weil representation. We construct these series following [6], and we relate them to traces of singular moduli. The goal of this section, Theorem 4.5, can be immediately derived as a special case of Alfes' result [1, Theorem 4.3], which applies the Kudla–Millson theta lift ([7]) to integer weight Poincaré series. However, we will also provide a direct proof that requires minimal advanced prior knowledge below.

4.1. The Whittaker functions

Let $M_{\mu,\nu}(z)$ and $W_{\mu,\nu}(z)$ be the Whittaker functions (for example, see [30, Ch. 16] and [17, 24]). The next two lemmas are crucial for constructing Maass–Poincaré series.

Lemma 4.1 [17, 7.2.1], [24, 13.15.19]. *For positive integers n, we have*

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(e^{-z/2}z^{-\nu-1/2}M_{\mu,\nu}(z)\right) = (-1)^n \frac{(\mu + \nu + 1/2)^{\overline{n}}}{(2\nu + 1)^{\overline{n}}} e^{-z/2}z^{-\nu-n/2-1/2}M_{\mu+n/2,\nu+n/2}(z).$$

Lemma 4.2 [17, 7.5.1], [24, 13.23.1]. *For* $\mathrm{Re}(\nu + \alpha + 1/2) > 0$ *and* $2\,\mathrm{Re}(z) > \beta > 0$, *we have*

$$\int_0^\infty e^{-zt}t^{\alpha-1}M_{\mu,\nu}(\beta t)\mathrm{d}t = \frac{\beta^{\nu+1/2}\Gamma\left(\nu + \alpha + \frac{1}{2}\right)}{\left(z + \frac{\beta}{2}\right)^{\nu+\alpha+1/2}} \cdot {}_2F_1\left(\nu - \mu + \frac{1}{2}, \nu + \alpha + \frac{1}{2}; 2\nu + 1; \frac{2\beta}{\beta + 2z}\right),$$

where ${}_2F_1(a, b; c; z)$ *is the Gaussian hypergeometric function.*

https://doi.org/10.1017/fms.2024.126 Published online by Cambridge University Press

For $n \in \mathbb{Z}$, $k \in \frac{1}{2}\mathbb{Z}$, $y > 0$, and $s \in \mathbb{C}$, we define the modified Whittaker functions

$$\mathcal{M}_{k,n}(y,s) := \begin{cases} \Gamma(2s)^{-1}(4\pi|n|y)^{-k/2} M_{\operatorname{sgn}(n)\frac{k}{2}, s-1/2}(4\pi|n|y) & \text{if } n \neq 0, \\ y^{s-k/2} & \text{if } n = 0, \end{cases} \tag{4.1}$$

$$\mathcal{W}_{k,n}(y,s) := \begin{cases} \Gamma(s+\operatorname{sgn}(n)\frac{k}{2})^{-1}|n|^{k-1}(4\pi|n|y)^{-k/2} W_{\operatorname{sgn}(n)\frac{k}{2}, s-1/2}(4\pi|n|y) & \text{if } n \neq 0, \\ \dfrac{(4\pi)^{1-k} y^{1-s-k/2}}{(2s-1)\Gamma(s-k/2)\Gamma(s+k/2)} & \text{if } n = 0. \end{cases} \tag{4.2}$$

The special values of these functions at $s = k/2$ play a crucial role in the construction of the Maass–Poincaré series. To this end, for $n < 0$, we have

$$\mathcal{M}_{k,n}\left(y, \frac{k}{2}\right) = \Gamma(k)^{-1} e^{-2\pi n y}. \tag{4.3}$$

As for the $\mathcal{W}$-function, we have

$$\mathcal{W}_{k,n}\left(y, \frac{k}{2}\right) = \begin{cases} \Gamma(k)^{-1} n^{k-1} e^{-2\pi n y} & \text{if } n > 0, \\ 0 & \text{if } n \leq 0 \end{cases} \tag{4.4}$$

(see [17, 7.2.4]). Moreover, we note that, ([17, 7.6.1], [24, 13.14]),

$$\begin{aligned} W_{\mu,\nu}(y) &\sim e^{-y/2} y^{\mu} \quad (y \to \infty), \\ M_{\mu,\nu}(y) &= y^{\nu+1/2}(1 + O(y)) \quad (y \to 0). \end{aligned} \tag{4.5}$$

4.2. Kloosterman sums

The Fourier expansions of the Maass–Poincaré series require Kloosterman sums, which we recall here. For $k \in \frac{1}{2}\mathbb{Z} \backslash \mathbb{Z}$, $m, n \in \mathbb{Z}$, and $c > 0$ with $c \equiv 0 \pmod 4$, we define the half-integral weight *Kloosterman sum* by

$$K_k(m,n,c) := \sum_{d\ (c)^*} \left(\frac{c}{d}\right) \epsilon_d^{2k} \mathbf{e}\left(\frac{m\overline{d} + nd}{c}\right), \tag{4.6}$$

where $\overline{d} \in \mathbb{Z}/c\mathbb{Z}$ satisfies that $d\overline{d} \equiv 1 \pmod c$. The condition $d\ (c)^*$ means that d runs over $d \in \mathbb{Z}/c\mathbb{Z}$ such that $(c,d) = 1$. We note that the Kloosterman sums satisfy

$$K_{k+2}(m,n,c) = K_k(m,n,c) \quad \text{and} \quad K_{3/2}(m,n,c) = -iK_{1/2}(-m,-n,c). \tag{4.7}$$

We now relate the Weil representation to such Kloosterman sums. For notational convenience, we let

$$\rho(\widetilde{\gamma}) = \begin{pmatrix} \rho(\widetilde{\gamma})_{00} & \rho(\widetilde{\gamma})_{01} \\ \rho(\widetilde{\gamma})_{10} & \rho(\widetilde{\gamma})_{11} \end{pmatrix}.$$

Then the following sum formula holds for each entry of $\rho(\widetilde{\gamma})$.

Proposition 4.3. *If $\alpha, \beta \in \{0,1\}$ and m and n satisfy $m \equiv -\alpha \pmod 4$ and $n \equiv -\beta \pmod 4$, then for every positive integer c, we have*

$$\frac{1}{4}\left(1 + \left(\frac{4}{c}\right)\right) K_{3/2}(m,n,4c) = \sum_{d\ (c)^*} \rho(\widetilde{\gamma})_{\alpha\beta} \mathbf{e}\left(\frac{ma+nd}{4c}\right),$$

where we take any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ for which (c,d) forms its bottom row.

https://doi.org/10.1017/fms.2024.126 Published online by Cambridge University Press

Proof. First, we check that the right-hand side is well defined. Let $R_{\alpha\beta}(\gamma)$ denote its summand. It suffices to show that $R_{\alpha\beta}(T^j\gamma T^l) = R_{\alpha\beta}(\gamma)$ holds for any $j, l \in \mathbb{Z}$. Since $\widetilde{T^j\gamma T^l} = \widetilde{T}^j\widetilde{\gamma}\widetilde{T}^l$ holds, we have

$$R_{\alpha\beta}(T^j\gamma T^l) = i^{\alpha j+\beta l}\rho(\widetilde{\gamma})_{\alpha\beta}\mathbf{e}\left(\frac{ma+nd}{4c}\right)\mathbf{e}\left(\frac{mj+nl}{4}\right) = R_{\alpha\beta}(\gamma).$$

Next, for each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $c > 0$, we prove the refined equation

$$\rho(\widetilde{\gamma})_{\alpha\beta} = \frac{1}{4}\left(1 + \left(\frac{4}{c}\right)\right)\sum_{\substack{\delta\ (4c) \\ \delta\equiv d\ (c) \\ \delta\equiv 1\ (2)}}\left(\frac{c}{\delta}\right)\epsilon_\delta^{-1}\mathbf{e}\left(\frac{a-\overline{\delta}}{4c}\right)^{\alpha}\mathbf{e}\left(\frac{d-\delta}{4c}\right)^{\beta}, \tag{4.8}$$

where $\overline{\delta}$ is the inverse of δ in $(\mathbb{Z}/4c\mathbb{Z})^\times$. This immediately implies the proposition.

To confirm (4.8), let $\delta = d_j := d + cj$ and $b_j := b + aj$ $(j = 0, 1, 2, 3)$. For simplicity, let $\rho''(\gamma)_{\alpha\beta}$ denote the right-hand side of (4.8) and show that $\rho''(\gamma)_{\alpha\beta} = \rho(\widetilde{\gamma})_{\alpha\beta}$. If δ is odd, then we can easily check that

$$\overline{\delta} = \begin{cases} a(1+b_jc) & \text{if } c \equiv 1 \pmod 2 \text{ and } a \equiv 1 \pmod 2, \\ (a+c)(1+(b_j+d_j)c) & \text{if } c \equiv 1 \pmod 2 \text{ and } a \equiv 0 \pmod 2, \\ a(1-ab_jcd_j) & \text{if } c \equiv 2 \pmod 4, \\ a(1-b_jc) & \text{if } c \equiv 0 \pmod 4. \end{cases}$$

We prove the case where $c \equiv 1 \pmod 2$ and $a \equiv 1 \pmod 2$, leaving the others to the reader. We have

$$\rho''(\gamma)_{\alpha\beta} = \frac{1}{2}\sum_{\substack{0\le j\le 3 \\ d_j\equiv 1\ (2)}}\left(\frac{c}{d_j}\right)\epsilon_{d_j}^{-1}\mathbf{e}\left(\frac{-ab_j}{4}\right)^{\alpha}\mathbf{e}\left(\frac{-j}{4}\right)^{\beta} = \frac{i^{-ab\alpha}}{2}\left(\frac{d}{c}\right)\sum_{\substack{0\le j\le 3 \\ d_j\equiv 1\ (2)}}(-1)^{\frac{(c-1)(d_j-1)}{4}}\epsilon_{d_j}^{-1}i^{-j(\alpha+\beta)}.$$

Since the value of the sum depends only on $c, d \pmod 4$, a direct calculation yields

$$\rho''(\gamma)_{\alpha\beta} = \frac{i^{-ab\alpha}}{2}\left(\frac{d}{c}\right)\frac{2\epsilon_c}{1+i}\times\begin{cases} 1 & \text{if } (\alpha,\beta) = (0,0), \\ i^{cd} & \text{if } (\alpha,\beta) = (0,1),(1,0), \\ (-1)^{d-1} & \text{if } (\alpha,\beta) = (1,1). \end{cases}$$

Combining simple calculations with Proposition 3.1, one obtains $\rho(\widetilde{\gamma})_{\alpha\beta}$. □

4.3. The Maass–Poincaré series

Using the two previous subsections, we now construct the Maass–Poincaré series. We let $\mathfrak{e}_0 := \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\mathfrak{e}_1 := \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Assume that $k \in \frac{1}{2}\mathbb{Z}$ satisfies $2k \equiv 3 \pmod 4$. For $\alpha \in \{0, 1\}$ and $m \equiv -\alpha \pmod 4$, we define the *Maass–Poincaré series* of weight k with respect to ρ^* by

$$\begin{aligned} P_{k,\rho^*}^{(\alpha,m)}(\tau, s) &:= \sum_{(\gamma,\phi)\in\widetilde{\Gamma}_\infty\backslash \mathrm{Mp}_2(\mathbb{Z})}\mathcal{M}_{k,m}\left(\frac{v}{4}, s\right)\mathbf{e}\left(\frac{mu}{4}\right)\mathfrak{e}_\alpha\bigg|_{k,\rho^*}(\gamma,\phi) \\ &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\mathcal{M}_{k,m}\left(\frac{v}{4}, s\right)\mathbf{e}\left(\frac{mu}{4}\right)\mathfrak{e}_\alpha\bigg|_{k,\rho^*}\widetilde{\gamma}. \end{aligned} \tag{4.9}$$

https://doi.org/10.1017/fms.2024.126 Published online by Cambridge University Press

This series converges absolutely and uniformly on compact subsets in $\mathrm{Re}(s) > 1$ [6, p.29], and we note that $\mathcal{M}_{k,m}(v/4,s)\mathbf{e}(mu/4)\mathfrak{e}_\alpha$ is invariant under $|_{k,\rho^*}(\gamma,\phi)$ for any $(\gamma,\phi) \in \widetilde{\Gamma}_\infty$ as $2k \equiv 3 \pmod 4$.

The Fourier expansions of the functions involve the Bessel functions (see [17, Ch. 3] and [30, Ch. 17])

$$I_\nu(z) := \sum_{m=0}^{\infty} \frac{(z/2)^{\nu+2m}}{m!\Gamma(\nu+m+1)}, \quad J_\nu(z) := \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\nu+2m}}{m!\Gamma(\nu+m+1)}.$$

Proposition 4.4. *For* $\mathrm{Re}(s) > 1$*, we have*

$$P_{k,\rho^*}^{(\alpha,m)}(\tau,s) = \mathcal{M}_{k,m}\Big(\frac{v}{4},s\Big)\mathbf{e}\Big(\frac{mu}{4}\Big)\mathfrak{e}_\alpha + \sum_{\beta\in\{0,1\}} \sum_{\substack{n\in\mathbb{Z}\\ n\equiv-\beta\ (4)}} b_{m,k}^{(\beta)}(n,s)\mathcal{W}_{k,n}\Big(\frac{v}{4},s\Big)\mathbf{e}\Big(\frac{nu}{4}\Big)\mathfrak{e}_\beta,$$

where

$$b_{m,k}^{(\beta)}(n,s) = 2\pi i^{-k} \sum_{c>0}\left(1+\left(\frac{4}{c}\right)\right)\frac{K_{3/2}(m,n,4c)}{4c} \times \begin{cases} |mn|^{\frac{1-k}{2}} J_{2s-1}\left(\frac{\pi\sqrt{|mn|}}{c}\right) & \text{if } mn > 0,\\ |mn|^{\frac{1-k}{2}} I_{2s-1}\left(\frac{\pi\sqrt{|mn|}}{c}\right) & \text{if } mn < 0,\\ 2^{k-1}\pi^{s+k/2-1}|m+n|^{s-k/2}(4c)^{1-2s} & \text{if } mn = 0, m+n \neq 0,\\ 2^{2k-2}\pi^{k-1}\Gamma(2s)(8c)^{1-2s} & \text{if } m = n = 0. \end{cases}$$

Proof. Dividing the sum of the Poincaré series into the identity class and the remaining part, we have

$$P_{k,\rho^*}^{(\alpha,m)}(\tau,s) = \mathcal{M}_{k,m}\Big(\frac{v}{4},s\Big)\mathbf{e}\Big(\frac{mu}{4}\Big)\mathfrak{e}_\alpha + \sum_{\substack{\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma\\ c>0}} \mathcal{M}_{k,m}\Big(\frac{v}{4},s\Big)\mathbf{e}\Big(\frac{mu}{4}\Big)\mathfrak{e}_\alpha\Big|_{k,\rho^*}\widetilde{\gamma}.$$

Let $H_{k,\rho^*}^{(\alpha,m)}(\tau,s)$ denote the sum of the second term. By following the exact same argument as in the proof of Theorem 1.9 in Bruinier's book [6], we obtain the Fourier expansion,

$$H_{k,\rho^*}^{(\alpha,m)}(\tau,s) = \sum_{\beta\in\{0,1\}}\sum_{n\in\mathbb{Z}}\left(\sum_{c>0}\sum_{d\in(\mathbb{Z}/c\mathbb{Z})^\times} \rho(\widetilde{\gamma})_{\alpha\beta}\mathbf{e}\left(\frac{ma+nd}{4c}\right)I_m(n)\right)\mathbf{e}\Big(\frac{nu}{4}\Big)\mathfrak{e}_\beta,$$

where $I_m(n)$ is given by

$$I_m(n) = \begin{cases} \dfrac{2\pi i^{-k}}{c}|mn|^{\frac{1-k}{2}} J_{2s-1}\left(\dfrac{\pi\sqrt{|mn|}}{c}\right)\mathcal{W}_{k,n}(v/4,s) & \text{if } mn > 0,\\ \dfrac{2\pi i^{-k}}{c}|mn|^{\frac{1-k}{2}} I_{2s-1}\left(\dfrac{\pi\sqrt{|mn|}}{c}\right)\mathcal{W}_{k,n}(v/4,s) & \text{if } mn < 0,\\ \dfrac{4^{1+k/2-2s}\pi^{s+k/2}i^{-k}|m+n|^{s-k/2}}{c^{2s}}\mathcal{W}_{k,n}(v/4,s) & \text{if } mn = 0, m+n\neq 0,\\ \dfrac{4^{-3s+k+1}\pi^k i^{-k}}{c^{2s}}\Gamma(2s)\mathcal{W}_{k,0}(v/4,s) & \text{if } m = n = 0. \end{cases}$$

By combining this with Proposition 4.3, we obtain Proposition 4.4. □

https://doi.org/10.1017/fms.2024.126 Published online by Cambridge University Press

4.4. Traces of singular moduli

The coefficients of these functions are related to traces of singular moduli, as shown in several previous works (for example, see [5, 8, 12]). To make this precise, we consider weight 3/2 modular forms h on $\Gamma_0(4)$ satisfying

$$h(\tau) = \sum_{n \equiv 0,3 \ (4)} c_n(v) q^n. \tag{4.10}$$

We define $h_i(\tau) = \sum_{n \equiv -i \ (4)} c_n(v/4) q^{n/4}$ for $i \in \{0, 1\}$, and then we have that

$$H(\tau) := \begin{pmatrix} h_0(\tau) \\ h_1(\tau) \end{pmatrix} \tag{4.11}$$

is a weight 3/2 vector-valued modular form with respect to ρ^* (see [13, Section 5] and [4, Ch. 2]).

We relate the $g_m(\tau)$ in (2.6) and $g_0(\tau) := \mathcal{H}(\tau)$ to the Maass–Poincaré expressions

$$G_m(\tau, s) := \begin{cases} -\dfrac{1}{12} P^{(0,0)}_{3/2,\rho^*}(\tau, s) & \text{if } m = 0, \\ -\dfrac{\sqrt{\pi}}{2} \displaystyle\sum_{n|m} n P^{(\alpha,-n^2)}_{3/2,\rho^*}(\tau, s) + 2\sigma_1(m) P^{(0,0)}_{3/2,\rho^*}(\tau, s) & \text{if } m > 0, \end{cases} \tag{4.12}$$

where $\alpha \equiv n^2 \pmod 4$ for each n. To be precise, we have the following theorem.

Theorem 4.5. *If m is a nonnegative integer, then we have*

$$\lim_{s \to 3/4} G_m(\tau, s) = \begin{pmatrix} g_{m,0}(\tau) \\ g_{m,1}(\tau) \end{pmatrix}.$$

Remark. We note that the case of $m = 0$ was stated by Williams [31, Example 5.1].

Sketch of the Proof. This result is standard, and so we sketch the proof. We first recall facts about Niebur–Poincaré series $F_m(\tau, s)$ (see [23] or [12, Section 4]), which are defined for $\mathrm{Re}(s) > 1$, and give alternative expressions for the $j_m(\tau)$. Specifically, as described in [12, (4.10)], it is known that

$$\operatorname*{Res}_{s=1} F_0(\tau, s) = \frac{3}{\pi}$$

and

$$\lim_{s \to 1} F_{-m}(\tau, s) = j_m(\tau) + 24\sigma_1(m) \quad (m > 0). \tag{4.13}$$

For nonnegative integers m, the trace functions

$$\mathrm{Tr}_d(F_{-m}(\cdot, s)) := \sum_{Q \in \mathcal{Q}_d/\Gamma} \frac{F_{-m}(\alpha_Q, s)}{\#\Gamma_Q}$$

have a direct connection to the coefficients of the earlier Maass–Poincaré series. Indeed, by combining the result of Duke, Imamoḡlu and Tóth in [12, Proposition 4] with our Proposition 4.4, for $\mathrm{Re}(s) > 1$, $m \geq 0$, and $d > 0$ with $d \equiv 0, 3 \pmod 4$, we obtain that

$$\mathrm{Tr}_d(F_{-m}(\cdot, s)) = \begin{cases} -d^{1/2} \displaystyle\sum_{n|m} n b^{(\beta)}_{-n^2,3/2}\left(d, \frac{s}{2} + \frac{1}{4}\right) & \text{if } m > 0, \\ -2^{s-2}\pi^{-s/2-1} d^{1/2} \zeta(s) b^{(\beta)}_{0,3/2}\left(d, \dfrac{s}{2} + \dfrac{1}{4}\right) & \text{if } m = 0. \end{cases}$$

Therefore, (4.13) implies that

$$\mathbf{t}_m(d) = \lim_{s\to 3/4}\begin{cases} -d^{1/2}\sum_{n|m} nb^{(\beta)}_{-n^2,3/2}(d,s) + \frac{4d^{1/2}}{\sqrt{\pi}}\sigma_1(m)b^{(\beta)}_{0,3/2}(d,s) & \text{if } m>0,\\ -\frac{d^{1/2}}{6\sqrt{\pi}}b^{(\beta)}_{0,3/2}(d,s) & \text{if } m=0. \end{cases} \tag{4.14}$$

By applying (4.3) and (4.4), we thereby conclude the proof of the theorem. □

Remark. We note that subtle technicalities arise in the proof of Theorem 4.5, which have been addressed in the aforementioned works but deserve commentary. The $G_m(\tau,s)$ are defined for $\mathrm{Re}(s)>1$, where they enjoy the Fourier series expansion in Proposition 4.4. As we can only be analytically continued up to $\mathrm{Re}(s)>3/4$, care is required when letting $s\to 3/4$. In fact, the Fourier coefficients $b^{(\beta)}_{-m^2,3/2}(-n^2,s)$ have a simple pole at $s=3/4$, which cancels out with a zero from $\mathcal{W}_{3/2,-n^2}(v/4,s)$, (for example, see [12, Lemma 3]). This issue is addressed by examining the growth of the Fourier coefficients of $G_m(\tau,s)$, including $\mathrm{Tr}_d(F_{-m}(\cdot,s))$, as $d\to\infty$ and the behavior as $s\to 3/4$. We refer the reader to [8, 11, 12] for these details.

5. Proof of Theorem 1.4

We have constructed the Poincaré series $G_m(\tau,s)$ whose Fourier coefficients give the traces of singular moduli. We turn to the problem of providing the Hecke decomposition of $\mathcal{G}_{m,\nu}(\tau)$. Specifically, we compute the Petersson inner product $\langle \mathcal{G}_{m,\nu}, f\rangle$ with a normalized Hecke eigenform f of $S_{2\nu+2}$. We first recall useful facts about Jacobi forms to relate the Rankin–Cohen brackets to these Poincaré series.

5.1. *Jacobi forms and the modified heat operator*

For a function $\varphi:\mathbb{H}\times\mathbb{C}\to\mathbb{C}$, $\gamma\in\mathrm{SL}_2(\mathbb{Z})$, and positive integers $k,m\in\mathbb{Z}_{>0}$, we define the slash operator

$$(\varphi|_{k,m}\gamma)(\tau,z) := (c\tau+d)^{-k}\mathbf{e}\left(\frac{-cmz^2}{c\tau+d}\right)\varphi\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right),$$

and the *weighted heat operator*

$$L_{k,m} := -D - \frac{1}{16\pi^2 m}\left(\frac{\partial^2}{\partial z^2} + \frac{2k-1}{z}\frac{\partial}{\partial z}\right),$$

where $D=\frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{1}{2\pi i}\frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)$. Then, we have

$$L_{k,m}(\varphi|_{k,m}\gamma) = (L_{k,m}\varphi)|_{k+2,m}\gamma \tag{5.1}$$

for any $\gamma\in\mathrm{SL}_2(\mathbb{Z})$ (see [13, (11) in Section 3]). For simplicity, we put $L_k := L_{k,1}$.

Lemma 5.1. *For a Poincaré series defined by*

$$G(\tau) = \sum_{(\gamma,\phi)\in\widetilde{\Gamma}_\infty\backslash \mathrm{Mp}_2(\mathbb{Z})} \begin{pmatrix}\psi_0(\tau)\\ \psi_1(\tau)\end{pmatrix}\Bigg|_{3/2,\rho^*}(\gamma,\phi),$$

with test functions $\psi_0,\psi_1:\mathbb{H}\to\mathbb{C}$*, we have*

$${}^t\Theta(\tau,z)G(\tau) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(\theta_0(\tau,z)\psi_0(\tau)+\theta_1(\tau,z)\psi_1(\tau))|_{2,1}\gamma.$$

https://doi.org/10.1017/fms.2024.126 Published online by Cambridge University Press

Proof. By a direct calculation with (3.4), we have

$$ {}^t\Theta(\tau,z)G(\tau) = \sum_{(\gamma,\phi)\in\widetilde{\Gamma}_\infty\backslash \mathrm{Mp}_2(\mathbb{Z})} \phi(\tau)^{-4}\mathbf{e}\left(\frac{-cz^2}{c\tau+d}\right){}^t\Theta\left(\gamma\tau, \frac{z}{c\tau+d}\right){}^t\rho((\gamma,\phi))^{-1}\rho^*((\gamma,\phi))^{-1}\begin{pmatrix}\psi_0(\gamma\tau)\\ \psi_1(\gamma\tau)\end{pmatrix}. $$

Since ${}^t\rho((\gamma,\phi))^{-1}\rho^*((\gamma,\phi))^{-1} = I$ and $\phi(\tau)^{-4} = (c\tau+d)^{-2}$, we obtain the result. □

We require the following proposition for the $p_k(r,n)$ in the Eichler–Selberg trace formula.

Proposition 5.2. *For $\nu, l \in \mathbb{Z}_{\geq 0}$ and $r \in \mathbb{Z}$, we define the differential operator by*

$$ p_{2\nu+2}(r,D,l) \coloneqq \sum_{0\leq j\leq \nu} (-1)^j \binom{2\nu+2l-j}{j} \frac{\binom{2l}{l}\binom{\nu+l-j}{l}}{\binom{2\nu+2l-j}{l}\binom{\nu+l}{l}} r^{2\nu-2j}D^j. \tag{5.2} $$

Then, for a function $: \mathbb{H} \to \mathbb{C}$, *we have the Taylor expansion*

$$ L_{2\nu}\circ\cdots\circ L_2 f(\tau)(\zeta^r+\zeta^{-r}) = 2\sum_{l=0}^{\infty} p_{2\nu+2}(r,D,l)f(\tau)\frac{(2\pi i r z)^{2l}}{(2l)!}. $$

In particular, letting $p_k(r,n)$ as in (1.4), we have that $p_{2\nu+2}(r,D,0) = p_{2\nu+2}(r,D)$ and

$$ \lim_{z\to 0} L_{2\nu}\circ\cdots\circ L_2 f(\tau)(\zeta^r+\zeta^{-r}) = 2p_{2\nu+2}(r,D)f(\tau). $$

Proof. We check that the Taylor coefficients of $L_{2\nu}\circ\cdots\circ L_2 f(\tau)(\zeta^r+\zeta^{-r})$ and the sequence (5.2) satisfy the same recursion. The claim is clear for $\nu=0$. For $\nu>0$, let

$$ S_{\nu,l,j} \coloneqq (-1)^j\binom{2\nu+2l-j}{j}\frac{\binom{2l}{l}\binom{\nu+l-j}{l}}{\binom{2\nu+2l-j}{l}\binom{\nu+l}{l}} r^{2\nu-2j}D^j. $$

Then $S_{\nu,l,j}$ satisfies the recursion

$$ S_{\nu,l,j} = -DS_{\nu-1,l,j-1} + \frac{r^2}{4}\left(1+\frac{4\nu-1}{2l+1}\right)S_{\nu-1,l+1,j}, $$

for $\nu\geq 1$ and $0\leq j\leq \nu$ with $S_{\nu,l,-1}=0$, which implies that

$$ p_{2\nu+2}(r,D,l) = -Dp_{2\nu}(r,D,l) + \frac{r^2}{4}\left(1+\frac{4\nu-1}{2l+1}\right)p_{2\nu}(r,D,l+1). $$

One can check that the Taylor coefficients also satisfy this recursion. □

We use this proposition to understand the combinatorial properties of the Rankin–Cohen bracket operators, which is a slight generalization of [13, Theorem 5.5].

Proposition 5.3. *Let $\nu\geq 0$ be a nonnegative integer. For a modular form h of weight $3/2$ on $\Gamma_0(4)$ of the form (4.10), we have*

$$ [h,\theta]_\nu|U_4 = \binom{2\nu}{\nu}\sum_{n\in\mathbb{Z}}\sum_{r\in\mathbb{Z}} p_{2\nu+2}(r,D)c_{4n-r^2}(v/4)q^n = \binom{2\nu}{\nu}\lim_{z\to 0} L_{2\nu}\circ\cdots\circ L_2{}^t\Theta(\tau,z)H(\tau). $$

https://doi.org/10.1017/fms.2024.126 Published online by Cambridge University Press

Proof. By definition, we have

$$[h,\theta]_\nu|U_4 = \sum_{\substack{r,s\geq 0\\ r+s=\nu}} (-1)^r \frac{\Gamma(3/2+\nu)\Gamma(1/2+\nu)}{s!\Gamma(3/2+r)r!\Gamma(1/2+s)} D^r\left(\sum_{n\equiv 0,3\ (4)} c_n(v)q^n\right) D^s\left(\sum_{m\in\mathbb{Z}} q^{m^2}\right)|U_4.$$

A direct calculation implies that

$$D^r\left(\sum_{n\equiv 0,3\ (4)} c_n(v)q^n\right) D^s\left(\sum_{m\in\mathbb{Z}} q^{m^2}\right)|U_4 = \sum_{N\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} m^{2s}(4D-m^2)^r c_{4N-m^2}(v/4)q^N$$

and

$$\sum_{\substack{r,s\geq 0\\ r+s=\nu}} (-1)^r \frac{\Gamma(3/2+\nu)\Gamma(1/2+\nu)}{s!\Gamma(3/2+r)r!\Gamma(1/2+s)} m^{2s}(4D-m^2)^r = \binom{2\nu}{\nu} p_{2\nu+2}(m,D).$$

The last equation immediately follows from Proposition 5.2. □

For each $n \geq 0$ and $\nu \geq 0$, we define

$$\Phi_{n,\nu}(\tau,s) := \lim_{z\to 0} L_{2\nu}\circ\cdots\circ L_2{}^t\Theta(\tau,z)P_{3/2,\rho^*}^{(\alpha,-n^2)}(\tau,s). \tag{5.3}$$

Combining Theorem 4.5 and Lemma 5.3, for $m \geq 1$, we obtain the following key expressions:

$$\mathcal{G}_{m,\nu}(\tau) = -\frac{1}{2\binom{2\nu}{\nu}}\cdot [g_m,\theta]_\nu|U_4 = -\frac{1}{2}\lim_{s\to 3/4}\left(-\frac{\sqrt{\pi}}{2}\sum_{n|m} n\Phi_{n,\nu}(\tau,s) + 2\sigma_1(m)\Phi_{0,\nu}(\tau,s)\right). \tag{5.4}$$

The order of limits of s and z is interchanged, which is justified by the Remark at the end of Section 4.4.

5.2. *The Selberg–Poincaré series*

To prove Theorem 1.4 using (5.4), we must calculate $\Phi_{n,\nu}(\tau,s)$ and $\langle\Phi_{n,\nu}(\cdot,s),f\rangle$ at $s = 3/4$ for Hecke eigenforms f. To this end, we use Selberg's generalization [27] of the Poincaré series in (1.7). For integers $k \geq 2$ and $m \in \mathbb{Z}$, they are defined by

$$P_{k,m}(\tau,s) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} v^s q^m|_k\gamma. \tag{5.5}$$

This series converges absolutely and uniformly on compact subsets for $\mathrm{Re}(s) > 1-k/2$ and admits meromorphic continuation. In particular, it is known that $P_{k,m}(\tau,s)$ is holomorphic at $s = 1-k/2$. This fact follows from comparing it with the Maass–Poincaré series defined by

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathcal{M}_{k,m}(v,s+k/2)\mathbf{e}(mu)|_k\gamma \quad (\mathrm{Re}(s) > 1-k/2).$$

Indeed, from (4.5), we have

$$(4\pi|m|v)^s - \Gamma(2s+k)\mathcal{M}_{k,m}(v,s+k/2) = O(v^{\mathrm{Re}(s)+1}).$$

Thus, for $\mathrm{Re}(s) > -k/2$, the poles of these two types of Poincaré series agree. However, the Fourier expansion of the Maass–Poincaré series (see [15, Theorem 3.2]) and the Weil bound for the Kloosterman sums imply its holomorphy at $s = 1-k/2$.

https://doi.org/10.1017/fms.2024.126 Published online by Cambridge University Press

The next lemma describes the Petersson inner product of cusp forms with these series.

Lemma 5.4. *For $f \in S_k$ and $m > 0$, we have*

$$\langle P_{k,m}(\cdot, s), f\rangle := \int_{\Gamma\backslash\mathbb{H}} P_{k,m}(\tau, s)\overline{f(\tau)} v^k \frac{\mathrm{d}u\mathrm{d}v}{v^2} = \frac{\Gamma(s+k-1)}{(4\pi m)^{s+k-1}} \overline{c_f(m)}.$$

Proof. It follows from the classical unfolding argument (see [4, Ch. 10.1], for instance). □

5.3. *The case of* $n = 0$

Here, we calculate $\langle \Phi_{0,\nu}(\cdot, s), f\rangle$ at $s = 3/4$ for a normalized Hecke eigenform *f*. To this end, we decompose $\Phi_{0,\nu}(\tau, s)$ in terms of the Selberg–Poincaré series.

Proposition 5.5. *We have that*

$$\Phi_{0,\nu}(\tau, s) = 4^{-s+3/4} \sum_{0\le l\le \nu} \frac{(s-3/4)^{\underline{l}}}{(4\pi)^l} \binom{2\nu+1}{2l+1} \sum_{r\in\mathbb{Z}} r^{2\nu-2l} P_{2\nu+2,r^2}\left(\tau, s - \frac{3}{4} - l\right).$$

Proof. By applying (5.1), Lemma 5.1 and Proposition 5.2,

$$\begin{aligned}\Phi_{0,\nu}(\tau, s) &= \lim_{z\to 0} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \sum_{\substack{r\in\mathbb{Z}\\ r\equiv 0\ (2)}} \left(L_{2\nu}\circ\cdots\circ L_2 \left(\frac{v}{4}\right)^{s-3/4} q^{r^2/4}\zeta^r \right)\Bigg|_{2\nu+2,1}\gamma \\ &= 4^{-s+3/4} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \sum_{r\in\mathbb{Z}} p_{2\nu+2}(2r, D) v^{s-3/4} q^{r^2}\Big|_{2\nu+2}\gamma.\end{aligned}$$

The summand is calculated as

$$p_{2\nu+2}(2r, D) v^{s-3/4} q^{r^2} = \sum_{0\le j\le\nu} (-1)^j \binom{2\nu-j}{j} (2r)^{2\nu-2j} D^j \left(v^{s-3/4} q^{r^2}\right).$$

Then the claim follows from the Leibniz rule, where $Dv^{s-3/4} = \frac{-1}{4\pi}\frac{\mathrm{d}}{\mathrm{d}v} v^{s-3/4}$, and the fact that

$$\sum_{l\le j\le\nu} (-1)^j 2^{2(\nu-j)} \binom{2\nu-j}{j}\binom{j}{l} = (-1)^l \binom{2\nu+1}{2l+1}.$$

□

The next result provides a formula for the Petersson norm of a cusp form *f*.

Theorem 5.6. *For a normalized Hecke eigenform $f \in S_{2\nu+2}$, we have*

$$\lim_{s\to 3/4} \langle \Phi_{0,\nu}(\cdot, s), f\rangle = 24\|f\|^2.$$

Proof. First, we note that the Fourier coefficients of a normalized Hecke eigenform are real. By Lemma 5.4 and Proposition 5.5, we find that

$$\langle \Phi_{0,\nu}(\cdot, s), f\rangle = 4^{-s+3/4} \sum_{0\le l\le\nu} \frac{(s-3/4)^{\underline{l}}}{(4\pi)^l} \frac{\Gamma(2\nu+s+1/4-l)}{(4\pi)^{2\nu+s+1/4-l}} \binom{2\nu+1}{2l+1} \cdot 2\sum_{r=1}^{\infty} \frac{c_f(r^2)}{r^{2s+2\nu+1/2}}.$$

https://doi.org/10.1017/fms.2024.126 Published online by Cambridge University Press

As in [10, Lemma 11.12.6], let

$$B(f,s) \coloneqq \sum_{n=1}^{\infty} \frac{c_f(n^2)}{n^s} = \frac{1}{\zeta(2(s-2\nu-1))} L(\mathrm{Sym}^2(f), s)$$

for $f \in S_{2\nu+2}$. Then, it is known that $B(f,s)$ admits the meromorphic continuation to the whole $\mathbb{C}$-plane, and $L(\mathrm{Sym}^2(f), s)$ has no poles (see [10, Remark 11.12.8]). In particular, $B(f, 2s+2\nu+1/2)$ has no pole at $s = 3/4$. Therefore, by [10, Corollary 11.12.7], we have

$$\begin{aligned}\lim_{s\to 3/4} \langle \Phi_{0,\nu}(\cdot, s), f\rangle &= \frac{\Gamma(2\nu+1)}{(4\pi)^{2\nu+1}}(2\nu+1)2B(f, 2\nu+2)\\ &= \frac{2(2\nu+1)!}{(4\pi)^{2\nu+1}}\frac{6}{\pi^2}\frac{\pi}{2}\frac{(4\pi)^{2\nu+2}}{(2\nu+1)!}\langle f, f\rangle\\ &= 24\|f\|^2.\end{aligned}$$

□

5.4. *The cases of* $n > 0$

We turn to the case of positive n. Again, we first decompose $\Phi_{n,\nu}(\tau, s)$.

Proposition 5.7. *For* $n > 0$*, we have*

$$\Phi_{n,\nu}(\tau,s) = \frac{1}{\Gamma(2s)} \sum_{\substack{r\in\mathbb{Z}\\ r\equiv n\ (2)}} \sum_{\substack{i_1,i_2\geq 0\\ i_1+i_2\leq \nu}} \frac{(-1)^{i_1}}{i_1! i_2!}\left(\frac{n^2}{4}\right)^{i_1+i_2} Q_{\nu,i_1+i_2}(n,r)\frac{(s-3/4)^{\underline{i_1}}(s-3/4)^{\overline{i_2}}}{(2s)^{\overline{i_2}}}\widetilde{P}^{i_1,i_2}_{n,r}(\tau,s),$$

where we let

$$\begin{aligned}Q_{\nu,i}(n,r) &\coloneqq \sum_{i\leq j\leq \nu} (-1)^j \binom{2\nu-j}{j} r^{2\nu-2j}\frac{j!}{(j-i)!}\left(\frac{r^2-n^2}{4}\right)^{j-i},\\ \widetilde{P}^{i_1,i_2}_{n,r}(\tau,s) &\coloneqq \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\pi n^2 v)^{-3/4-i_1-i_2/2} M_{-3/4+i_2/2, s-1/2+i_2/2}(\pi n^2 v)\mathbf{e}\left(\frac{r^2-n^2}{4}u\right)e^{-\frac{\pi r^2 v}{2}}\bigg|_{2\nu+2}\gamma.\end{aligned}$$

Proof. Arguing as above, by applying (5.1), Lemma 5.1 and Proposition 5.2, we obtain

$$\begin{aligned}\Phi_{n,\nu}(\tau,s) &= \lim_{z\to 0}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sum_{\substack{r\in\mathbb{Z}\\ r\equiv n\ (2)}}\left(L_{2\nu}\circ\cdots\circ L_2\mathcal{M}_{3/2,-n^2}\left(\frac{v}{4},s\right)\mathbf{e}\left(\frac{-n^2u}{4}\right)q^{r^2/4}\zeta^r\right)\bigg|_{2\nu+2,1}\gamma\\ &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sum_{\substack{r\in\mathbb{Z}\\ r\equiv n\ (2)}} p_{2\nu+2}(r,D)\mathcal{M}_{3/2,-n^2}\left(\frac{v}{4},s\right)\mathbf{e}\left(\frac{-n^2u}{4}\right)q^{r^2/4}\bigg|_{2\nu+2}\gamma.\end{aligned}$$

https://doi.org/10.1017/fms.2024.126 Published online by Cambridge University Press

The summand is calculated as

$$\begin{aligned}
&p_{2\nu+2}(r,D)\mathcal{M}_{3/2,-n^2}\left(\frac{v}{4},s\right)\mathbf{e}\left(\frac{-n^2u}{4}\right)q^{r^2/4}\\
&=\frac{1}{\Gamma(2s)}\sum_{0\le j\le \nu}(-1)^j\binom{2\nu-j}{j}r^{2\nu-2j}D^j\left[(\pi n^2 v)^{s-3/4}\cdot(\pi n^2 v)^{-s}M_{-3/4,s-1/2}(\pi n^2 v)e^{-\frac{\pi n^2 v}{2}}\cdot q^{\frac{r^2-n^2}{4}}\right]\\
&=\frac{1}{\Gamma(2s)}\sum_{0\le j\le \nu}(-1)^j\binom{2\nu-j}{j}r^{2\nu-2j}\\
&\quad\times\sum_{\substack{i_1,i_2,i_3\ge 0\\ i_1+i_2+i_3=j}}\frac{j!}{i_1!i_2!i_3!}D^{i_1}(\pi n^2 v)^{s-3/4}\cdot D^{i_2}\left[(\pi n^2 v)^{-s}M_{-3/4,s-1/2}(\pi n^2 v)e^{-\frac{\pi n^2 v}{2}}\right]\cdot D^{i_3}q^{\frac{r^2-n^2}{4}}.
\end{aligned}$$

Similar to the case of $n=0$, direct calculation utilizing $Df(v)=\frac{-1}{4\pi}\frac{\mathrm{d}}{\mathrm{d}v}f(v)$ yields

$$\begin{aligned}
D^{i_1}(\pi n^2 v)^{s-3/4}&=\left(-\frac{n^2}{4}\right)^{i_1}(s-3/4)^{\underline{i_1}}(\pi n^2 v)^{s-3/4-i_1},\\
D^{i_3}q^{\frac{r^2-n^2}{4}}&=\left(\frac{r^2-n^2}{4}\right)^{i_3}q^{\frac{r^2-n^2}{4}}.
\end{aligned}$$

For the second term, by Lemma 4.1, we find that

$$\begin{aligned}
&D^{i_2}\left[(\pi n^2 v)^{-s}M_{-3/4,s-1/2}(\pi n^2 v)e^{-\frac{\pi n^2 v}{2}}\right]\\
&=\left(\frac{n^2}{4}\right)^{i_2}\frac{(s-3/4)^{\overline{i_2}}}{(2s)^{\overline{i_2}}}e^{-\frac{\pi n^2 v}{2}}(\pi n^2 v)^{-s-i_2/2}M_{-3/4+i_2/2,s-1/2+i_2/2}(\pi n^2 v).
\end{aligned}$$

The claim follows by combining these results. □

We split the sum defining $\Phi_{n,\nu}(\tau,s)$ into $\Phi^+_{n,\nu}(\tau,s)$ and $\Phi^-_{n,\nu}(\tau,s)$, based on the inequalities $r^2>n^2$ or $r^2\le n^2$. We consider them as $s\to 3/4$. By (4.5), the summand of the Poincaré series $\widetilde{P}^{i_1,i_2}_{n,r}(\tau,s)$ satisfies

$$v^{-3/4-i_1-i_2/2}M_{-3/4+i_2/2,s-1/2+i_2/2}(\pi n^2 v)\mathbf{e}\left(\frac{r^2-n^2}{4}u\right)e^{-\frac{\pi r^2 v}{2}}=O(v^{\mathrm{Re}(s)-3/4-i_1})$$

as $v\to 0$. Therefore, for $\mathrm{Re}(s)>-\nu+i_1+3/4$, the Poincaré series is holomorphic (in s). In particular, $\widetilde{P}^{i_1,i_2}_{n,r}(\tau,s)$ is holomorphic at $s=3/4$ for $0\le i_1<\nu$. Regarding the case of $i_1=\nu$, by a similar argument as in Section 5.2 – that is, by comparing it with the Selberg–Poincaré series or the Maass–Poincaré series – we see that it is also holomorphic at $s=3/4$. Therefore, we have

$$\lim_{s\to 3/4}\Phi^-_{n,\nu}(\tau,s)=\frac{1}{\Gamma(3/2)}\sum_{\substack{r\in\mathbb{Z}\\ r\equiv n\ (2)\\ r^2\le n^2}}Q_{\nu,0}(n,r)\widetilde{P}^{0,0}_{n,r}(\tau,3/4).$$

Since $Q_{\nu,0}(n,r)=p_{2\nu+2}(r,(r^2-n^2)/4)$ and $\widetilde{P}^{0,0}_{n,r}(\tau,3/4)=P_{2\nu+2,\frac{r^2-n^2}{4}}(\tau)$, by (2.8), we have

$$\lim_{s\to 3/4}\Phi^-_{n,\nu}(\tau,s)=\frac{4}{n\sqrt{\pi}}\sum_{0<r\le n}r^{2\nu+1}P_{2\nu+2,-r(n-r)}(\tau).\tag{5.6}$$

https://doi.org/10.1017/fms.2024.126 Published online by Cambridge University Press

As a counterpart to Theorem 5.6, the Petersson inner product of $\Phi^+_{n,\nu}(\tau, s)$ with a Hecke eigenform is expressed in terms of the symmetrized shifted convolution L-functions.

Theorem 5.8. *For a normalized Hecke eigenform $f \in S_{2\nu+2}$, we have*

$$\lim_{s\to 3/4} \langle \Phi^+_{n,\nu}(\cdot, s), f\rangle = \frac{4}{n\sqrt{\pi}} \frac{\Gamma(2\nu+1)}{(4\pi)^{2\nu+1}} \sum_{d|n} \mu(d) \widehat{L}(f, n/d; 2\nu+1).$$

Proof. By Proposition 5.7, we have

$$\begin{aligned}
&\langle \Phi^+_{n,\nu}(\cdot, s), f\rangle \\
&= \frac{1}{\Gamma(2s)} \sum_{\substack{r\in\mathbb{Z} \\ r\equiv n\ (2) \\ r^2>n^2}} \sum_{\substack{i_1,i_2\geq 0 \\ i_1+i_2\leq \nu}} \frac{(-1)^{i_1}}{i_1! i_2!} \left(\frac{n^2}{4}\right)^{i_1+i_2} Q_{\nu,i_1+i_2}(n,r) \frac{(s-3/4)\underline{^{i_1}}(s-3/4)^{\overline{i_2}}}{(2s)^{\overline{i_2}}} \langle \widetilde{P}^{i_1,i_2}_{n,r}(\cdot, s), f\rangle.
\end{aligned}$$

The unfolding argument, combined with Lemma 4.2, gives

$$\begin{aligned}
&(\pi n^2)^{\frac{3}{4}+i_1+\frac{i_2}{2}} \langle \widetilde{P}^{i_1,i_2}_{n,r}(\cdot, s), f\rangle \\
&= \sum_{m=1}^{\infty} c_f(m) \int_0^\infty \int_0^1 v^{2\nu-\frac{3}{4}-i_1-\frac{i_2}{2}} M_{-\frac{3}{4}+\frac{i_2}{2}, s-\frac{1}{2}+\frac{i_2}{2}}(\pi n^2 v) \mathbf{e}\left(\left(\frac{r^2-n^2}{4} - m\right)u\right) e^{-2\pi\left(\frac{r^2}{4}+m\right)v} \mathrm{d}u\mathrm{d}v \\
&= c_f\left(\frac{r^2-n^2}{4}\right) \int_0^\infty e^{-\pi\left(r^2-\frac{n^2}{2}\right)v} v^{2\nu-\frac{3}{4}-i_1-\frac{i_2}{2}} M_{-\frac{3}{4}+\frac{i_2}{2}, s-\frac{1}{2}+\frac{i_2}{2}}(\pi n^2 v)\mathrm{d}v \\
&= c_f\left(\frac{r^2-n^2}{4}\right) \frac{(\pi n^2)^{s+\frac{i_2}{2}} \Gamma\left(s+2\nu+\frac{1}{4}-i_1\right)}{(\pi r^2)^{s+2\nu+\frac{1}{4}-i_1}} \cdot {}_2F_1\left(s+\frac{3}{4}, s+2\nu+\frac{1}{4}-i_1; 2s+i_2; \frac{n^2}{r^2}\right).
\end{aligned}$$

By changing variables $r = 2m+n$ for $r > n$ and $r = -2m-n$ for $r < -n$, we have

$$\begin{aligned}
&\langle \Phi^+_{n,\nu}(\cdot, s), f\rangle \\
&= \frac{2}{\Gamma(2s)} \sum_{\substack{i_1,i_2\geq 0 \\ i_1+i_2\leq \nu}} \frac{(-1)^{i_1}}{i_1! i_2!} \left(\frac{n^2}{4}\right)^{i_1+i_2} \frac{(s-3/4)\underline{^{i_1}}(s-3/4)^{\overline{i_2}}}{(2s)^{\overline{i_2}}} (\pi n^2)^{s-\frac{3}{4}-i_1} \Gamma\left(s+2\nu+\frac{1}{4}-i_1\right) \\
&\times \sum_{m=1}^{\infty} \frac{Q_{\nu,i_1+i_2}(n, 2m+n) c_f(m(m+n))}{(\pi(2m+n)^2)^{s+2\nu+\frac{1}{4}-i_1}} {}_2F_1\left(s+\frac{3}{4}, s+2\nu+\frac{1}{4}-i_1; 2s+i_2; \frac{n^2}{(2m+n)^2}\right),
\end{aligned}$$

where we note that $Q_{\nu,i}(n, -r) = Q_{\nu,i}(n, r)$ holds. For a normalized Hecke eigenform $f \in S_{2\nu+2}$, since

$$c_f(m(m+n)) = \sum_{d|(m,m+n)} \mu(d) d^{2\nu+1} c_f\left(\frac{m}{d}\right) c_f\left(\frac{m+n}{d}\right),$$

the last sum becomes

$$\begin{aligned}
&\sum_{d|n} \mu(d) d^{2\nu+1} \sum_{m=1}^{\infty} \frac{Q_{\nu,i_1+i_2}(n, 2dm+n) c_f(m) c_f(m+n/d)}{(\pi(2dm+n)^2)^{s+2\nu+\frac{1}{4}-i_1}} \\
&\times {}_2F_1\left(s+\frac{3}{4}, s+2\nu+\frac{1}{4}-i_1; 2s+i_2; \frac{n^2}{(2dm+n)^2}\right).
\end{aligned}$$

https://doi.org/10.1017/fms.2024.126 Published online by Cambridge University Press

Then, this Dirichlet series is holomorphic at $s = 3/4$. Indeed, since $Q_{\nu,i_1+i_2}(n, 2dm + n)$ has degree $2(\nu - i_1 - i_2)$ in m, it suffices to show that

$$\sum_{m=1}^{\infty} \frac{c_f(m)c_f(m+n/d)}{m^{2(s+\nu+1/4+i_2)}}$$

(conditionally) converges at $s = 3/4$. This can be seen by partial summation using the estimate

$$\sum_{1\le m\le x} c_f(m)c_f(m+n/d) \ll x^{2\nu+2-\delta},$$

with some $\delta > 0$, (see [2, Corollary 1.4]). Therefore, all terms corresponding to nonzero (i_1, i_2) vanish as $s \to 3/4$, and we obtain

$$\lim_{s\to 3/4} \langle \Phi^+_{n,\nu}(\cdot, s), f\rangle = \frac{4}{\sqrt{\pi}}\Gamma(2\nu+1)\sum_{d|n}\mu(d)d^{2\nu+1} \times \sum_{m=1}^{\infty} \frac{Q_{\nu,0}(n, 2dm+n)c_f(m)c_f(m+n/d)}{(\pi(2dm+n)^2)^{2\nu+1}}\, {}_2F_1\left(\frac{3}{2}, 2\nu+1; \frac{3}{2}; \frac{n^2}{(2dm+n)^2}\right).$$

Since we have

$$\frac{1}{r^{2(2\nu+1)}}\, {}_2F_1\left(\frac{3}{2}, 2\nu+1; \frac{3}{2}; \frac{n^2}{r^2}\right) = \frac{1}{(r^2-n^2)^{2\nu+1}}$$

and $Q_{\nu,0}(n, r) = p_{2\nu+2}(r, (r^2 - n^2)/4)$ with (2.8) again, the proof is complete as

$$\lim_{s\to 3/4} \langle \Phi^+_{n,\nu}(\cdot, s), f\rangle = \frac{4}{n\sqrt{\pi}}\frac{\Gamma(2\nu+1)}{(4\pi)^{2\nu+1}}\sum_{d|n}\mu(d)\sum_{m=1}^{\infty} c_f(m)c_f(m+n/d)\left(\frac{1}{m^{2\nu+1}} - \frac{1}{(m+n/d)^{2\nu+1}}\right).$$

□

5.5. *Proof of Theorem 1.4*

We apply the results from the two previous subsections. For $m > 0$ and $\nu \ge 0$, let

$$\Phi(\tau, s) := \frac{\sqrt{\pi}}{4}\sum_{n|m} n\Phi_{n,\nu}(\tau, s) - \sigma_1(m)\Phi_{0,\nu}(\tau, s).$$

As stated in (5.4), we have

$$\lim_{s\to 3/4} \Phi(\tau, s) = \mathcal{G}_{m,\nu}(\tau).$$

However, from (5.6), the minus part

$$\lim_{s\to 3/4} \Phi^-(\tau, s) := \lim_{s\to 3/4} \frac{\sqrt{\pi}}{4}\sum_{n|m} n\Phi^-_{n,\nu}(\tau, s) = \sum_{n|m}\sum_{0<r\le n} r^{2\nu+1}P_{2\nu+2,-r(n-r)}(\tau).$$

https://doi.org/10.1017/fms.2024.126 Published online by Cambridge University Press

For the plus part, by Theorem 5.6 and Theorem 5.8 and the Möbius inversion formula, we have

$$\begin{aligned}\lim_{s\to 3/4}\langle \Phi^{+}(\cdot,s),f\rangle &= \sum_{n|m}\frac{\Gamma(2\nu+1)}{(4\pi)^{2\nu+1}}\sum_{d|n}\mu(d)\widehat{L}(f,n/d;2\nu+1)-24\sigma_1(m)\|f\|^2\\ &= \frac{\Gamma(2\nu+1)}{(4\pi)^{2\nu+1}}\widehat{L}(f,m;2\nu+1)-24\sigma_1(m)\|f\|^2.\end{aligned}$$

Combining these facts, we are pleased to obtain the conclusion of the theorem

$$\lim_{s\to 3/4}\Phi(\tau,s)=\sum_{n|m}\sum_{0<r\le n} r^{2\nu+1}P_{2\nu+2,-r(n-r)}(\tau)-\sum_{j=1}^{d_{2\nu+2}}\left(24\sigma_1(m)-\frac{\Gamma(2\nu+1)}{(4\pi)^{2\nu+1}}\frac{\widehat{L}(f,m;2\nu+1)}{\|f_j\|^2}\right)f_j.$$

Acknowledgements. The authors thank Masanobu Kaneko and Gyucheol Shin for comments that improved this paper.

Competing interest. The authors have no competing interest to declare.

Funding statement. This work was supported by the MEXT Initiative for Realizing Diversity in the Research Environment through Kyushu University's Diversity and Super Global Training Program for Female and Young Faculty (SENTAN-Q). The second author was supported by JSPS KAKENHI (JP20K14292, JP21K18141 and JP24K16901). The third author thanks the Thomas Jefferson Fund and grants from the NSF (DMS-2002265 and DMS-2055118).

References

[1] C. Alfes, 'Formulas for the coefficients of half-integral weight harmonic Maaß forms', *Math. Z.* **277**(3–4), (2014), 769–795.
[2] V. Blomer, 'Shifted convolution sums and subconvexity bounds for automorphic L-functions', *Int. Math. Res. Not.* (**73**) (2004), 3905–3926.
[3] R. E. Borcherds, 'Automorphic forms with singularities on Grassmannians', *Invent. Math.* **132**(3) (1998), 491–562.
[4] K. Bringmann, A. Folsom, K. Ono and L. Rolen, *Harmonic Maass Forms and Mock Modular Forms: Theory and Applications* (American Mathematical Society Colloquium Publications) vol. 64 (American Mathematical Society, Providence, RI, 2017).
[5] K. Bringmann and K. Ono, 'Arithmetic properties of coefficients of half-integral weight Maass-Poincaré series', *Math. Ann.* **337**(3) (2007), 591–612.
[6] J. H. Bruinier, *Borcherds Products on $O(2,l)$ and Chern Classes of Heegner Divisors* (Lecture Notes in Mathematics) vol. 1780 (Springer-Verlag, Berlin, 2002).
[7] J. H. Bruinier and J. Funke, 'On two geometric theta lifts', *Duke Math. J.* **125**(1) (2004), 45–90.
[8] J. H. Bruinier, P. Jenkins and K. Ono, 'Hilbert class polynomials and traces of singular moduli', *Math. Ann.* **334**(2) (2006), 373–393.
[9] J. H. Bruinier and K. Ono, 'Heegner divisors, L-functions and harmonic weak Maass forms', *Ann. of Math. (2)* **172**(3) (2010), 2135–2181.
[10] H. Cohen and F. Strömberg, *Modular Forms* (Graduate Studies in Mathematics) vol. 179 (American Mathematical Society, Providence, RI, 2017). A classical approach.
[11] W. Duke, 'Modular functions and the uniform distribution of CM points', *Math. Ann.* **334**(2) (2006), 241–252.
[12] W. Duke, Ö. Imamoḡlu, and Á. Tóth, 'Cycle integrals of the j-function and mock modular forms', *Ann. of Math. (2)* **173**(2) (2011), 947–981.
[13] M. Eichler and D. Zagier, *The Theory of Jacobi Forms* (Progress in Mathematics) vol. 55 (Birkhäuser Boston, Inc., Boston, MA, 1985).
[14] F. Hirzebruch and D. Zagier, 'Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus', *Invent. Math.* **36** (1976), 57–113.
[15] D. Jeon, S.-Y. Kang and C. H. Kim, 'Weak Maass-Poincaré series and weight 3/2 mock modular forms', *J. Number Theory* **133**(8) (2013), 2567–2587.
[16] M. Kaneko, 'The Fourier coefficients and the singular moduli of the elliptic modular function $j(\tau)$', *Mem. Fac. Engrg. Design Kyoto Inst. Tech. Ser. Sci. Tech.* **44** (1996), 1–5.
[17] W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Die Grundlehren der mathematischen Wissenschaften) Band 52, enlarged edn. (Springer-Verlag New York, Inc., New York, 1966).
[18] T. Matsusaka, 'The Fourier coefficients of the McKay-Thompson series and the traces of CM values', *Res. Number Theory* **3** (2017), Paper No. 23, 16.

https://doi.org/10.1017/fms.2024.126 Published online by Cambridge University Press

[19] T. Matsusaka and R. Osanai, 'Arithmetic formulas for the Fourier coefficients of Hauptmoduln of level 2, 3, and 5', *Proc. Amer. Math. Soc.* **145**(4) (2017), 1383–1392.
[20] M. H. Mertens,'Mock modular forms and class number relations', *Res. Math. Sci.* **1** (2014), Art. 6.
[21] M. H. Mertens, 'Mock modular forms and class numbers of quadratic forms', PhD thesis, Universität zu Köln, 2014.
[22] M. H. Mertens and K. Ono, 'Special values of shifted convolution Dirichlet series', *Mathematika* **62**(1) (2016), 47–66.
[23] D. Niebur, 'A class of nonanalytic automorphic functions', *Nagoya Math. J.* **52** (1973), 133–145.
[24] NIST, Digital Library of Mathematical Functions, https://dlmf.nist.gov.
[25] K. Ohta, 'Formulas for the Fourier coefficients of some genus zero modular functions', *Kyushu J. Math.* **63**(1) (2009), 1–15.
[26] K. Ono and H. Saad, 'Some Eichler-Selberg trace formulas', *Hardy-Ramanujan J.* **45** (2022), 94–107.
[27] A. Selberg, 'On the estimation of Fourier coefficients of modular forms', in *Proc. Sympos. Pure Math., Vol. VIII* (Amer. Math. Soc., Providence, RI, 1965), 1–15.
[28] J.-P. Serre, *A Course in Arithmetic* (Graduate Texts in Mathematics) no. 7 (Springer-Verlag, New York-Heidelberg, 1973). Translated from the French.
[29] T. Shintani, 'On construction of holomorphic cusp forms of half integral weight', *Nagoya Math. J.* **58** (1975), 83–126.
[30] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis. An Introduction to the General Theory of Infinite Processes and of Analytic Functions: With an Account of the Principal Transcendental Functions*, fourth edn. (Cambridge University Press, New York, 1962).
[31] B. Williams, 'Vector-valued Eisenstein series of small weight', *Int. J. Number Theory* **15**(2) (2019), 265–287.
[32] D. Zagier, 'Modular forms of one variable', Notes based on a course given in Utrecht, Spring 1991, available in https://people.mpim-bonn.mpg.de/zagier/files/tex/UtrechtLectures/UtBook.pdf.
[33] D. Zagier, 'Nombres de classes et formes modulaires de poids 3/2', *C. R. Acad. Sci. Paris Sér. A-B* **281**(21) (1975), Ai, A883–A886.
[34] D. Zagier, 'Traces of singular moduli', in *Motives, Polylogarithms and Hodge Theory, Part I (Irvine, CA, 1998)* (Int. Press Lect. Ser.) vol. 3, (Int. Press, Somerville, MA, 2002), 211–244.

https://doi.org/10.1017/fms.2024.126 Published online by Cambridge University Press